IB Diploma

MATHEMATICS AA

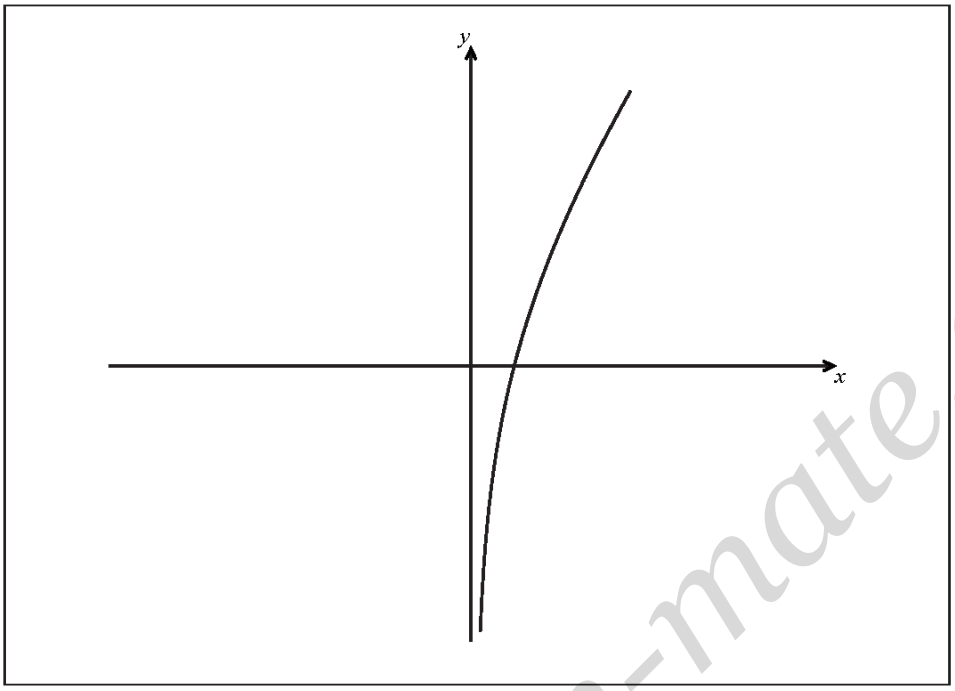
HL P1

2012 — 2022

1 - (MAT/11_HL_Summer_2012_Q4) - *Graphs, Functions - Roots*

The graph below shows $y = f(x)$, where $f(x) = x + \ln x$.

(a) On the graph below, sketch the curve $y = f^{-1}(x)$. *[2 marks]*



(b) Find the coordinates of the point of intersection of the graph of $y = f(x)$ and the graph of $y = f^{-1}(x)$. *[4 marks]*

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2 – (MAT/12_HL_Summer_2012_Q10) – *Functions - Roots, Graphs, Integration*

The function f is defined on the domain $\left[0, \frac{3\pi}{2}\right]$ by $f(x) = e^{-x}\cos x$.

(a) State the two zeros of f. *[1 mark]*

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(b) Sketch the graph of f. *[1 mark]*

(c) The region bounded by the graph, the x-axis and the y-axis is denoted by A and the region bounded by the graph and the x-axis is denoted by B. Show that the ratio of the area of A to the area of B is

$$\frac{e^{\pi}\left(e^{\frac{\pi}{2}}+1\right)}{e^{\pi}+1}.$$

[7 marks]

3 – (MAT/12_HL_Summer_2012_Q11) – *Functions - Roots, Probability*

Consider the following functions:

$$f(x) = \frac{2x^2+3}{75},\ x \geq 0$$

$$g(x) = \frac{|3x-4|}{10},\ x \in \mathbb{R}.$$

(a) State the range of f and of g. *[2 marks]*

(b) Find an expression for the composite function $f \circ g(x)$ in the form $\frac{ax^2+bx+c}{3750}$, where a, b and $c \in \mathbb{Z}$. *[4 marks]*

(c) (i) Find an expression for the inverse function $f^{-1}(x)$.

(ii) State the domain and range of f^{-1}. *[4 marks]*

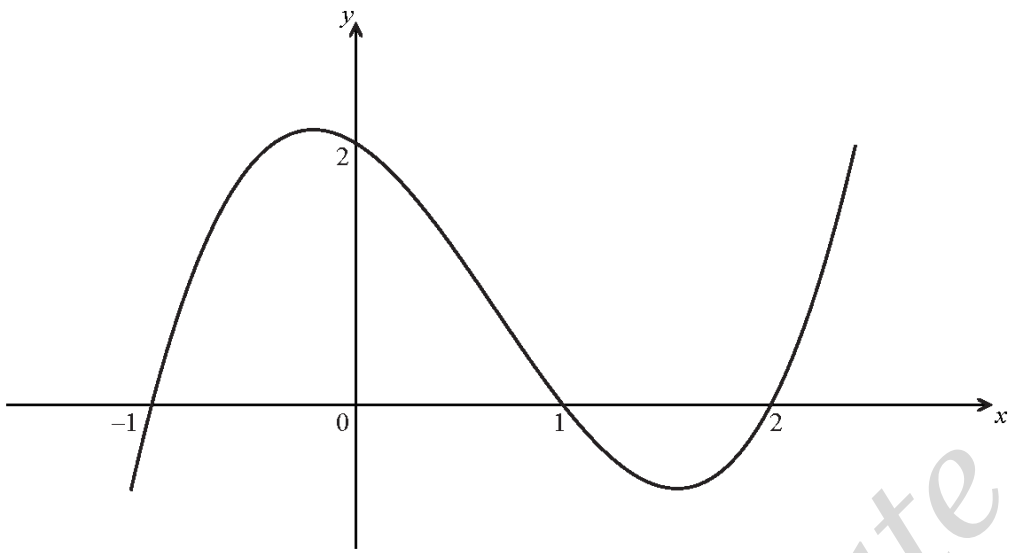
The domains of f and g are now restricted to $\{0, 1, 2, 3, 4\}$.

(d) By considering the values of f and g on this new domain, determine which of f and g could be used to find a probability distribution for a discrete random variable X, stating your reasons clearly. *[6 marks]*

(e) Using this probability distribution, calculate the mean of X. *[2 marks]*

4 – (MAT/10_HL_Winter_2012_Q3) – *Functions - Roots, Graphs*

Let $f(x) = x^3 + ax^2 + bx + c$, where a, b, $c \in \mathbb{Z}$. The diagram shows the graph of $y = f(x)$.



(a) Using the information shown in the diagram, find the values of a, b and c. *[4 marks]*

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(b) If $g(x) = 3f(x-2)$,

(i) state the coordinates of the points where the graph of g intercepts the x-axis.

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(ii) Find the y-intercept of the graph of g. *[3 marks]*

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5 - (MAT/10_HL_Winter_2012_Q12) - *Functions - Roots, Mathematical Induction*

Consider a function f, defined by $f(x)=\dfrac{x}{2-x}$ for $0 \le x \le 1$.

(a) Find an expression for $(f \circ f)(x)$. *[3 marks]*

Let $F_n(x)=\dfrac{x}{2^n-(2^n-1)x}$, where $0 \le x \le 1$.

(b) Use mathematical induction to show that for any $n \in \mathbb{Z}^+$

$$\underbrace{(f \circ f \circ \ldots \circ f)}_{n \text{ times}}(x)=F_n(x).$$ *[8 marks]*

(c) Show that $F_{-n}(x)$ is an expression for the inverse of F_n. *[6 marks]*

(d) (i) State $F_n(0)$ and $F_n(1)$.

(ii) Show that $F_n(x)<x$, given $0<x<1$, $n \in \mathbb{Z}^+$.

(iii) For $n \in \mathbb{Z}^+$, let A_n be the area of the region enclosed by the graph of F_n^{-1}, the x-axis and the line $x=1$. Find the area B_n of the region enclosed by F_n and F_n^{-1} in terms of A_n. *[6 marks]*

6 – (MAT/12_HL_Summer_2013_Q12) – *Functions - Roots, Graphs*

The function f is defined by $f(x)=\dfrac{2x-1}{x+2}$, with domain $D=\{x:-1\le x\le 8\}$.

(a) Express $f(x)$ in the form $A+\dfrac{B}{x+2}$, where A and $B\in\mathbb{Z}$. *[2 marks]*

(b) Hence show that $f'(x)>0$ on D. *[2 marks]*

(c) State the range of f. *[2 marks]*

(d) (i) Find an expression for $f^{-1}(x)$.

(ii) Sketch the graph of $y=f(x)$, showing the points of intersection with both axes.

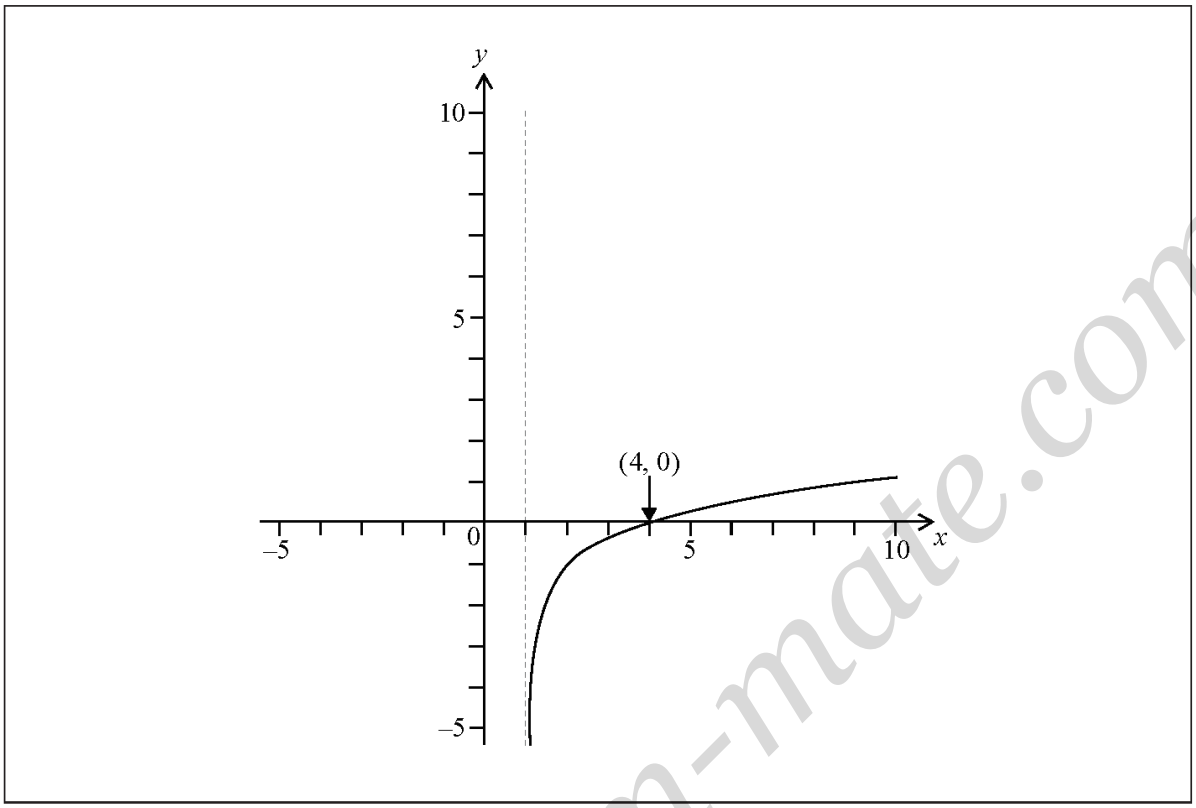
(iii) On the same diagram, sketch the graph of $y=f^{-1}(x)$. *[8 marks]*

(e) (i) On a different diagram, sketch the graph of $y=f(|x|)$ where $x\in D$.

(ii) Find all solutions of the equation $f(|x|)=-\dfrac{1}{4}$. *[7 marks]*

7 – (MAT/10_HL_Winter_2013_Q3) – *Functions - Roots, Graphs*

The diagram below shows a sketch of the graph of $y = f(x)$.



(a) Sketch the graph of $y = f^{-1}(x)$ on the same axes. *[2]*

(b) State the range of f^{-1}. *[1]*

(c) Given that $f(x) = \ln(ax + b)$, $x > 1$, find the value of a and the value of b. *[4]*

8 – (MAT/11_HL_Summer_2014_Q4) – *Functions - Roots*

[Maximum mark: 6]

The equation $5x^3 + 48x^2 + 100x + 2 = a$ has roots r_1, r_2 and r_3.
Given that $r_1 + r_2 + r_3 + r_1 r_2 r_3 = 0$, find the value of a.

9 – (MAT/12_HL_Summer_2014_Q8) – *Functions - Roots*

The function f is defined by

$$f(x)=\begin{cases} 1-2x, & x\le 2 \\ \frac{3}{4}(x-2)^2-3, & x>2 \end{cases}$$

(a) Determine whether or not f is continuous. *[2]*

The graph of the function g is obtained by applying the following transformations to the graph of f:

a reflection in the y–axis followed by a translation by the vector $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

(b) Find $g(x)$. *[4]*

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10 – (MAT/12_HL_Summer_2014_Q14) – *Functions - Roots*

Consider the following functions:

$$h(x) = \arctan(x), \quad x \in \mathbb{R}$$
$$g(x) = \frac{1}{x}, \quad x \in \mathbb{R}, \quad x \neq 0$$

(a) Sketch the graph of $y = h(x)$. *[2]*

(b) Find an expression for the composite function $h \circ g(x)$ and state its domain. *[2]*

Given that $f(x) = h(x) + h \circ g(x)$,

(c) (i) find $f'(x)$ in simplified form;

(ii) show that $f(x) = \frac{\pi}{2}$ for $x > 0$. *[7]*

Nigel states that f is an odd function and Tom argues that f is an even function.

(d) (i) State who is correct and justify your answer.

(ii) Hence find the value of $f(x)$ for $x < 0$. *[3]*

11 - (MAT/10_HL_Winter_2014_Q1) - *Functions - Roots, Graphs*

The function f is defined by $f(x)=\frac{1}{x}$, $x \neq 0$.

The graph of the function $y=g(x)$ is obtained by applying the following transformations to the graph of $y=f(x)$:

a translation by the vector $\begin{pmatrix}-3\\0\end{pmatrix}$;

a translation by the vector $\begin{pmatrix}0\\1\end{pmatrix}$.

(a) Find an expression for $g(x)$. *[2]*

(b) State the equations of the asymptotes of the graph of g. *[2]*

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12 – (MAT/10_HL_Winter_2014_Q11) – *Functions - Roots, Differentiation, Integration*

The function f is defined as $f(x) = \mathrm{e}^{3x+1}$, $x \in \mathbb{R}$.

(a) (i) Find $f^{-1}(x)$.

(ii) State the domain of f^{-1}. *[4]*

The function g is defined as $g(x) = \ln x$, $x \in \mathbb{R}^+$.
The graph of $y = g(x)$ and the graph of $y = f^{-1}(x)$ intersect at the point P.

(b) Find the coordinates of P. *[5]*

The graph of $y = g(x)$ intersects the x-axis at the point Q.

(c) Show that the equation of the tangent T to the graph of $y = g(x)$ at the point Q is $y = x - 1$. *[3]*

A region R is bounded by the graphs of $y = g(x)$, the tangent T and the line $x = \mathrm{e}$.

(d) Find the area of the region R. *[5]*

(e) (i) Show that $g(x) \le x - 1$, $x \in \mathbb{R}^+$.

(ii) By replacing x with $\frac{1}{x}$ in part (e)(i), show that $\frac{x-1}{x} \le g(x)$, $x \in \mathbb{R}^+$. *[6]*

13 - (MAT/11_HL_Summer_2015_Q5) - *Functions - Roots*

The functions f and g are defined by $f(x) = ax^2 + bx + c$, $x \in \mathbb{R}$ and $g(x) = p\sin x + qx + r$, $x \in \mathbb{R}$ where a, b, c, p, q, r are real constants.

(a) Given that f is an even function, show that $b = 0$. [2]

(b) Given that g is an odd function, find the value of r. [2]

The function h is both odd and even, with domain $\mathbb{R}$.

(c) Find $h(x)$. [2]

14 – (MAT/11_HL_Summer_2015_Q6) – *Functions - Roots, Integration*

A function f is defined by $f(x)=\dfrac{3x-2}{2x-1}$, $x\in\mathbb{R}$, $x\neq\dfrac{1}{2}$.

(a) Find an expression for $f^{-1}(x)$. [4]

(b) Given that $f(x)$ can be written in the form $f(x)=A+\dfrac{B}{2x-1}$, find the values of the constants A and B. [2]

(c) Hence, write down $\int\dfrac{3x-2}{2x-1}\,\mathrm{d}x$. [1]

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15 - (MAT/11_HL_Summer_2015_Q7) - *Functions - Roots*

Let $p(x) = 2x^5 + x^4 - 26x^3 - 13x^2 + 72x + 36$, $x \in \mathbb{R}$.

(a) For the polynomial equation $p(x) = 0$, state

(i) the sum of the roots;

(ii) the product of the roots. [3]

A new polynomial is defined by $q(x) = p(x+4)$.

(b) Find the sum of the roots of the equation $q(x) = 0$. [2]

16 - (MAT/11_HL_Summer_2015_Q9) - *Functions - Roots*

The functions f and g are defined by $f(x) = 2x + \frac{\pi}{5}$, $x \in \mathbb{R}$ and $g(x) = 3\sin x + 4$, $x \in \mathbb{R}$.

(a) Show that $g \circ f(x) = 3\sin\left(2x + \frac{\pi}{5}\right) + 4$. [1]

(b) Find the range of $g \circ f$. [2]

(c) Given that $g \circ f\left(\frac{3\pi}{20}\right) = 7$, find the next value of x, greater than $\frac{3\pi}{20}$, for which $g \circ f(x) = 7$. [2]

(d) The graph of $y = g \circ f(x)$ can be obtained by applying four transformations to the graph of $y = \sin x$. State what the four transformations represent geometrically and give the order in which they are applied. [4]

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17 – (MAT/12_HL_Summer_2015_Q10) – *Functions - Roots*

The function f is defined by $f(x)=\dfrac{3x}{x-2}$, $x\in\mathbb{R}$, $x\neq 2$.

(a) Sketch the graph of $y=f(x)$, indicating clearly any asymptotes and points of intersection with the x and y axes. [4]

(b) Find an expression for $f^{-1}(x)$. [4]

(c) Find all values of x for which $f(x)=f^{-1}(x)$. [3]

(d) Solve the inequality $|f(x)|<\dfrac{3}{2}$. [4]

(e) Solve the inequality $f(|x|)<\dfrac{3}{2}$. [2]

18 – (MAT/12_HL_Summer_2015_Q11) – *Functions - Roots, Differentiation, Integration*

Consider the functions $f(x) = \tan x,\ 0 \le x < \frac{\pi}{2}$ and $g(x) = \frac{x+1}{x-1},\ x \in \mathbb{R},\ x \ne 1$.

(a) Find an expression for $g \circ f(x)$, stating its domain. [2]

(b) Hence show that $g \circ f(x) = \frac{\sin x + \cos x}{\sin x - \cos x}$. [2]

(c) Let $y = g \circ f(x)$, find an exact value for $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point on the graph of $y = g \circ f(x)$ where $x = \frac{\pi}{6}$, expressing your answer in the form $a + b\sqrt{3},\ a,\ b \in \mathbb{Z}$. [6]

(d) Show that the area bounded by the graph of $y = g \circ f(x)$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{6}$ is $\ln\left(1 + \sqrt{3}\right)$. [6]

19 – (MAT/12_HL_Summer_2015_Q12) – *Functions - Roots*

The cubic equation $x^3 + px^2 + qx + c = 0$, has roots α, β, γ. By expanding $(x-\alpha)(x-\beta)(x-\gamma)$ show that

(a) (i) $p = -(\alpha + \beta + \gamma)$;

(ii) $q = \alpha\beta + \beta\gamma + \gamma\alpha$;

(iii) $c = -\alpha\beta\gamma$. [3]

It is now given that $p = -6$ and $q = 18$ for parts (b) and (c) below.

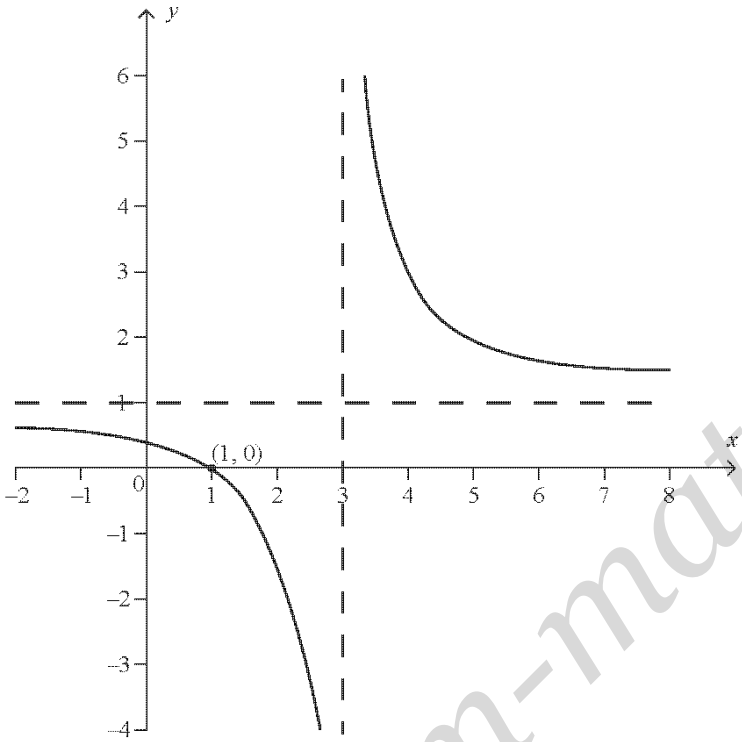
(b) (i) In the case that the three roots α, β, γ form an arithmetic sequence, show that one of the roots is 2.

(ii) Hence determine the value of c. [5]

(c) In another case the three roots α, β, γ form a geometric sequence. Determine the value of c. [6]

20 – (MAT/10_HL_Winter_2016_Q3) – *Functions - Roots*

A rational function is defined by $f(x) = a + \dfrac{b}{x-c}$ where the parameters $a, b, c \in \mathbb{Z}$ and $x \in \mathbb{R} \setminus \{c\}$. The following diagram represents the graph of $y = f(x)$.



Using the information on the graph,

(a) state the value of a and the value of c; [2]

(b) find the value of b. [2]

21 – (MAT/10_HL_Winter_2016_Q5) – *Functions - Roots*

[Maximum mark: 6]

The quadratic equation $x^2 - 2kx + (k-1) = 0$ has roots α and β such that $\alpha^2 + \beta^2 = 4$.
Without solving the equation, find the possible values of the real number k.

22 – (MAT/12_HL_Summer_2017_Q2) – *Functions - Roots*

The function f is defined by $f(x) = 2x^3 + 5$, $-2 \le x \le 2$.

(a) Write down the range of f. [2]

(b) Find an expression for $f^{-1}(x)$. [2]

(c) Write down the domain and range of f^{-1}. [2]

23 – (MAT/12_HL_Summer_2017_Q9) – *Functions - Roots, Differentiation, Integration*

Consider the function f defined by $f(x) = x^2 - a^2,\ x \in \mathbb{R}$ where a is a positive constant.

(a) Showing any x and y intercepts, any maximum or minimum points and any asymptotes, sketch the following curves on separate axes.

(i) $y = f(x)$;

(ii) $y = \dfrac{1}{f(x)}$;

(iii) $y = \left|\dfrac{1}{f(x)}\right|$. [8]

(b) Find $\int f(x)\cos x\,\mathrm{d}x$. [5]

The function g is defined by $g(x) = x\sqrt{f(x)}$ for $|x| > a$.

(c) By finding $g'(x)$ explain why g is an increasing function. [4]

24 – (MAT/11_HL_Summer_2017_Q11) – *Functions - Roots, Graphs, Integration*

(a) (i) Express x^2+3x+2 in the form $(x+h)^2+k$.

(ii) Factorize x^2+3x+2. [2]

Consider the function $f(x)=\dfrac{1}{x^2+3x+2}$, $x\in\mathbb{R}$, $x\neq-2$, $x\neq-1$.

(b) Sketch the graph of $f(x)$, indicating on it the equations of the asymptotes, the coordinates of the y-intercept and the local maximum. [5]

(c) Show that $\dfrac{1}{x+1}-\dfrac{1}{x+2}=\dfrac{1}{x^2+3x+2}$. [1]

(d) Hence find the value of p if $\int_0^1 f(x)\,\mathrm{d}x=\ln(p)$. [4]

(e) Sketch the graph of $y=f(|x|)$. [2]

(f) Determine the area of the region enclosed between the graph of $y=f(|x|)$, the x-axis and the lines with equations $x=-1$ and $x=1$. [3]

25 – (MAT/11_HL_Summer_2017_Q12) – *Functions - Roots, Graphs*

Consider the polynomial $P(z) = z^5 - 10z^2 + 15z - 6$, $z \in \mathbb{C}$.

(a) Write down the sum and the product of the roots of $P(z) = 0$. [2]

(b) Show that $(z - 1)$ is a factor of $P(z)$. [2]

The polynomial can be written in the form $P(z) = (z - 1)^3\left(z^2 + bz + c\right)$.

(c) Find the value of b and the value of c. [5]

(d) Hence find the complex roots of $P(z) = 0$. [3]

Consider the function $q(x) = x^5 - 10x^2 + 15x - 6$, $x \in \mathbb{R}$.

(e) (i) Show that the graph of $y = q(x)$ is concave up for $x > 1$.

(ii) Sketch the graph of $y = q(x)$ showing clearly any intercepts with the axes. [6]

26 – (MAT/12_HL_Summer_2018_Q10) – *Functions - Roots*

The function f is defined by $f(x) = \dfrac{ax + b}{cx + d}$, for $x \in \mathbb{R}, x \neq -\dfrac{d}{c}$.

(a) Find the inverse function f^{-1}, stating its domain. [5]

The function g is defined by $g(x) = \dfrac{2x - 3}{x - 2}$, $x \in \mathbb{R}, x \neq 2$.

(b) (i) Express $g(x)$ in the form $A + \dfrac{B}{x - 2}$ where A, B are constants.

(ii) Sketch the graph of $y = g(x)$. State the equations of any asymptotes and the coordinates of any intercepts with the axes. [5]

The function h is defined by $h(x) = \sqrt{x}$, for $x \geq 0$.

(c) State the domain and range of $h \circ g$. [4]

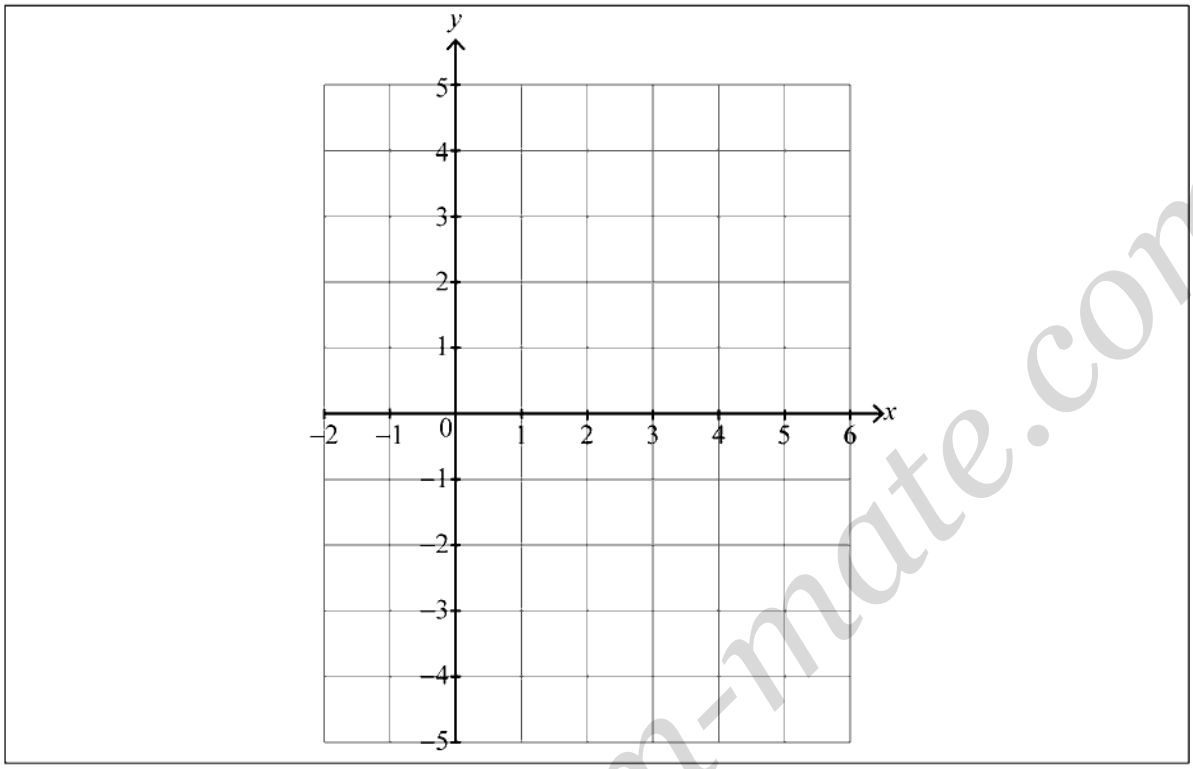
27 – (MAT/12_HL_Summer_2019_Q3) – *Functions - Roots*

[Maximum mark: 5]

Consider the function $f(x) = x^4 - 6x^2 - 2x + 4$, $x \in \mathbb{R}$.
The graph of f is translated two units to the left to form the function $g(x)$.
Express $g(x)$ in the form $ax^4 + bx^3 + cx^2 + dx + e$ where $a, b, c, d, e \in \mathbb{Z}$.

28 – (MAT/12_HL_Summer_2019_Q5) – *Graphs, Functions - Roots*

(a) Sketch the graph of $y = \frac{x-4}{2x-5}$, stating the equations of any asymptotes and the coordinates of any points of intersection with the axes. [5]



(b) Consider the function $f : x \to \sqrt{\frac{x-4}{2x-5}}$.

Write down

(i) the largest possible domain of f;

(ii) the corresponding range of f. [3]

29 – (MathAA/12_HL_Summer_2021_Q1) – *Algebra-functions - Roots*

[Maximum mark: 4]

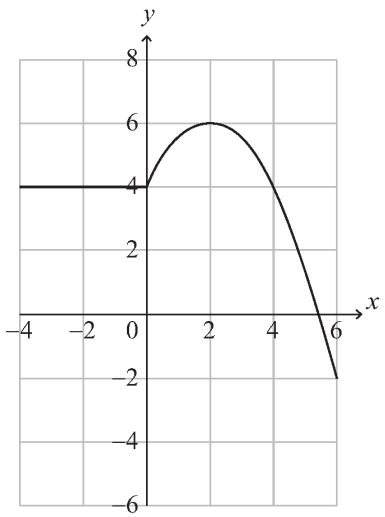
Consider two consecutive positive integers, n and $n+1$.

Show that the difference of their squares is equal to the sum of the two integers.

30 – (MathAA/11_HL_Summer_2021_Q1) – *Algebra-functions - Roots*

[Maximum mark: 5]

The graph of $y = f(x)$ for $-4 \le x \le 6$ is shown in the following diagram.



(a) Write down the value of

(i) $f(2)$;

(ii) $(f \circ f)(2)$. [2]

(b) Let $g(x) = \frac{1}{2} f(x) + 1$ for $-4 \le x \le 6$. On the axes above, sketch the graph of g. [3]

31 - (MathAA/12_HL_Summer_2021_Q7) - *Algebra-functions - Roots*

[Maximum mark: 5]

The cubic equation $x^3 - kx^2 + 3k = 0$ where $k > 0$ has roots α, β and $\alpha + \beta$.

Given that $\alpha\beta = \frac{k^2}{4}$, find the value of k.

32 – (MathAA/10_HL_Winter_2021_Q2) – *Algebra-functions - Roots, Graphs*

[Maximum mark: 9]

The function f is defined by $f(x) = \frac{2x+4}{3-x}$, where $x \in \mathbb{R}$, $x \neq 3$.

(a) Write down the equation of

(i) the vertical asymptote of the graph of f;

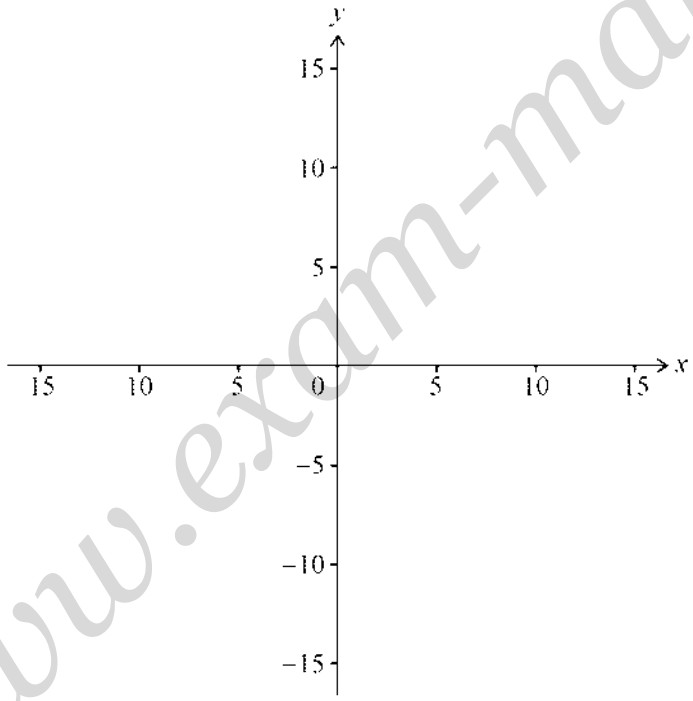
(ii) the horizontal asymptote of the graph of f. [2]

(b) Find the coordinates where the graph of f crosses

(i) the x-axis;

(ii) the y-axis. [2]

(c) Sketch the graph of f on the axes below. [1]



The function g is defined by $g(x) = \frac{ax+4}{3-x}$, where $x \in \mathbb{R}$, $x \neq 3$ and $a \in \mathbb{R}$.

(d) Given that $g(x) = g^{-1}(x)$, determine the value of a. [4]

33 – (MathAA/12_HL_Summer_2022_Q11) – *Algebra-functions - Roots*

[Maximum mark: 20]

A function f is defined by $f(x)=\dfrac{1}{x^2-2x-3}$, where $x \in \mathbb{R}$, $x \neq -1$, $x \neq 3$.

(a) Sketch the curve $y=f(x)$, clearly indicating any asymptotes with their equations. State the coordinates of any local maximum or minimum points and any points of intersection with the coordinate axes. [6]

A function g is defined by $g(x)=\dfrac{1}{x^2-2x-3}$, where $x \in \mathbb{R}$, $x > 3$.

(b) The inverse of g is g^{-1}.

(i) Show that $g^{-1}(x)=1+\dfrac{\sqrt{4x^2+x}}{x}$.

(ii) State the domain of g^{-1}. [7]

A function h is defined by $h(x)=\arctan\dfrac{x}{2}$, where $x \in \mathbb{R}$.

(c) Given that $(h \circ g)(a)=\dfrac{\pi}{4}$, find the value of a.

Give your answer in the form $p+\dfrac{q}{2}\sqrt{r}$, where $p, q, r \in \mathbb{Z}^+$. [7]

34 – (MathAA/12_HL_Summer_2022_Q9) – *Algebra-functions - Roots*

[Maximum mark: 5]

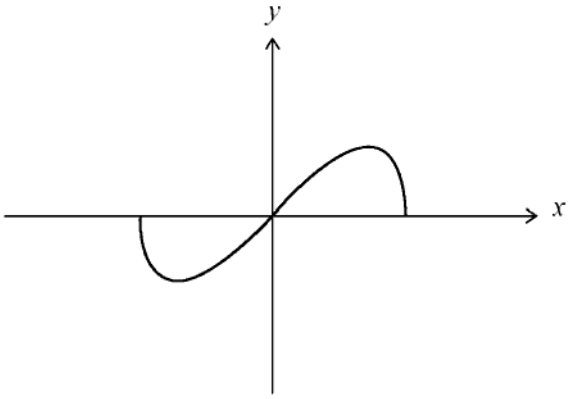
Prove by contradiction that the equation $2x^3 + 6x - 1 = 0$ has no integer roots.

35 – (MathAA/12_HL_Summer_2022_Q6) – *Algebra-functions - Roots*

[Maximum mark: 8]

A function f is defined by $f(x) = x\sqrt{1-x^2}$ where $-1 \le x \le 1$.

The graph of $y = f(x)$ is shown below.



(a) Show that f is an odd function. [2]

The range of f is $a \le y \le b$, where $a, b \in \mathbb{R}$.

(b) Find the value of a and the value of b. [6]

36 – (MathAA/12_HL_Summer_2022_Q2) – *Algebra-functions - Roots*

[Maximum mark: 6]

Consider any three consecutive integers, $n-1$, n and $n-1$.

(a) Prove that the sum of these three integers is always divisible by 3. [2]

(b) Prove that the sum of the squares of these three integers is never divisible by 3. [4]

37 – (MathAA/11_HL_Summer_2022_Q8) – *Algebra-functions - Roots*

[Maximum mark: 6]

Consider integers a and b such that $a^2 + b^2$ is exactly divisible by 4. Prove by contradiction that a and b cannot both be odd.

38 – (MathAA/11_HL_Summer_2022_Q3) – *Algebra-functions - Roots*

[Maximum mark: 7]

Consider the functions $f(x) = \sqrt{3}\sin x + \cos x$ where $0 \le x \le \pi$ and $g(x) = 2x$ where $x \in \mathbb{R}$.

(a) Find $(f \circ g)(x)$. [2]

(b) Solve the equation $(f \circ g)(x) = 2\cos 2x$ where $0 \le x \le \pi$. [5]

39 – (MathAA/10_HL_Winter_2022_Q8) – *Trigonometry, Algebra-functions - Roots*

[Maximum mark: 5]

Let $f(x) = \cos(x - k)$, where $0 \le x \le a$ and $a, k \in \mathbb{R}$.

(a) Consider the case where $k = \frac{\pi}{2}$.

By sketching a suitable graph, or otherwise, find the largest value of a for which the inverse function f^{-1} exists. [2]

(b) Find the largest value of a for which the inverse function f^{-1} exists in the case where $k = \pi$. [1]

(c) Find the largest value of a for which the inverse function f^{-1} exists in the case where $\pi < k < 2\pi$. Give your answer in terms of k. [2]

40 – (MathAA/10_HL_Winter_2022_Q3) – *Algebra-functions - Roots*

[Maximum mark: 5]

Let a be a constant, where $a > 1$.

(a) Show that $a^2 + \left(\frac{a^2-1}{2}\right)^2 = \left(\frac{a^2+1}{2}\right)^2$. [3]

Consider a right-angled triangle with sides of length a, $\left(\frac{a^2-1}{2}\right)$ and $\left(\frac{a^2+1}{2}\right)$.

(b) Find an expression for the area of the triangle in terms of a. [2]

1 – (MAT/11_HL_Summer_2013_Q12) – *Quadratics, Graphs, Integration*

(a) Express $4x^2 - 4x + 5$ in the form $a(x-h)^2 + k$ where $a, h, k \in \mathbb{Q}$. *[2 marks]*

(b) The graph of $y = x^2$ is transformed onto the graph of $y = 4x^2 - 4x + 5$. Describe a sequence of transformations that does this, making the order of transformations clear. *[3 marks]*

The function f is defined by $f(x) = \dfrac{1}{4x^2 - 4x + 5}$.

(c) Sketch the graph of $y = f(x)$. *[2 marks]*

(d) Find the range of f. *[2 marks]*

(e) By using a suitable substitution show that $\int f(x)\,\mathrm{d}x = \dfrac{1}{4}\int \dfrac{1}{u^2+1}\,\mathrm{d}u$. *[3 marks]*

(f) Prove that $\displaystyle\int_1^{3.5} \frac{1}{4x^2 - 4x + 5}\,\mathrm{d}x = \frac{\pi}{16}$. *[7 marks]*

2 – (MAT/12_HL_Summer_2014_Q4) – *Quadratics*

The roots of a quadratic equation $2x^2 + 4x - 1 = 0$ are α and β.

Without solving the equation,

(a) find the value of $\alpha^2 + \beta^2$; *[4]*

(b) find a quadratic equation with roots α^2 and β^2. *[2]*

3 – (MAT/10_HL_Winter_2014_Q2) – *Quadratics*

The quadratic equation $2x^2 - 8x + 1 = 0$ has roots α and β.

(a) Without solving the equation, find the value of

(i) $\alpha + \beta$;

(ii) $\alpha\beta$. *[2]*

Another quadratic equation $x^2 + px + q = 0$, $p, q \in \mathbb{Z}$ has roots $\frac{2}{\alpha}$ and $\frac{2}{\beta}$.

(b) Find the value of p and the value of q. *[4]*

4 – (MAT/10_HL_Winter_2020_Q6) – *Quadratics, Trigonometry*

[Maximum mark: 4]

Consider the equation $ax^2 + bx + c = 0$, where $a \neq 0$. Given that the roots of this equation are $x = \sin\theta$ and $x = \cos\theta$, show that $b^2 = a^2 + 2ac$.

5 – (MathAA/11_SL_Summer_2021_Q7) – *Quadratics*

[Maximum mark: 14]

Let $f(x) = mx^2 - 2mx$, where $x \in \mathbb{R}$ and $m \in \mathbb{R}$. The line $y = mx - 9$ meets the graph of f at exactly one point.

(a) Show that $m = 4$. [6]

The function f can be expressed in the form $f(x) = 4(x - p)(x - q)$, where $p, q \in \mathbb{R}$.

(b) Find the value of p and the value of q. [2]

The function f can also be expressed in the form $f(x) = 4(x - h)^2 - k$, where $h, k \in \mathbb{R}$.

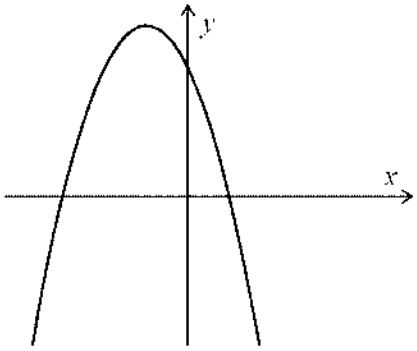
(c) Find the value of h and the value of k. [3]

(d) Hence find the values of x where the graph of f is both negative and increasing. [3]

6 – (MathAA/10_SL_Winter_2021_Q1) – *Quadratics*

[Maximum mark: 7]

Consider the function $f(x) = -2(x-1)(x+3)$, for $x \in \mathbb{R}$. The following diagram shows part of the graph of f.



(a) For the graph of f

(i) find the x-coordinates of the x-intercepts;

(ii) find the coordinates of the vertex. [5]

The function f can be written in the form $f(x) = -2(x-h)^2 + k$.

(b) Write down the value of h and the value of k. [2]

7 – (MathAA/10_HL_Winter_2021_Q7) – *Quadratics*

[Maximum mark: 7]

The equation $3px^2 + 2px + 1 - p$ has two real, distinct roots.

(a) Find the possible values for p. [5]

(b) Consider the case when $p = 4$. The roots of the equation can be expressed in the form $x = \frac{a \pm \sqrt{13}}{6}$, where $a \in \mathbb{Z}$. Find the value of a. [2]

1 – (MAT/20_HL_Winter_2013_Q3) – *Graphs, Exponentials - Logarithms*

Consider $f(x) = \ln x - e^{\cos x}$, $0 < x \leq 10$.

(a) Sketch the graph of $y = f(x)$, stating the coordinates of any maximum and minimum points and points of intersection with the x-axis. *[5]*

(b) Solve the inequality $\ln x \leq e^{\cos x}$, $0 < x \leq 10$. *[2]*

2 – (MAT/20_HL_Winter_2013_Q13) – *Functions - Roots, Exponentials - Logarithms, Integration*

A function f is defined by $f(x)=\frac{1}{2}\left(e^{x}+e^{-x}\right)$, $x \in \mathbb{R}$.

(a) (i) Explain why the inverse function f^{-1} does not exist.

(ii) Show that the equation of the normal to the curve at the point P where $x=\ln 3$ is given by $9x+12y-9\ln 3-20=0$.

(iii) Find the x-coordinates of the points Q and R on the curve such that the tangents at Q and R pass through $(0, 0)$. *[14]*

(b) The domain of f is now restricted to $x \geq 0$.

(i) Find an expression for $f^{-1}(x)$.

(ii) Find the volume generated when the region bounded by the curve $y=f(x)$ and the lines $x=0$ and $y=5$ is rotated through an angle of 2π radians about the y-axis. *[8]*

3 – (MAT/22_HL_Summer_2016_Q3) – *Exponentials - Logarithms*

[Maximum mark: 6]

Solve the simultaneous equations

$$\ln \frac{y}{x} = 2$$

$$\ln x^2 + \ln y^3 = 7 .$$

4 – (MAT/21_HL_Summer_2016_Q7) – *Exponentials - Logarithms*

It has been suggested that in rowing competitions the time, T seconds taken to complete a $2000\,\mathrm{m}$ race can be modelled by an equation of the form $T = aN^b$, where N is the number of rowers in the boat and a and b are constants for rowers of a similar standard.

To test this model the times for the finalists in all the $2000\,\mathrm{m}$ men's races at a recent Olympic games were recorded and the mean calculated.

The results are shown in the following table for $N = 1$ and $N = 2$.

N	T (seconds)
1	420.65
2	390.94

(a) Use these results to find estimates for the value of a and the value of b. Give your answers to five significant figures. [4]

(b) Use this model to estimate the mean time for the finalists in an Olympic race for boats with 8 rowers. Give your answer correct to two decimal places. [1]

It is now given that the mean time in the final for boats with 8 rowers was 342.08 seconds.

(c) Calculate the error in your estimate as a percentage of the actual value. [1]

(d) Comment on the likely validity of the model as N increases beyond 8. [2]

5 – (MAT/22_HL_Summer_2017_Q6) – *Exponentials - Logarithms*

[Maximum mark: 5]

Given that $\log_{10}\left(\frac{1}{2\sqrt{2}}(p+2q)\right)=\frac{1}{2}\left(\log_{10}p+\log_{10}q\right)$, $p>0$, $q>0$, find p in terms of q.

6 – (MAT/21_HL_Summer_2019_Q11) – *Exponentials - Logarithms*

Consider the equation $x^5 - 3x^4 + mx^3 + nx^2 + px + q = 0$, where $m, n, p, q \in \mathbb{R}$.
The equation has three distinct real roots which can be written as $\log_2 a$, $\log_2 b$ and $\log_2 c$.
The equation also has two imaginary roots, one of which is $d\mathrm{i}$ where $d \in \mathbb{R}$.

(a) Show that $abc = 8$. [5]

The values a, b, and c are consecutive terms in a geometric sequence.

(b) Show that one of the real roots is equal to 1. [3]

(c) Given that $q = 8d^2$, find the other two real roots. [9]

7 – (MAT/10_HL_Winter_2020_Q5) – *Exponentials - Logarithms, Sequences - Series*

[Maximum mark: 5]

The first term in an arithmetic sequence is 4 and the fifth term is $\log_2 625$.

Find the common difference of the sequence, expressing your answer in the form $\log_2 p$, where $p \in \mathbb{Q}$.

8 – (MathAA/10_HL_Winter_2021_Q3) – *Exponentials - Logarithms*

[Maximum mark: 5]

Solve the equation $\log_3 \sqrt{x} = \frac{1}{2\log_2 3} + \log_3\left(4x^3\right)$, where $x > 0$.

1 – (MAT/21_HL_Summer_2016_Q5) – *Graphs*

The function f is given by $f(x) = \frac{3x^2+10}{x^2-4}$, $x \in \mathbb{R}$, $x \neq 2$, $x \neq -2$.

(a) Prove that f is an even function. [2]

(b) (i) Sketch the graph $y = f(x)$.

(ii) Write down the range of f. [5]

2 - (MAT/21_HL_Summer_2017_Q6) - *Graphs*

Consider the graph of $y = \tan\left(x + \frac{\pi}{4}\right)$, where $-\pi \le x \le 2\pi$.

(a) Write down the equations of the vertical asymptotes of the graph. [2]

The graph is reflected in the y-axis, then stretched parallel to the y-axis by a factor $\frac{1}{2}$,

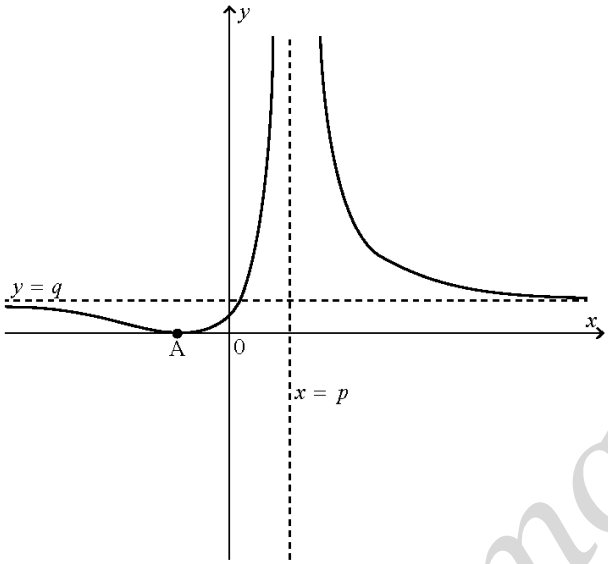
then translated by $\begin{pmatrix} \frac{\pi}{4} \\ -3 \end{pmatrix}$.

(b) Give the equation of the transformed graph. [4]

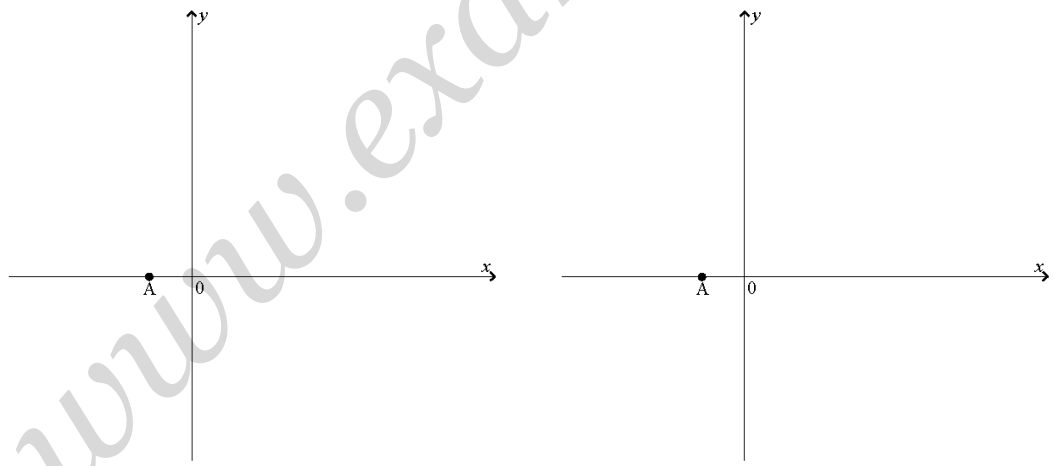
3 – (MAT/20_HL_Winter_2018_Q8) – *Graphs*

Consider the function $f(x) = \dfrac{ax+1}{bx+c}$, $x \neq -\dfrac{c}{b}$, where $a, b, c \in \mathbb{Z}$.

The following graph shows the curve $y = (f(x))^2$. It has asymptotes at $x = p$ and $y = q$ and meets the x-axis at A.



(a) On the following axes, sketch the two possible graphs of $y = f(x)$ giving the equations of any asymptotes in terms of p and q. [4]



(b) Given that $p = \dfrac{4}{3}$, $q = \dfrac{4}{9}$ and A has coordinates $\left(-\dfrac{1}{2}, 0\right)$, determine the possible sets of values for a, b and c. [4]

4 – (MAT/22_HL_Summer_2019_Q9) – *Graphs*

Consider the polynomial $P(z) \equiv z^4 - 6z^3 - 2z^2 + 58z - 51$. $z \in \mathbb{C}$.

(a) Express $P(z)$ in the form $(z^2 + az + b)(z^2 + cz + d)$ where $a, b, c, d \in \mathbb{R}$. [7]

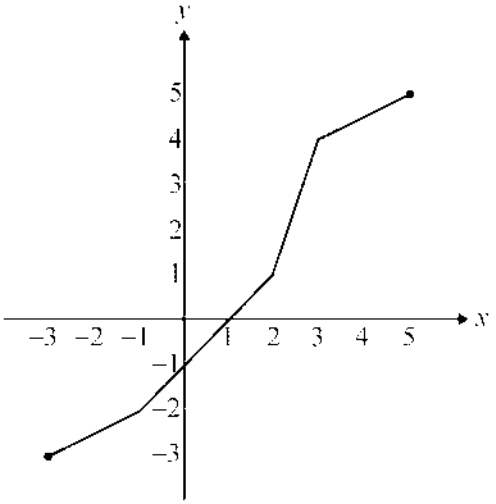
(b) Sketch the graph of $y = x^4 - 6x^3 - 2x^2 + 58x - 51$, stating clearly the coordinates of any maximum and minimum points and intersections with axes. [6]

(c) Hence, or otherwise, state the condition on $k \in \mathbb{R}$ such that all roots of the equation $P(z) = k$ are real. [2]

5 – (MAT/20_HL_Winter_2019_Q3) – *Graphs*

[Maximum mark: 6]

The following diagram shows the graph of $y = f(x)$, $-3 \le x \le 5$.



(a) Find the value of $(f \circ f)(1)$. [2]

(b) Given that $f^{-1}(a) = 3$, determine the value of a. [2]

(c) Given that $g(x) = 2f(x-1)$, find the domain and range of g. [2]

6 – (MAT/10_HL_Winter_2020_Q10) – *Differentiation, Graphs*

Consider the function $f(x) = ax^3 + bx^2 + cx + d$, where $x \in \mathbb{R}$ and $a, b, c, d \in \mathbb{R}$.

(a) (i) Write down an expression for $f'(x)$.

(ii) Hence, given that f^{-1} does not exist, show that $b^2 - 3ac > 0$. [4]

(b) Consider the function $g(x) = \frac{1}{2}x^3 - 3x^2 + 6x - 8$, where $x \in \mathbb{R}$.

(i) Show that g^{-1} exists.

(ii) $g(x)$ can be written in the form $p(x-2)^3 + q$, where $p, q \in \mathbb{R}$.
Find the value of p and the value of q.

(iii) Hence find $g^{-1}(x)$. [8]

The graph of $y = g(x)$ may be obtained by transforming the graph of $y = x^3$ using a sequence of three transformations.

(c) State each of the transformations in the order in which they are applied. [3]

(d) Sketch the graphs of $y = g(x)$ and $y = g^{-1}(x)$ on the same set of axes, indicating the points where each graph crosses the coordinate axes. [5]

7 – (MAT/10_HL_Winter_2020_Q12) – *Graphs, Integration*

Consider the function defined by $f(x) = \dfrac{kx-5}{x-k}$, where $x \in \mathbb{R} \setminus \{k\}$ and $k^2 \neq 5$.

(a) State the equation of the vertical asymptote on the graph of $y = f(x)$. [1]

(b) State the equation of the horizontal asymptote on the graph of $y = f(x)$. [1]

(c) Use an algebraic method to determine whether f is a self-inverse function. [4]

Consider the case where $k = 3$.

(d) Sketch the graph of $y = f(x)$, stating clearly the equations of any asymptotes and the coordinates of any points of intersections with the coordinate axes. [3]

(e) The region bounded by the x-axis, the curve $y = f(x)$, and the lines $x = 5$ and $x = 7$ is rotated through 2π about the x-axis. Find the volume of the solid generated, giving your answer in the form $\pi(a + b\ln 2)$, where $a, b \in \mathbb{Z}$. [6]

8 – (MathAA/10_HL_Winter_2021_Q2) – *Algebra-functions - Roots, Graphs*

[Maximum mark: 9]

The function f is defined by $f(x)=\dfrac{2x+4}{3-x}$, where $x \in \mathbb{R}$, $x \neq 3$.

(a) Write down the equation of

(i) the vertical asymptote of the graph of f;

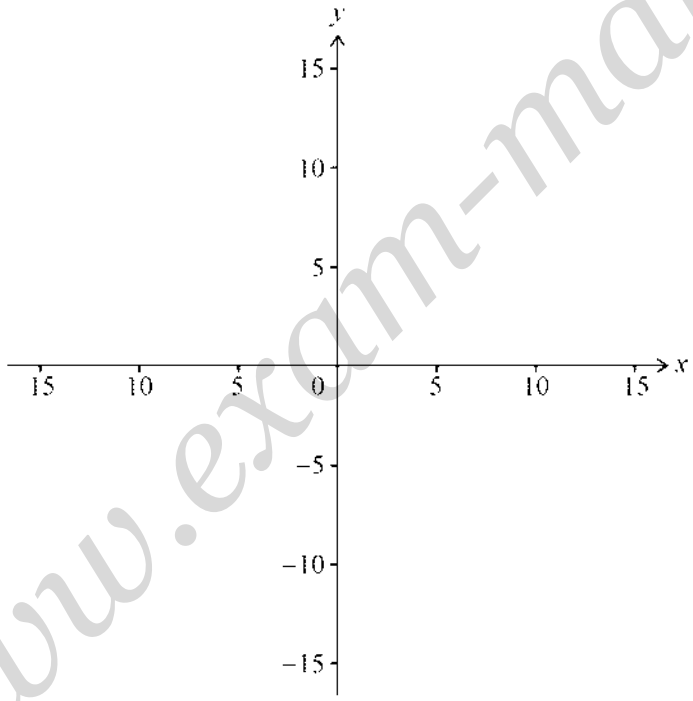
(ii) the horizontal asymptote of the graph of f. [2]

(b) Find the coordinates where the graph of f crosses

(i) the x-axis;

(ii) the y-axis. [2]

(c) Sketch the graph of f on the axes below. [1]



The function g is defined by $g(x)=\dfrac{ax+4}{3-x}$, where $x \in \mathbb{R}$, $x \neq 3$ and $a \in \mathbb{R}$.

(d) Given that $g(x)=g^{-1}(x)$, determine the value of a. [4]

9 – (MathAA/20_HL_Winter_2021_Q10) – *Algebra-functions - Roots, Graphs, Integration*

[Maximum mark: 18]

Consider the function $f(x) = \dfrac{x^2 - x - 12}{2x - 15}$, $x \in \mathbb{R}$, $x \neq \dfrac{15}{2}$.

(a) Find the coordinates where the graph of f crosses the

(i) x-axis;

(ii) y-axis. [3]

(b) Write down the equation of the vertical asymptote of the graph of f. [1]

(c) The oblique asymptote of the graph of f can be written as $y = ax + b$ where $a, b \in \mathbb{Z}$.

Find the value of a and the value of b. [4]

(d) Sketch the graph of f for $-30 \leq x \leq 30$, clearly indicating the points of intersection with each axis and any asymptotes. [3]

(e) (i) Express $\dfrac{1}{f(x)}$ in partial fractions.

(ii) Hence find the exact value of $\displaystyle\int_0^3 \frac{1}{f(x)}\,\mathrm{d}x$, expressing your answer as a single logarithm. [7]

10 – (MathAA/21_HL_Summer_2021_Q11) – *Algebra-functions - Roots, Graphs*

[Maximum mark: 20]

The function f is defined by $f(x) = \dfrac{3x+2}{4x^2-1}$, for $x \in \mathbb{R}$, $x \neq p$, $x \neq q$.

(a) Find the value of p and the value of q. [2]

(b) Find an expression for $f'(x)$. [3]

The graph of $y = f(x)$ has exactly one point of inflexion.

(c) Find the x-coordinate of the point of inflexion. [2]

(d) Sketch the graph of $y = f(x)$ for $-3 \leq x \leq 3$, showing the values of any axes intercepts, the coordinates of any local maxima and local minima, and giving the equations of any asymptotes. [5]

The function g is defined by $g(x) = \dfrac{4x^2-1}{3x+2}$, for $x \in \mathbb{R}$, $x \neq -\dfrac{2}{3}$.

(e) Find the equations of all the asymptotes on the graph of $y = g(x)$. [4]

(f) By considering the graph of $y = g(x) - f(x)$, or otherwise, solve $f(x) < g(x)$ for $x \in \mathbb{R}$. [4]

11 – (MathAA/11_HL_Summer_2022_Q5) – *Graphs*

[Maximum mark: 7]

Consider $f(x) = 4\sin x + 2.5$ and $g(x) = 4\sin\left(x - \frac{3\pi}{2}\right) + 2.5 + q$, where $x \in \mathbb{R}$ and $q > 0$.

The graph of g is obtained by two transformations of the graph of f.

(a) Describe these two transformations. [2]

The y-intercept of the graph of g is at $(0, r)$.

(b) Given that $g(x) \geq 7$, find the smallest value of r. [5]

12 – (MathAA/12_HL_Summer_2022_Q3) – *Graphs*

[Maximum mark: 8]

A function f is defined by $f(x)=\dfrac{2x-1}{x+1}$, where $x \in \mathbb{R}$, $x \neq -1$.

(a) The graph of $y=f(x)$ has a vertical asymptote and a horizontal asymptote.

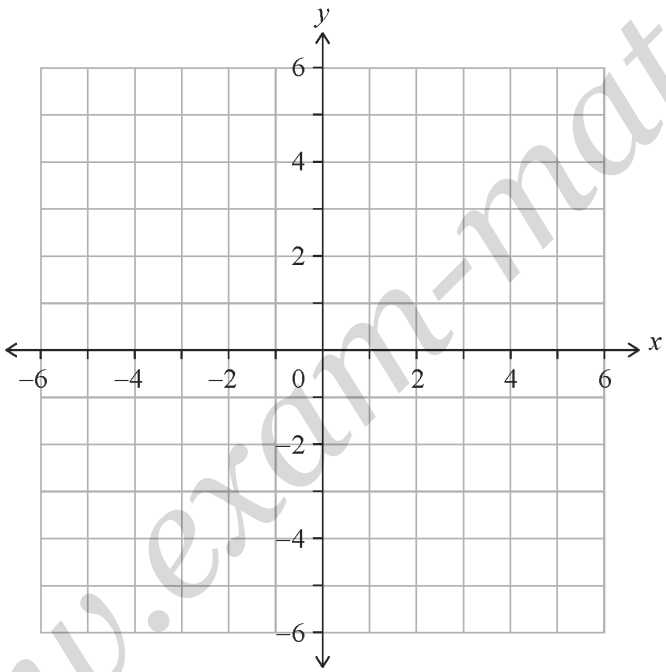
Write down the equation of

(i) the vertical asymptote;

(ii) the horizontal asymptote. [2]

(b) On the set of axes below, sketch the graph of $y=f(x)$.

On your sketch, clearly indicate the asymptotes and the position of any points of intersection with the axes. [3]



(c) Hence, solve the inequality $0<\dfrac{2x-1}{x+1}<2$. [1]

(d) Solve the inequality $0<\dfrac{2|x|-1}{|x|+1}<2$. [2]

13 – (MathAA/10_HL_Winter_2022_Q10) – *Differentiation, Graphs*

[Maximum mark: 20]

The function f is defined by $f(x) = \cos^2 x - 3\sin^2 x$, $0 \le x \le \pi$.

(a) Find the roots of the equation $f(x) = 0$. [5]

(b) (i) Find $f'(x)$.

(ii) Hence find the coordinates of the points on the graph of $y = f(x)$ where $f'(x) = 0$. [7]

(c) Sketch the graph of $y = |f(x)|$, clearly showing the coordinates of any points where $f'(x) = 0$ and any points where the graph meets the coordinate axes. [4]

(d) Hence or otherwise, solve the inequality $|f(x)| > 1$. [4]

1 - (MAT/11_HL_Summer_2012_Q3) - *Complex Numbers*

[Maximum mark: 7]

If $z_1 = a + a\sqrt{3}\,\mathrm{i}$ and $z_2 = 1 - \mathrm{i}$, where a is a real constant, express z_1 and z_2 in the form $r\operatorname{cis}\theta$, and hence find an expression for $\left(\frac{z_1}{z_2}\right)^6$ in terms of a and i.

2 – (MAT/12_HL_Summer_2012_Q6) – *Complex Numbers*

[Maximum mark: 7]

Given that $(4-5\mathrm{i})m+4n=16+15\mathrm{i}$, where $\mathrm{i}^2=-1$, find m and n if

(a) m and n are real numbers; *[3 marks]*

(b) m and n are conjugate complex numbers. *[4 marks]*

3 – (MAT/11_HL_Summer_2012_Q7) – *Complex Numbers*

[Maximum mark: 6]

Given that z is the complex number $x+\mathrm{i}y$ and that $|z|+z=6-2\mathrm{i}$, find the value of x and the value of y.

4 – (MAT/12_HL_Summer_2012_Q12) – *Complex Numbers*

Part A *[Maximum mark: 12]*

(a) Given that $(x+\mathrm{i}y)^2 = -5+12\mathrm{i}$, x, $y \in \mathbb{R}$. Show that

(i) $x^2 - y^2 = -5$;

(ii) $xy = 6$. *[2 marks]*

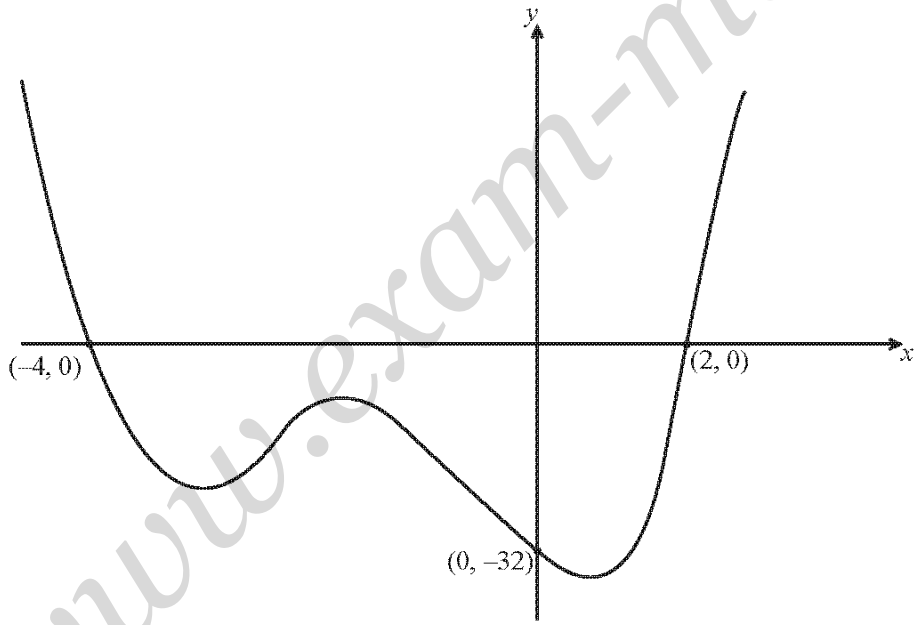
(b) Hence find the two square roots of $-5+12\mathrm{i}$. *[5 marks]*

(c) For any complex number z, show that $(z^*)^2 = (z^2)^*$. *[3 marks]*

(d) Hence write down the two square roots of $-5-12\mathrm{i}$. *[2 marks]*

Part B *[Maximum mark: 17]*

The graph of a polynomial function f of degree 4 is shown below.



(a) Explain why, of the four roots of the equation $f(x) = 0$, two are real and two are complex. *[2 marks]*

(b) The curve passes through the point $(-1, -18)$. Find $f(x)$ in the form $f(x) = (x-a)(x-b)(x^2+cx+d)$, where $a, b, c, d \in \mathbb{Z}$. *[5 marks]*

(c) Find the two complex roots of the equation $f(x) = 0$ in Cartesian form. *[2 marks]*

(d) Draw the four roots on the complex plane (the Argand diagram). *[2 marks]*

(e) Express each of the four roots of the equation in the form $re^{i\theta}$. *[6 marks]*

5 - (MAT/10_HL_Winter_2012_Q10) - *Complex Numbers*

Consider the complex numbers

$$z_1 = 2\sqrt{3}\,\text{cis}\,\frac{3\pi}{2} \text{ and } z_2 = -1+\sqrt{3}\text{i}.$$

(a) (i) Write down z_1 in Cartesian form.

(ii) Hence determine $(z_1 + z_2)^*$ in Cartesian form. *[3 marks]*

(b) (i) Write z_2 in modulus-argument form.

(ii) Hence solve the equation $z^3 = z_2$. *[6 marks]*

(c) Let $z = r\,\text{cis}\,\theta$, where $r \in \mathbb{R}^+$ and $0 \le \theta < 2\pi$. Find all possible values of r and θ,

(i) if $z^2 = (1+z_2)^2$;

(ii) if $z = -\dfrac{1}{z_2}$. *[6 marks]*

(d) Find the smallest positive value of n for which $\left(\dfrac{z_1}{z_2}\right)^n \in \mathbb{R}^+$. *[4 marks]*

6 – (MAT/11_HL_Summer_2013_Q1) – *Complex Numbers*

(a) If $w = 2 + 2\mathrm{i}$, find the modulus and argument of w. *[2 marks]*

(b) Given $z = \cos\left(\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(\frac{5\pi}{6}\right)$, find in its simplest form $w^4 z^6$. *[4 marks]*

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7 – (MAT/12_HL_Summer_2013_Q7) – *Complex Numbers*

Given the complex numbers $z_1 = 1 + 3\mathrm{i}$ and $z_2 = -1 - \mathrm{i}$.

(a) Write down the exact values of $|z_1|$ and $\arg(z_2)$. *[2 marks]*

(b) Find the minimum value of $|z_1 + \alpha z_2|$, where $\alpha \in \mathbb{R}$. *[5 marks]*

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8 – (MAT/12_HL_Summer_2013_Q13) – *Complex Numbers*

(a) (i) Express each of the complex numbers $z_1 = \sqrt{3} + \mathrm{i}$, $z_2 = -\sqrt{3} + \mathrm{i}$ and $z_3 = -2\mathrm{i}$ in modulus-argument form.

(ii) Hence show that the points in the complex plane representing z_1, z_2 and z_3 form the vertices of an equilateral triangle.

(iii) Show that $z_1^{3n} + z_2^{3n} = 2z_3^{3n}$ where $n \in \mathbb{N}$. *[9 marks]*

(b) (i) State the solutions of the equation $z^7 = 1$ for $z \in \mathbb{C}$, giving them in modulus-argument form.

(ii) If w is the solution to $z^7 = 1$ with least positive argument, determine the argument of $1 + w$. Express your answer in terms of π.

(iii) Show that $z^2 - 2z\cos\left(\frac{2\pi}{7}\right) + 1$ is a factor of the polynomial $z^7 - 1$. State the two other quadratic factors with real coefficients. *[9 marks]*

9 – (MAT/10_HL_Winter_2013_Q12) – *Complex Numbers, Trigonometry, Integration*

Consider the complex number $z = \cos\theta + \mathrm{i}\sin\theta$.

(a) Use De Moivre's theorem to show that $z^n + z^{-n} = 2\cos n\theta$, $n \in \mathbb{Z}^+$. *[2]*

(b) Expand $\left(z + z^{-1}\right)^4$. *[1]*

(c) Hence show that $\cos^4\theta = p\cos 4\theta + q\cos 2\theta + r$, where p, q and r are constants to be determined. *[4]*

(d) Show that $\cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$. *[3]*

(e) Hence find the value of $\int_0^{\frac{\pi}{2}} \cos^6\theta \,\mathrm{d}\theta$. *[3]*

The region S is bounded by the curve $y = \sin x\cos^2 x$ and the x-axis between $x = 0$ and $x = \frac{\pi}{2}$.

(f) S is rotated through 2π radians about the x-axis. Find the value of the volume generated. *[4]*

(g) (i) Write down an expression for the constant term in the expansion of $\left(z + z^{-1}\right)^{2k}$, $k \in \mathbb{Z}^+$.

(ii) Hence determine an expression for $\int_0^{\frac{\pi}{2}} \cos^{2k}\theta \,\mathrm{d}\theta$ in terms of k. *[3]*

10 - (MAT/12_HL_Summer_2014_Q7) - *Complex Numbers*

Consider the complex numbers $u = 2 + 3\mathrm{i}$ and $v = 3 + 2\mathrm{i}$.

(a) Given that $\frac{1}{u} + \frac{1}{v} = \frac{10}{w}$, express w in the form $a + b\mathrm{i}$, $a, b \in \mathbb{R}$. *[4]*

(b) Find w^* and express it in the form $re^{\mathrm{i}\theta}$. *[3]*

11 - (MAT/11_HL_Summer_2014_Q13) - *Complex Numbers, Sequences - Series*

A geometric sequence $\{u_n\}$, with complex terms, is defined by $u_{n+1} = (1+\mathrm{i})u_n$ and $u_1 = 3$.

(a) Find the fourth term of the sequence, giving your answer in the form $x + y\mathrm{i},\ x,\ y \in \mathbb{R}$. *[3]*

(b) Find the sum of the first 20 terms of $\{u_n\}$, giving your answer in the form $a \times (1+2^m)$ where $a \in \mathbb{C}$ and $m \in \mathbb{Z}$ are to be determined. *[4]*

A second sequence $\{v_n\}$ is defined by $v_n = u_n u_{n+k},\ k \in \mathbb{N}$.

(c) (i) Show that $\{v_n\}$ is a geometric sequence.

(ii) State the first term.

(iii) Show that the common ratio is independent of k. *[5]*

A third sequence $\{w_n\}$ is defined by $w_n = |u_n - u_{n+1}|$.

(d) (i) Show that $\{w_n\}$ is a geometric sequence.

(ii) State the geometrical significance of this result with reference to points on the complex plane. *[5]*

12 – (MAT/10_HL_Winter_2014_Q13) – *Complex Numbers, Trigonometry, Integration*

(a) (i) Show that $(1+\mathrm{i}\tan\theta)^n+(1-\mathrm{i}\tan\theta)^n=\dfrac{2\cos n\theta}{\cos^n\theta}$, $\cos\theta\neq 0$.

(ii) Hence verify that $\mathrm{i}\tan\dfrac{3\pi}{8}$ is a root of the equation $(1+z)^4+(1-z)^4=0$, $z\in\mathbb{C}$.

(iii) State another root of the equation $(1+z)^4+(1-z)^4=0$, $z\in\mathbb{C}$. *[10]*

(b) (i) Use the double angle identity $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}$ to show that $\tan\dfrac{\pi}{8}=\sqrt{2}-1$.

(ii) Show that $\cos 4x=8\cos^4 x-8\cos^2 x+1$.

(iii) Hence find the value of $\displaystyle\int_0^{\frac{\pi}{8}}\frac{2\cos 4x}{\cos^2 x}\,\mathrm{d}x$. *[13]*

13 – (MAT/12_HL_Summer_2015_Q7) – *Complex Numbers*

(a) Find three distinct roots of the equation $8z^3 + 27 = 0$, $z \in \mathbb{C}$ giving your answers in modulus-argument form. [6]

The roots are represented by the vertices of a triangle in an Argand diagram.

(b) Show that the area of the triangle is $\frac{27\sqrt{3}}{16}$. [3]

14 – (MAT/10_HL_Winter_2015_Q11) – *Complex Numbers*

(a) Solve the equation $z^3 = 8\mathrm{i}$, $z \in \mathbb{C}$ giving your answers in the form $z = r(\cos\theta + \mathrm{i}\sin\theta)$ **and** in the form $z = a + b\mathrm{i}$ where $a, b \in \mathbb{R}$. [6]

(b) Consider the complex numbers $z_1 = 1 + \mathrm{i}$ and $z_2 = 2\left(\cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right)\right)$.

(i) Write z_1 in the form $r(\cos\theta + \mathrm{i}\sin\theta)$.

(ii) Calculate $z_1 z_2$ and write in the form $z = a + b\mathrm{i}$ where $a, b \in \mathbb{R}$.

(iii) Hence find the value of $\tan\frac{5\pi}{12}$ in the form $c + d\sqrt{3}$, where $c, d \in \mathbb{Z}$.

(iv) Find the smallest value $p > 0$ such that $(z_2)^p$ is a positive real number. [11]

15 – (MAT/11_HL_Summer_2016_Q12) – *Complex Numbers*

(a) Use de Moivre's theorem to find the value of $\left(\cos\left(\frac{\pi}{3}\right)+\mathrm{i}\sin\left(\frac{\pi}{3}\right)\right)^3$. [2]

(b) Use mathematical induction to prove that

$$(\cos\theta-\mathrm{i}\sin\theta)^n=\cos n\theta-\mathrm{i}\sin n\theta \text{ for } n\in\mathbb{Z}^+.$$ [6]

Let $z=\cos\theta+\mathrm{i}\sin\theta$.

(c) Find an expression in terms of θ for $(z)^n+(z^*)^n$, $n\in\mathbb{Z}^+$ where z^* is the complex conjugate of z. [2]

(d) (i) Show that $zz^*=1$.

(ii) Write down the binomial expansion of $(z+z^*)^3$ in terms of z and z^*.

(iii) Hence show that $\cos 3\theta=4\cos^3\theta-3\cos\theta$. [5]

(e) Hence solve $4\cos^3\theta-2\cos^2\theta-3\cos\theta+1=0$ for $0\le\theta<\pi$. [6]

16 – (MAT/12_HL_Summer_2016_Q12) – *Complex Numbers*

Let $w = \cos\frac{2\pi}{7} + \mathrm{i}\sin\frac{2\pi}{7}$.

(a) Verify that w is a root of the equation $z^7 - 1 = 0$, $z \in \mathbb{C}$. [3]

(b) (i) Expand $(w-1)\left(1 + w + w^2 + w^3 + w^4 + w^5 + w^6\right)$.

(ii) Hence deduce that $1 + w + w^2 + w^3 + w^4 + w^5 + w^6 = 0$. [3]

(c) Write down the roots of the equation $z^7 - 1 = 0$, $z \in \mathbb{C}$ in terms of w and plot these roots on an Argand diagram. [3]

Consider the quadratic equation $z^2 + bz + c = 0$ where $b, c \in \mathbb{R}$, $z \in \mathbb{C}$. The roots of this equation are α and α^* where α^* is the complex conjugate of α.

(d) (i) Given that $\alpha = w + w^2 + w^4$, show that $\alpha^* = w^6 + w^5 + w^3$.

(ii) Find the value of b and the value of c. [10]

(e) Using the values for b and c obtained in part (d)(ii), find the imaginary part of α, giving your answer in surd form. [4]

17 – (MAT/10_HL_Winter_2016_Q12) – *Complex Numbers*

Let ω be one of the non-real solutions of the equation $z^3 = 1$.

(a) Determine the value of

(i) $1 + \omega + \omega^2$;

(ii) $1 + \omega^* + (\omega^*)^2$. [4]

(b) Show that $(\omega - 3\omega^2)(\omega^2 - 3\omega) = 13$. [4]

Consider the complex numbers $p = 1 - 3\mathrm{i}$ and $q = x + (2x + 1)\mathrm{i}$, where $x \in \mathbb{R}$.

(c) Find the values of x that satisfy the equation $|p| = |q|$. [5]

(d) Solve the inequality $\mathrm{Re}(pq) + 8 < (\mathrm{Im}(pq))^2$. [6]

18 – (MAT/11_HL_Summer_2017_Q2) – *Complex Numbers*

Consider the complex numbers $z_1 = 1 + \sqrt{3}\,\mathrm{i}$, $z_2 = 1 + \mathrm{i}$ and $w = \dfrac{z_1}{z_2}$.

(a) By expressing z_1 and z_2 in modulus-argument form write down

(i) the modulus of w;

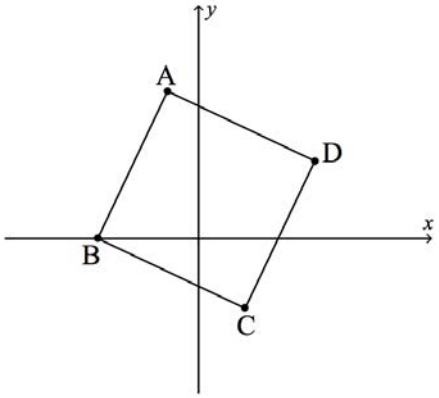
(ii) the argument of w. [4]

(b) Find the smallest positive integer value of n, such that w^n is a real number. [2]

19 – (MAT/12_HL_Summer_2017_Q5) – *Complex Numbers*

[Maximum mark: 4]

In the following Argand diagram the point A represents the complex number $-1+4\mathrm{i}$ and the point B represents the complex number $-3+0\mathrm{i}$. The shape of ABCD is a square. Determine the complex numbers represented by the points C and D.



20 – (MAT/12_HL_Summer_2017_Q11) – *Trigonometry, Complex Numbers*

(a) Solve $2\sin(x+60^\circ)=\cos(x+30^\circ)$, $0^\circ \le x \le 180^\circ$. [5]

(b) Show that $\sin 105^\circ + \cos 105^\circ = \frac{1}{\sqrt{2}}$. [3]

(c) Let $z = 1 - \cos 2\theta - \mathrm{i}\sin 2\theta$, $z \in \mathbb{C}$, $0 \le \theta \le \pi$.

(i) Find the modulus and argument of z in terms of θ. Express each answer in its simplest form.

(ii) Hence find the cube roots of z in modulus-argument form. [14]

21 - (MAT/10_HL_Winter_2017_Q8) - *Complex Numbers*

[Maximum mark: 7]

Determine the roots of the equation $(z+2\mathrm{i})^3 = 216\mathrm{i}$, $z \in \mathbb{C}$, giving the answers in the form $z = a\sqrt{3} + b\mathrm{i}$ where $a, b \in \mathbb{Z}$.

22 - (MAT/12_HL_Summer_2018_Q7) - *Complex Numbers*

Consider the distinct complex numbers $z = a + \mathrm{i}b$, $w = c + \mathrm{i}d$, where $a, b, c, d \in \mathbb{R}$.

(a) Find the real part of $\dfrac{z + w}{z - w}$. [4]

(b) Find the value of the real part of $\dfrac{z + w}{z - w}$ when $|z| = |w|$. [2]

23 – (MAT/11_HL_Summer_2018_Q11) – *Complex Numbers*

Consider $w = 2\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$.

(a) (i) Express w^2 and w^3 in modulus-argument form.

(ii) Sketch on an Argand diagram the points represented by w^0, w^1, w^2 and w^3. [5]

These four points form the vertices of a quadrilateral, Q.

(b) Show that the area of the quadrilateral Q is $\frac{21\sqrt{3}}{2}$. [3]

Let $z = 2\left(\cos\frac{\pi}{n} + \mathrm{i}\sin\frac{\pi}{n}\right)$, $n \in \mathbb{Z}^+$. The points represented on an Argand diagram by $z^0, z^1, z^2, \ldots, z^n$ form the vertices of a polygon P_n.

(c) Show that the area of the polygon P_n can be expressed in the form $a\left(b^n - 1\right)\sin\frac{\pi}{n}$, where $a, b \in \mathbb{R}$. [6]

24 – (MAT/10_HL_Winter_2018_Q8) – *Complex Numbers*

[Maximum mark: 7]

Consider the equation $z^4 + az^3 + bz^2 + cz + d = 0$, where $a, b, c, d \in \mathbb{R}$ and $z \in \mathbb{C}$.
Two of the roots of the equation are $\log_2 6$ and $\mathrm{i}\sqrt{3}$ and the sum of all the roots is $3 + \log_2 3$.
Show that $6a + d + 12 = 0$.

25 – (MAT/10_HL_Winter_2018_Q11) – *Complex Numbers*

(a) Find the roots of $z^{24} = 1$ which satisfy the condition $0 < \arg(z) < \frac{\pi}{2}$, expressing your answers in the form $re^{i\theta}$, where $r, \theta \in \mathbb{R}^+$. [5]

(b) Let S be the sum of the roots found in part (a).

(i) Show that $\operatorname{Re} S = \operatorname{Im} S$.

(ii) By writing $\frac{\pi}{12}$ as $\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$, find the value of $\cos\frac{\pi}{12}$ in the form $\frac{\sqrt{a} + \sqrt{b}}{c}$, where a, b and c are integers to be determined.

(iii) Hence, or otherwise, show that $S = \frac{1}{2}\left(1 + \sqrt{2}\right)\left(1 + \sqrt{3}\right)(1 + \mathrm{i})$. [11]

26 – (MAT/10_HL_Winter_2019_Q5) – *Complex Numbers*

[Maximum mark: 7]

Consider the equation $z^4 = -4$, where $z \in \mathbb{C}$.

(a) Solve the equation, giving the solutions in the form $a + \mathrm{i}b$, where $a, b \in \mathbb{R}$. [5]

(b) The solutions form the vertices of a polygon in the complex plane. Find the area of the polygon. [2]

27 – (MAT/10_HL_Winter_2020_Q4) – *Complex Numbers*

[Maximum mark: 5]

Consider the equation $\frac{2z}{3-z^*} = \mathrm{i}$, where $z = x + \mathrm{i}y$ and $x, y \in \mathbb{R}$.

Find the value of x and the value of y.

28 – (MAT/10_HL_Winter_2020_Q7) – *Complex Numbers*

Consider the complex numbers $z_1 = \cos\frac{11\pi}{12} + \mathrm{i}\sin\frac{11\pi}{12}$ and $z_2 = \cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}$.

(a) (i) Find $\frac{z_1}{z_2}$

(ii) Find $\frac{z_2}{z_1}$ [3]

(b) 0, $\frac{z_1}{z_2}$ and $\frac{z_2}{z_1}$ are represented by three points O, A and B respectively on an Argand diagram. Determine the area of the triangle OAB. [2]

29 – (MathAA/11_HL_Summer_2021_Q7) – *Complex Numbers*

[Maximum mark: 8]

Consider the quartic equation $z^4 + 4z^3 + 8z^2 + 80z - 400 = 0$, $z \in \mathbb{C}$.

Two of the roots of this equation are $a + b\mathrm{i}$ and $b + a\mathrm{i}$, where $a, b \in \mathbb{Z}$.

Find the possible values of a.

30 – (MathAA/10_HL_Winter_2021_Q12) – *Complex Numbers*

[Maximum mark: 22]

Consider the equation $(z-1)^3 = \mathrm{i}$, $z \in \mathbb{C}$. The roots of this equation are ω_1, ω_2 and ω_3, where $\mathrm{Im}(\omega_2) > 0$ and $\mathrm{Im}(\omega_3) < 0$.

(a) (i) Verify that $\omega_1 = 1 + \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ is a root of this equation.

(ii) Find ω_2 and ω_3, expressing these in the form $a + \mathrm{e}^{\mathrm{i}\theta}$, where $a \in \mathbb{R}$ and $\theta > 0$. [6]

The roots ω_1, ω_2 and ω_3 are represented by the points A, B and C respectively on an Argand diagram.

(b) Plot the points A, B and C on an Argand diagram. [4]

(c) Find AC. [3]

Consider the equation $(z-1)^3 = \mathrm{i}z^3$, $z \in \mathbb{C}$.

(d) By using de Moivre's theorem, show that $\alpha = \dfrac{1}{1 - \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}$ is a root of this equation. [3]

(e) Determine the value of $\mathrm{Re}(\alpha)$. [6]

31 – (MathAA/11_HL_Summer_2022_Q9) – *Complex Numbers*

[Maximum mark: 6]

Consider the complex numbers $z_1 = 1 + b\mathrm{i}$ and $z_2 = (1 - b^2) - 2b\mathrm{i}$, where $b \in \mathbb{R}$, $b \neq 0$.

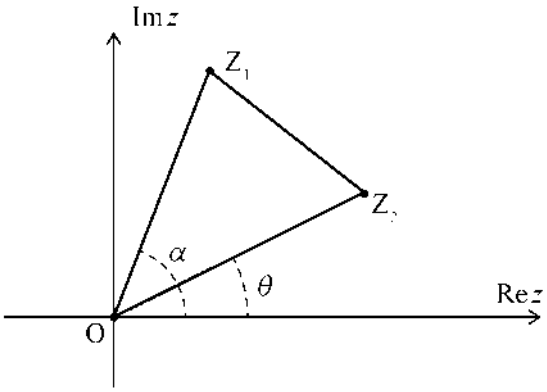
(a) Find an expression for $z_1 z_2$ in terms of b. [3]

(b) Hence, given that $\arg(z_1 z_2) = \frac{\pi}{4}$, find the value of b. [3]

32 – (MathAA/12_HL_Summer_2022_Q12) – *Complex Numbers*

[Maximum mark: 18]

In the following Argand diagram, the points Z_1, O and Z_2 are the vertices of triangle Z_1OZ_2 described anticlockwise.



The point Z_1 represents the complex number $z_1 = r_1e^{i\alpha}$, where $r_1 > 0$. The point Z_2 represents the complex number $z_2 = r_2e^{i\theta}$, where $r_2 > 0$.

Angles α, θ are measured anticlockwise from the positive direction of the real axis such that $0 \le \alpha, \theta < 2\pi$ and $0 < \alpha - \theta < \pi$.

(a) Show that $z_1z_2^* = r_1r_2e^{i(\alpha-\theta)}$ where z_2^* is the complex conjugate of z_2. [2]

(b) Given that $\mathrm{Re}(z_1z_2^*) = 0$, show that Z_1OZ_2 is a right-angled triangle. [2]

In parts (c), (d) and (e), consider the case where Z_1OZ_2 is an equilateral triangle.

(c) (i) Express z_1 in terms of z_2.

(ii) Hence show that $z_1^2 + z_2^2 = z_1z_2$. [6]

Let z_1 and z_2 be the distinct roots of the equation $z^2 + az + b = 0$ where $z \in \mathbb{C}$ and $a, b \in \mathbb{R}$.

(d) Use the result from part (c)(ii) to show that $a^2 = 3b$. [5]

Consider the equation $z^2 + az + 12 = 0$, where $z \in \mathbb{C}$ and $a \in \mathbb{R}$.

(e) Given that $0 < \alpha - \theta < \pi$, deduce that only one equilateral triangle Z_1OZ_2 can be formed from the point O and the roots of this equation. [3]

33 - (MathAA/10_HL_Winter_2022_Q5) - *Complex Numbers*

[Maximum mark: 7]

Consider the equation $z^4 + pz^3 + 54z^2 - 108z + 80 = 0$ where $z \in \mathbb{C}$ and $p \in \mathbb{R}$.

Three of the roots of the equation are $3 + \mathrm{i}$, α and α^2, where $\alpha \in \mathbb{R}$.

(a) By considering the product of all the roots of the equation, find the value of α. [4]

(b) Find the value of p. [3]

34 – (MathAA/10_HL_Winter_2022_Q12) – *Complex Numbers*

[Maximum mark: 18]

Let z_n be the complex number defined as $z_n = (n^2 + n - 1) + \mathrm{i}$ for $n \in \mathbb{N}$.

(a) (i) Find $\arg(z_n)$.

(ii) Write down an expression for $\arg(z_n)$ in terms of n. [3]

Let $w_n = z_n z_1 z_2 z_3 \ldots z_{n-1} z_n$ for $n \in \mathbb{N}$.

(b) (i) Show that $\arctan(a) + \arctan(b) = \arctan\left(\frac{a+b}{1-ab}\right)$ for $a, b \in \mathbb{R}^+$, $ab < 1$.

(ii) Hence or otherwise, show that $\arg(w_1) = \arctan(2)$. [5]

(c) Prove by mathematical induction that $\arg(w_n) = \arctan(n+1)$ for $n \in \mathbb{N}$. [10]

1 – (MAT/10_HL_Winter_2014_Q10) – *Permutation - Combination*

[Maximum mark: 7]

A set of positive integers $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is used to form a pack of nine cards. Each card displays one positive integer without repetition from this set. Grace wishes to select four cards at random from this pack of nine cards.

(a) Find the number of selections Grace could make if the largest integer drawn among the four cards is either a 5, a 6 or a 7. *[3]*

(b) Find the number of selections Grace could make if at least two of the four integers drawn are even. *[4]*

2 – (MAT/10_HL_Winter_2018_Q2) – *Permutation - Combination*

A team of four is to be chosen from a group of four boys and four girls.

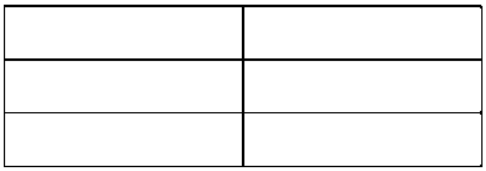
(a) Find the number of different possible teams that could be chosen. [3]

(b) Find the number of different possible teams that could be chosen, given that the team must include at least one girl and at least one boy. [2]

3 – (MathAA/11_HL_Summer_2021_Q9) – *Permutation - Combination*

[Maximum mark: 8]

A farmer has six sheep pens, arranged in a grid with three rows and two columns as shown in the following diagram.



Five sheep called Amber, Brownie, Curly, Daisy and Eden are to be placed in the pens. Each pen is large enough to hold all of the sheep. Amber and Brownie are known to fight.

Find the number of ways of placing the sheep in the pens in each of the following cases:

(a) Each pen is large enough to contain five sheep. Amber and Brownie must not be placed in the same pen. [4]

(b) Each pen may only contain one sheep. Amber and Brownie must not be placed in pens which share a boundary. [4]

4 – (MathAA/10_HL_Winter_2022_Q11) – *Permutation - Combination, Remainder & Factor Theorem*

[Maximum mark: 16]

Consider a three-digit code abc, where each of a, b and c is assigned one of the values 1, 2, 3, 4 or 5.

(a) Find the total number of possible codes

(i) assuming that each value can be repeated (for example, 121 or 444);

(ii) assuming that no value is repeated. [4]

Let $P(x) = x^3 + ax^2 + bx + c$, where each of a, b and c is assigned one of the values 1, 2, 3, 4 or 5. Assume that no value is repeated.

Consider the case where $P(x)$ has a factor of $(x^2 + 3x + 2)$.

(b) (i) Find an expression for b in terms of a.

(ii) Hence show that the only way to assign the values is $a = 4$, $b = 5$ and $c = 2$.

(iii) Express $P(x)$ as a product of linear factors.

(iv) Hence or otherwise, sketch the graph of $y = P(x)$, clearly showing the coordinates of any intercepts with the axes. [12]

1 – (MAT/12_HL_Summer_2012_Q4) – *Binomial Theorem*

(a) Expand and simplify $\left(x-\frac{2}{x}\right)^4$. *[3 marks]*

(b) Hence determine the constant term in the expansion $(2x^2+1)\left(x-\frac{2}{x}\right)^4$. *[2 marks]*

2 – (MAT/10_HL_Winter_2012_Q2) – *Binomial Theorem*

[Maximum mark: 4]

Expand and simplify $\left(\frac{x}{y}-\frac{y}{x}\right)^4$.

3 – (MAT/12_HL_Summer_2013_Q3) – *Binomial Theorem*

[Maximum mark: 4]

Expand $(2-3x)^5$ in ascending powers of x, simplifying coefficients.

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4 – (MAT/11_HL_Summer_2013_Q13) – *Probability, Binomial Theorem*

On Saturday, Alfred and Beatrice play 6 different games against each other. In each game, one of the two wins. The probability that Alfred wins any one of these games is $\frac{2}{3}$.

(a) Show that the probability that Alfred wins exactly 4 of the games is $\frac{80}{243}$. *[3 marks]*

(b) (i) Explain why the total number of possible outcomes for the results of the 6 games is 64.

(ii) By expanding $(1+x)^6$ and choosing a suitable value for x, prove

$$64 = \binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}.$$

(iii) State the meaning of this equality in the context of the 6 games played. *[4 marks]*

(c) The following day Alfred and Beatrice play the 6 games again. Assume that the probability that Alfred wins any one of these games is still $\frac{2}{3}$.

(i) Find an expression for the probability Alfred wins 4 games on the first day and 2 on the second day. Give your answer in the form $\binom{6}{r}^2\left(\frac{2}{3}\right)^s\left(\frac{1}{3}\right)^t$ where the values of r, s and t are to be found.

(ii) Using your answer to (c)(i) and 6 similar expressions write down the probability that Alfred wins a total of 6 games over the two days as the sum of 7 probabilities.

(iii) Hence prove that $\binom{12}{6} = \binom{6}{0}^2 + \binom{6}{1}^2 + \binom{6}{2}^2 + \binom{6}{3}^2 + \binom{6}{4}^2 + \binom{6}{5}^2 + \binom{6}{6}^2$. *[9 marks]*

(d) Alfred and Beatrice play n games. Let A denote the number of games Alfred wins. The expected value of A can be written as $\mathrm{E}(A) = \sum_{r=0}^{n} r\binom{n}{r}\frac{a^r}{b^n}$.

(i) Find the values of a and b.

(ii) By differentiating the expansion of $(1+x)^n$, prove that the expected number of games Alfred wins is $\frac{2n}{3}$. *[6 marks]*

5 – (MAT/12_HL_Summer_2015_Q2) – *Binomial Theorem*

[Maximum mark: 4]

Expand $(3-x)^4$ in ascending powers of x and simplify your answer.

6 – (MAT/11_HL_Summer_2015_Q4) – *Binomial Theorem, Differentiation*

(a) Expand $(x+h)^3$. [2]

(b) Hence find the derivative of $f(x)=x^3$ from first principles. [3]

7 – (MAT/10_HL_Winter_2015_Q3) – *Binomial Theorem*

(a) Write down and simplify the expansion of $(2+x)^4$ in ascending powers of x. [3]

(b) Hence find the exact value of $(2.1)^4$. [3]

8 – (MAT/12_HL_Summer_2016_Q6) – *Binomial Theorem*

Consider the expansion of $(1+x)^n$ in ascending powers of x, where $n \geq 3$.

(a) Write down the first four terms of the expansion. [2]

The coefficients of the second, third and fourth terms of the expansion are consecutive terms of an arithmetic sequence.

(b) (i) Show that $n^3 - 9n^2 + 14n = 0$.

(ii) Hence find the value of n. [6]

9 – (MAT/12_HL_Summer_2017_Q1) – *Binomial Theorem*

[Maximum mark: 5]

Find the term independent of x in the binomial expansion of $\left(2x^2 + \frac{1}{2x^3}\right)^{10}$.

10 – (MAT/10_HL_Winter_2017_Q4) – *Binomial Theorem*

[Maximum mark: 4]

Find the coefficient of x^8 in the expansion of $\left(x^2 - \frac{2}{x}\right)^7$.

11 - (MAT/11_HL_Summer_2019_Q2) - *Binomial Theorem*

[Maximum mark: 4]

Determine the first three terms of $(1-2x)^{11}$ in ascending powers of x, giving each term in its simplest form.

12 – (MathAA/12_HL_Summer_2021_Q3) – *Binomial Theorem*

[Maximum mark: 5]

In the expansion of $(x+k)^7$, where $k \in \mathbb{R}$, the coefficient of the term in x^5 is 63.

Find the possible values of k.

13 – (MathAA/10_HL_Winter_2021_Q9) – *Binomial Theorem*

[Maximum mark: 7]

Consider the expression $\frac{1}{\sqrt{1+ax}} - \sqrt{1-x}$ where $a \in \mathbb{Q}$, $a \neq 0$.

The binomial expansion of this expression, in ascending powers of x, as far as the term in x^2 is $4bx + bx^2$, where $b \in \mathbb{Q}$.

(a) Find the value of a and the value of b. [6]

(b) State the restriction which must be placed on x for this expansion to be valid. [1]

14 – (MathAA/11_HL_Summer_2022_Q6) – *Binomial Theorem*

[Maximum mark: 5]

Consider the expansion of $\left(8x^3 - \frac{1}{2x}\right)^n$ where $n \in \mathbb{Z}^+$. Determine all possible values of n for which the expansion has a non-zero constant term.

15 – (MathAA/12_HL_Summer_2022_Q5) – *Binomial Theorem*

[Maximum mark: 7]

Consider the binomial expansion $(x+1)^7 = x^7 + ax^6 + bx^5 + 35x^4 + \ldots + 1$ where $x \neq 0$ and $a, b \in \mathbb{Z}$.

(a) Show that $b = 21$. [2]

The third term in the expansion is the mean of the second term and the fourth term in the expansion.

(b) Find the possible values of x. [5]

1 - (MAT/12_HL_Summer_2012_Q1) - *Remainder & Factor Theorem*

The same remainder is found when $2x^3+kx^2+6x+32$ and $x^4-6x^2-k^2x+9$ are divided by $x+1$. Find the possible values of k.

2 – (MAT/10_HL_Winter_2013_Q1) – *Remainder & Factor Theorem*

[Maximum mark: 5]

The cubic polynomial $3x^3 + px^2 + qx - 2$ has a factor $(x+2)$ and leaves a remainder 4 when divided by $(x+1)$. Find the value of p and the value of q.

3 – (MAT/11_HL_Summer_2014_Q1) – *Remainder & Factor Theorem*

[Maximum mark: 5]

When the polynomial $3x^3 + ax + b$ is divided by $(x-2)$, the remainder is 2, and when divided by $(x+1)$, it is 5. Find the value of a and the value of b.

4 – (MAT/10_HL_Winter_2017_Q3) – *Remainder & Factor Theorem*

[Maximum mark: 6]

Consider the polynomial $q(x) = 3x^3 - 11x^2 + kx + 8$.

(a) Given that $q(x)$ has a factor $(x - 4)$, find the value of k. [3]

(b) Hence or otherwise, factorize $q(x)$ as a product of linear factors. [3]

5 - (MAT/11_HL_Summer_2018_Q1) - *Remainder & Factor Theorem*

[Maximum mark: 5]

Let $f(x) = x^4 + px^3 + qx + 5$ where p, q are constants.
The remainder when $f(x)$ is divided by $(x+1)$ is 7, and the remainder when $f(x)$ is divided by $(x-2)$ is 1. Find the value of p and the value of q.

6 – (MAT/11_HL_Summer_2019_Q10) – *Remainder & Factor Theorem, Probability*

The function $p(x)$ is defined by $p(x) = x^3 - 3x^2 + 8x - 24$ where $x \in \mathbb{R}$.

(a) Find the remainder when $p(x)$ is divided by

(i) $(x-2)$

(ii) $(x-3)$. [3]

(b) Prove that $p(x)$ has only one real zero. [4]

(c) Write down the transformation that will transform the graph of $y = p(x)$ onto the graph of $y = 8x^3 - 12x^2 + 16x - 24$. [2]

The random variable X follows a Poisson distribution with a mean of λ and $6\mathrm{P}(X=3) = 3\mathrm{P}(X=2) - 2\mathrm{P}(X=1) + 3\mathrm{P}(X=0)$.

(d) Find the value of λ. [6]

7 – (MathAA/10_HL_Winter_2022_Q11) – *Permutation - Combination, Remainder & Factor Theorem*

[Maximum mark: 16]

Consider a three-digit code abc, where each of a, b and c is assigned one of the values 1, 2, 3, 4 or 5.

(a) Find the total number of possible codes

(i) assuming that each value can be repeated (for example, 121 or 444);

(ii) assuming that no value is repeated. [4]

Let $P(x) = x^3 + ax^2 + bx + c$, where each of a, b and c is assigned one of the values 1, 2, 3, 4 or 5. Assume that no value is repeated.

Consider the case where $P(x)$ has a factor of $(x^2 + 3x + 2)$.

(b) (i) Find an expression for b in terms of a.

(ii) Hence show that the only way to assign the values is $a = 4$, $b = 5$ and $c = 2$.

(iii) Express $P(x)$ as a product of linear factors.

(iv) Hence or otherwise, sketch the graph of $y = P(x)$, clearly showing the coordinates of any intercepts with the axes. [12]

1 - (MAT/11_HL_Summer_2012_Q11) - *Mathematical Induction*

(a) A and U are square matrices, and $\boldsymbol{X} = \boldsymbol{U}^{-1}\boldsymbol{A}\boldsymbol{U}$. Use mathematical induction to prove that $\boldsymbol{X}^n = \boldsymbol{U}^{-1}\boldsymbol{A}^n\boldsymbol{U}$, for $n \in \mathbb{Z}^+$. *[7 marks]*

(b) Let $A = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}$ and $\boldsymbol{U} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$.

(i) Find the matrix $\boldsymbol{D}$ such that $\boldsymbol{AU} = \boldsymbol{UD}$.

(ii) Write down the matrix $\boldsymbol{D}^2$.

(iii) **Hence** prove that $A^{2n} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, for $n \in \mathbb{Z}^+$.

(iv) Using the result from part (iii), show that $(A^n)^{-1} = A^n$, for $n \in \mathbb{Z}^+$. *[10 marks]*

2 – (MAT/12_HL_Summer_2012_Q13) – *Differentiation, Mathematical Induction*

(a) Using the definition of a derivative as $f'(x) = \lim_{h \to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$, show that the derivative of $\frac{1}{2x+1}$ is $\frac{-2}{(2x+1)^2}$. *[4 marks]*

(b) Prove by induction that the nth derivative of $(2x+1)^{-1}$ is $(-1)^n \frac{2^n n!}{(2x+1)^{n+1}}$. *[9 marks]*

3 – (MAT/10_HL_Winter_2012_Q12) – *Functions - Roots, Mathematical Induction*

Consider a function f, defined by $f(x)=\dfrac{x}{2-x}$ for $0 \le x \le 1$.

(a) Find an expression for $(f \circ f)(x)$. *[3 marks]*

Let $F_n(x)=\dfrac{x}{2^n-(2^n-1)x}$, where $0 \le x \le 1$.

(b) Use mathematical induction to show that for any $n \in \mathbb{Z}^+$

$$\underbrace{(f \circ f \circ \ldots \circ f)}_{n \text{ times}}(x)=F_n(x).$$ *[8 marks]*

(c) Show that $F_{-n}(x)$ is an expression for the inverse of F_n. *[6 marks]*

(d) (i) State $F_n(0)$ and $F_n(1)$.

(ii) Show that $F_n(x)<x$, given $0<x<1$, $n \in \mathbb{Z}^+$.

(iii) For $n \in \mathbb{Z}^+$, let A_n be the area of the region enclosed by the graph of F_n^{-1}, the x-axis and the line $x=1$. Find the area B_n of the region enclosed by F_n and F_n^{-1} in terms of A_n. *[6 marks]*

4 – (MAT/10_HL_Winter_2013_Q6) – *Mathematical Induction*

[Maximum mark: 7]

Prove by mathematical induction that $n^3 + 11n$ is divisible by 3 for all $n \in \mathbb{Z}^+$.

5 – (MAT/10_HL_Winter_2014_Q8) – *Mathematical Induction*

[Maximum mark: 7]

Use mathematical induction to prove that $(2n)! \geq 2^n (n!)^2$, $n \in \mathbb{Z}^+$.

6 – (MAT/12_HL_Summer_2015_Q13) – *Mathematical Induction*

(a) Show that $\frac{1}{\sqrt{n}+\sqrt{n+1}}=\sqrt{n+1}-\sqrt{n}$ where $n \geq 0$, $n \in \mathbb{Z}$. [2]

(b) Hence show that $\sqrt{2}-1<\frac{1}{\sqrt{2}}$. [2]

(c) Prove, by mathematical induction, that $\sum_{r=1}^{r=n} \frac{1}{\sqrt{r}}>\sqrt{n}$ for $n \geq 2$, $n \in \mathbb{Z}$. [9]

7 – (MAT/10_HL_Winter_2015_Q8) – *Mathematical Induction*

(a) Show that $\sin\left(\theta+\frac{\pi}{2}\right)=\cos\theta$. [1]

(b) Consider $f(x)=\sin(ax)$ where a is a constant. Prove by mathematical induction that $f^{(n)}(x)=a^n\sin\left(ax+\frac{n\pi}{2}\right)$ where $n\in\mathbb{Z}^+$ and $f^{(n)}(x)$ represents the nth derivative of $f(x)$. [7]

8 – (MAT/12_HL_Summer_2016_Q8) – *Mathematical Induction*

[Maximum mark: 8]

Use mathematical induction to prove that $n\left(n^2+5\right)$ is divisible by 6 for $n \in \mathbb{Z}^+$.

9 – (MAT/11_HL_Summer_2017_Q8) – *Mathematical Induction*

[Maximum mark: 6]

Use the method of mathematical induction to prove that $4^n + 15n - 1$ is divisible by 9 for $n \in \mathbb{Z}^+$.

10 – (MAT/12_HL_Summer_2017_Q8) – *Mathematical Induction*

[Maximum mark: 9]

Prove by mathematical induction that $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{n-1}{2}=\binom{n}{3}$,

where $n \in \mathbb{Z}, n \geq 3$.

11 – (MAT/10_HL_Winter_2017_Q11) – *Mathematical Induction*

[Maximum mark: 21]

Consider the function $f_n(x) = (\cos 2x)(\cos 4x)\ldots(\cos 2^n x)$, $n \in \mathbb{Z}^+$.

(a) Determine whether f_n is an odd or even function, justifying your answer. [2]

(b) By using mathematical induction, prove that

$$f_n(x) = \frac{\sin 2^{n+1} x}{2^n \sin 2x},\ x \neq \frac{m\pi}{2} \text{ where } m \in \mathbb{Z}.$$ [8]

(c) Hence or otherwise, find an expression for the derivative of $f_n(x)$ with respect to x. [3]

(d) Show that, for $n > 1$, the equation of the tangent to the curve $y = f_n(x)$ at $x = \frac{\pi}{4}$ is $4x - 2y - \pi = 0$. [8]

12 - (MAT/11_HL_Summer_2018_Q6) - *Mathematical Induction*

[Maximum mark: 7]

Use the principle of mathematical induction to prove that

$$1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+\ldots+n\left(\frac{1}{2}\right)^{n-1}=4-\frac{n+2}{2^{n-1}}, \text{ where } n\in\mathbb{Z}^+.$$

13 – (MAT/10_HL_Winter_2018_Q6) – *Mathematical Induction*

[Maximum mark: 6]

Use mathematical induction to prove that $\sum_{r=1}^{n} r(r!) = (n+1)! - 1$, for $n \in \mathbb{Z}^+$.

14 – (MAT/10_HL_Winter_2019_Q6) – *Mathematical Induction*

[Maximum mark: 7]

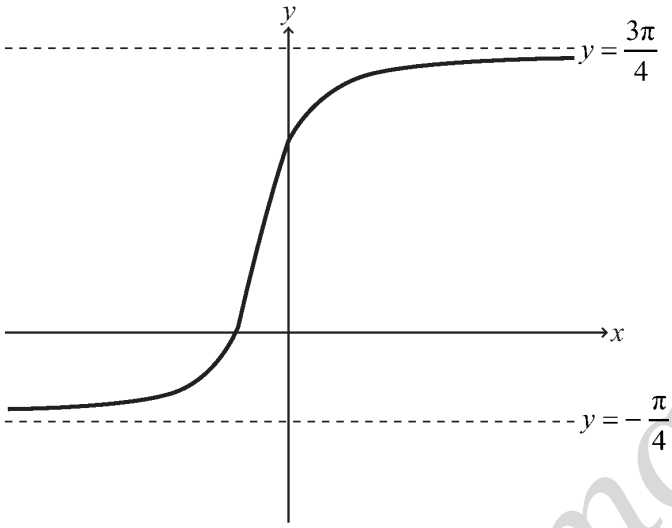
Consider the function $f(x) = x\mathrm{e}^{2x}$, where $x \in \mathbb{R}$. The nth derivative of $f(x)$ is denoted by $f^{(n)}(x)$.

Prove, by mathematical induction, that $f^{(n)}(x) = (2^n x + n2^{n-1})\mathrm{e}^{2x}$, $n \in \mathbb{Z}^+$.

15 – (MathAA/12_HL_Summer_2021_Q12) – *Trigonometry, Mathematical Induction*

[Maximum mark: 19]

The following diagram shows the graph of $y = \arctan(2x+1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$, with asymptotes at $y = -\frac{\pi}{4}$ and $y = \frac{3\pi}{4}$.



(a) Describe a sequence of transformations that transforms the graph of $y = \arctan x$ to the graph of $y = \arctan(2x+1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$. [3]

(b) Show that $\arctan p + \arctan q \equiv \arctan\left(\frac{p+q}{1-pq}\right)$ where p, $q > 0$ and $pq < 1$. [4]

(c) Verify that $\arctan(2x+1) = \arctan\left(\frac{x}{x+1}\right) + \frac{\pi}{4}$ for $x \in \mathbb{R}$, $x > 0$. [3]

(d) Using mathematical induction and the result from part (b), prove that $\sum_{r=1}^{n} \arctan\left(\frac{1}{2r^2}\right) = \arctan\left(\frac{n}{n+1}\right)$ for $n \in \mathbb{Z}^+$. [9]

16 – (MathAA/11_HL_Summer_2021_Q12) – *Differentiation, Mathematical Induction*

[Maximum mark: 20]

Let $f(x)=\sqrt{1+x}$ for $x>-1$.

(a) Show that $f''(x)=-\dfrac{1}{4\sqrt{(1+x)^3}}$. [3]

(b) Use mathematical induction to prove that $f^{(n)}(x)=\left(-\dfrac{1}{4}\right)^{n-1}\dfrac{(2n-3)!}{(n-2)!}(1+x)^{\frac{1}{2}-n}$

for $n\in\mathbb{Z},\ n\geq 2$. [9]

Let $g(x)=\mathrm{e}^{mx},\ m\in\mathbb{Q}$.

Consider the function h defined by $h(x)=f(x)\times g(x)$ for $x>-1$.

It is given that the x^2 term in the Maclaurin series for $h(x)$ has a coefficient of $\dfrac{7}{4}$.

(c) Find the possible values of m. [8]

17 – (MathAA/10_HL_Winter_2021_Q11) – *Mathematical Induction*

[Maximum mark: 14]

(a) Prove by mathematical induction that $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2\mathrm{e}^x\right)=\left[x^2+2nx+n\left(n-1\right)\right]\mathrm{e}^x$ for $n\in\mathbb{Z}$. [7]

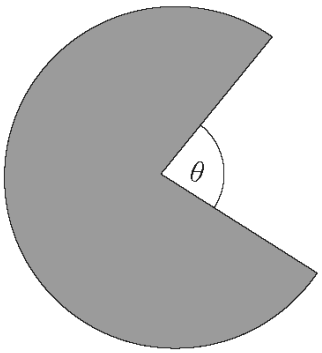
(b) Hence or otherwise, determine the Maclaurin series of $f(x)=x^2\mathrm{e}^x$ in ascending powers of x, up to and including the term in x^4. [3]

(c) Hence or otherwise, determine the value of $\lim\limits_{x\to 0}\left[\frac{\left(x^2\mathrm{e}^x-x^2\right)^3}{x^9}\right]$. [4]

1 – (MAT/11_HL_Summer_2015_Q1) – *Radian*

The logo, for a company that makes chocolate, is a sector of a circle of radius $2\,\text{cm}$, shown as shaded in the diagram. The area of the logo is $3\pi\,\text{cm}^2$.

diagram not to scale



(a) Find, in radians, the value of the angle θ, as indicated on the diagram. [3]

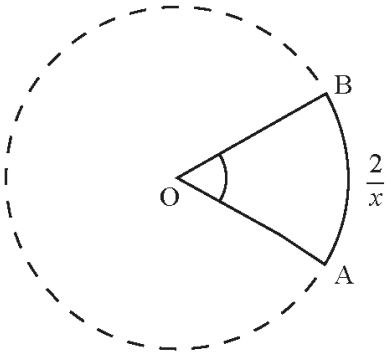
(b) Find the total length of the perimeter of the logo. [2]

2 – (MAT/10_HL_Winter_2015_Q1) – *Radian*

[Maximum mark: 4]

The following diagram shows a sector of a circle where $\mathrm{A\hat{O}B} = x$ radians and the length of the arc $\mathrm{AB} = \frac{2}{x}\,\mathrm{cm}$.

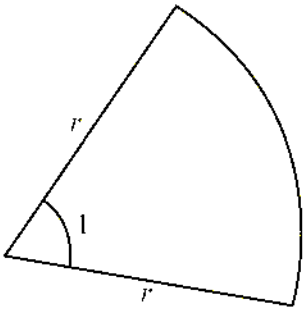
Given that the area of the sector is $16\,\mathrm{cm}^2$, find the length of the arc AB.



3 – (MAT/11_HL_Summer_2019_Q3) – *Radian*

[Maximum mark: 4]

A sector of a circle with radius r cm, where $r > 0$, is shown on the following diagram.
The sector has an angle of 1 radian at the centre.



Let the area of the sector be A cm^2 and the perimeter be P cm. Given that $A = P$, find the value of r.

1 – (MAT/11_HL_Summer_2012_Q5) – *Trigonometry*

Let $f(x)=\dfrac{\sin 3x}{\sin x}-\dfrac{\cos 3x}{\cos x}$.

(a) For what values of x does $f(x)$ not exist? *[2 marks]*

(b) Simplify the expression $\dfrac{\sin 3x}{\sin x}-\dfrac{\cos 3x}{\cos x}$. *[5 marks]*

2 – (MAT/12_HL_Summer_2012_Q9) – *Trigonometry*

[Maximum mark: 6]

Show that $\dfrac{\cos A + \sin A}{\cos A - \sin A} = \sec 2A + \tan 2A$.

3 - (MAT/11_HL_Summer_2012_Q10) - *Trigonometry*

In the triangle ABC, $\mathrm{A\hat{B}C} = 90^\circ$, $\mathrm{AC} = \sqrt{2}$ and $\mathrm{AB} = \mathrm{BC} + 1$.

(a) Show that $\cos\hat{A} - \sin\hat{A} = \dfrac{1}{\sqrt{2}}$. *[3 marks]*

(b) By squaring both sides of the equation in part (a), solve the equation to find the angles in the triangle. *[8 marks]*

(c) Apply Pythagoras' theorem in the triangle ABC to find BC, and hence show that $\sin\hat{A} = \dfrac{\sqrt{6} - \sqrt{2}}{4}$. *[6 marks]*

(d) Hence, or otherwise, calculate the length of the perpendicular from B to [AC]. *[4 marks]*

4 – (MAT/10_HL_Winter_2012_Q1) – *Trigonometry*

[Maximum mark: 4]

Given that $\frac{\pi}{2} < \alpha < \pi$ and $\cos\alpha = -\frac{3}{4}$, find the value of $\sin 2\alpha$.

5 – (MAT/10_HL_Winter_2012_Q7) – *Trigonometry*

In the triangle PQR, $\mathrm{PQ} = 6$, $\mathrm{PR} = k$ and $\mathrm{P\hat{Q}R} = 30^\circ$.

(a) For the case $k = 4$, find the two possible values of QR. *[4 marks]*

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(b) Determine the values of k for which the conditions above define a unique triangle. *[3 marks]*

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6 – (MAT/12_HL_Summer_2013_Q10) – *Trigonometry*

(a) Given that $\arctan\left(\frac{1}{5}\right)+\arctan\left(\frac{1}{8}\right)=\arctan\left(\frac{1}{p}\right)$, where $p \in \mathbb{Z}^+$, find p. *[3 marks]*

(b) Hence find the value of $\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{1}{5}\right)+\arctan\left(\frac{1}{8}\right)$. *[3 marks]*

7 – (MAT/11_HL_Summer_2013_Q11) – *Trigonometry*

(a) (i) Express $\cos\left(\frac{\pi}{6}+x\right)$ in the form $a\cos x - b\sin x$ where $a, b \in \mathbb{R}$.

(ii) Hence solve $\sqrt{3}\cos x - \sin x = 1$ for $0 \le x \le 2\pi$. *[7 marks]*

(b) Let $p(x) = 2x^3 - x^2 - 2x + 1$.

(i) Show that $x = 1$ is a zero of p.

(ii) Hence find all the solutions of $2x^3 - x^2 - 2x + 1 = 0$.

(iii) Express $\sin 2\theta \cos\theta + \sin^2\theta$ in terms of $\sin\theta$.

(iv) Hence solve $\sin 2\theta \cos\theta + \sin^2\theta = 1$ for $0 \le \theta \le 2\pi$. *[12 marks]*

8 – (MAT/10_HL_Winter_2013_Q8) – *Trigonometry*

(a) Prove the trigonometric identity $\sin(x+y)\sin(x-y)=\sin^2 x-\sin^2 y$. *[4]*

(b) Given $f(x)=\sin\left(x+\frac{\pi}{6}\right)\sin\left(x-\frac{\pi}{6}\right)$, $x\in[0,\ \pi]$, find the range of f. *[2]*

(c) Given $g(x)=\csc\left(x+\frac{\pi}{6}\right)\csc\left(x-\frac{\pi}{6}\right)$, $x\in[0,\ \pi]$, $x\neq\frac{\pi}{6}$, $x\neq\frac{5\pi}{6}$, find the range of g. *[2]*

9 – (MAT/10_HL_Winter_2013_Q12) – *Complex Numbers, Trigonometry, Integration*

Consider the complex number $z = \cos\theta + \mathrm{i}\sin\theta$.

(a) Use De Moivre's theorem to show that $z^n + z^{-n} = 2\cos n\theta$, $n \in \mathbb{Z}^+$. *[2]*

(b) Expand $\left(z + z^{-1}\right)^4$. *[1]*

(c) Hence show that $\cos^4\theta = p\cos 4\theta + q\cos 2\theta + r$, where p, q and r are constants to be determined. *[4]*

(d) Show that $\cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$. *[3]*

(e) Hence find the value of $\int_0^{\frac{\pi}{2}} \cos^6\theta \,\mathrm{d}\theta$. *[3]*

The region S is bounded by the curve $y = \sin x\cos^2 x$ and the x-axis between $x = 0$ and $x = \frac{\pi}{2}$.

(f) S is rotated through 2π radians about the x-axis. Find the value of the volume generated. *[4]*

(g) (i) Write down an expression for the constant term in the expansion of $\left(z + z^{-1}\right)^{2k}$, $k \in \mathbb{Z}^+$.

(ii) Hence determine an expression for $\int_0^{\frac{\pi}{2}} \cos^{2k}\theta \,\mathrm{d}\theta$ in terms of k. *[3]*

10 – (MAT/11_HL_Summer_2014_Q5) – *Trigonometry*

(a) Use the identity $\cos 2\theta = 2\cos^2\theta - 1$ to prove that $\cos\frac{1}{2}x = \sqrt{\frac{1+\cos x}{2}}$, $0 \le x \le \pi$. *[2]*

(b) Find a similar expression for $\sin\frac{1}{2}x$, $0 \le x \le \pi$. *[2]*

(c) Hence find the value of $\int_0^{\frac{\pi}{2}}\left(\sqrt{1+\cos x}+\sqrt{1-\cos x}\right)\mathrm{d}x$. *[4]*

11 – (MAT/12_HL_Summer_2014_Q5) – *Graphs, Trigonometry*

(a) Sketch the graph of $y = \left|\cos\left(\frac{x}{4}\right)\right|$ for $0 \le x \le 8\pi$. *[2]*

(b) Solve $\left|\cos\left(\frac{x}{4}\right)\right| = \frac{1}{2}$ for $0 \le x \le 8\pi$. *[3]*

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12 – (MAT/11_HL_Summer_2014_Q7) – *Trigonometry*

[Maximum mark: 5]

The triangle ABC is equilateral of side 3 cm. The point D lies on [BC] such that $BD = 1\,\text{cm}$.
Find $\cos D\hat{A}C$.

13 – (MAT/12_HL_Summer_2014_Q9) – *Sequences - Series, Trigonometry*

The first three terms of a geometric sequence are $\sin x$, $\sin 2x$ and $4\sin x\cos^2 x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

(a) Find the common ratio r. *[1]*

(b) Find the set of values of x for which the geometric series $\sin x + \sin 2x + 4\sin x\cos^2 x + \ldots$ converges. *[3]*

Consider $x = \arccos\left(\frac{1}{4}\right)$, $x > 0$.

(c) Show that the sum to infinity of this series is $\frac{\sqrt{15}}{2}$. *[3]*

14 – (MAT/11_HL_Summer_2014_Q10) – *Trigonometry*

[Maximum mark: 6]

Given that $\sin x + \cos x = \frac{2}{3}$, find $\cos 4x$.

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15 – (MAT/10_HL_Winter_2014_Q13) – *Complex Numbers, Trigonometry, Integration*

(a) (i) Show that $(1+\mathrm{i}\tan\theta)^n+(1-\mathrm{i}\tan\theta)^n=\dfrac{2\cos n\theta}{\cos^n\theta}$, $\cos\theta\neq 0$.

(ii) Hence verify that $\mathrm{i}\tan\dfrac{3\pi}{8}$ is a root of the equation $(1+z)^4+(1-z)^4=0$, $z\in\mathbb{C}$.

(iii) State another root of the equation $(1+z)^4+(1-z)^4=0$, $z\in\mathbb{C}$. *[10]*

(b) (i) Use the double angle identity $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}$ to show that $\tan\dfrac{\pi}{8}=\sqrt{2}-1$.

(ii) Show that $\cos 4x=8\cos^4 x-8\cos^2 x+1$.

(iii) Hence find the value of $\displaystyle\int_0^{\frac{\pi}{8}}\frac{2\cos 4x}{\cos^2 x}\,\mathrm{d}x$. *[13]*

16 – (MAT/12_HL_Summer_2015_Q3) – *Trigonometry*

[Maximum mark: 6]

Find all solutions to the equation $\tan x + \tan 2x = 0$ where $0^\circ \le x < 360^\circ$.

17 – (MAT/12_HL_Summer_2015_Q6) – *Trigonometry*

In triangle ABC, $\mathrm{BC} = \sqrt{3}\,\mathrm{cm}$, $\mathrm{A\hat{B}C} = \theta$ and $\mathrm{B\hat{C}A} = \frac{\pi}{3}$.

(a) Show that length $\mathrm{AB} = \frac{3}{\sqrt{3}\cos\theta + \sin\theta}$. [4]

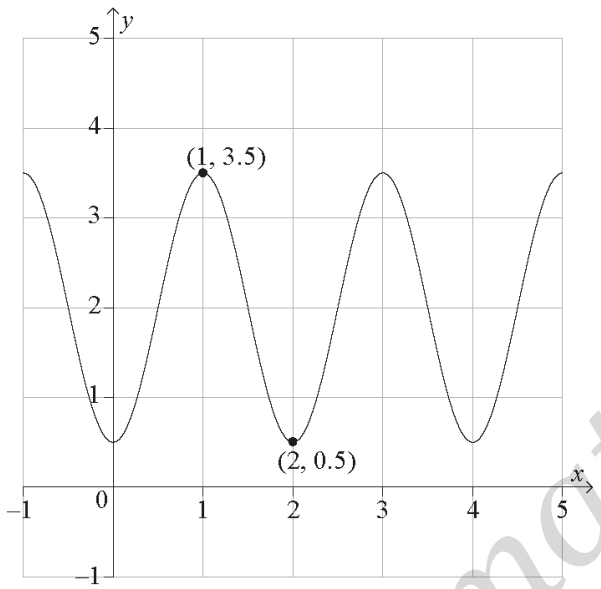
(b) Given that AB has a minimum value, determine the value of θ for which this occurs. [4]

18 – (MAT/10_HL_Winter_2015_Q9) – *Trigonometry*

Solve the equation $\sin 2x - \cos 2x = 1 + \sin x - \cos x$ for $x \in [-\pi, \pi]$.

19 – (MAT/11_HL_Summer_2016_Q3) – *Trigonometry*

The following diagram shows the curve $y = a\sin\left(b(x+c)\right) + d$, where a, b, c and d are all positive constants. The curve has a maximum point at $(1, 3.5)$ and a minimum point at $(2, 0.5)$.



(a) Write down the value of a and the value of d. [2]

(b) Find the value of b. [2]

(c) Find the smallest possible value of c, given $c > 0$. [2]

20 – (MAT/12_HL_Summer_2016_Q3) – *Trigonometry, Integration*

(a) Show that $\cot\alpha = \tan\left(\frac{\pi}{2} - \alpha\right)$ for $0 < \alpha < \frac{\pi}{2}$. [1]

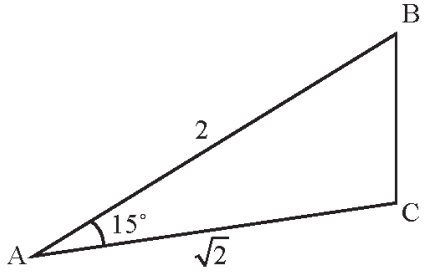
(b) Hence find $\int_{\tan\alpha}^{\cot\alpha} \frac{1}{1+x^2}\,dx$, $0 < \alpha < \frac{\pi}{2}$. [4]

21 – (MAT/11_HL_Summer_2016_Q5) – *Trigonometry*

(a) Expand and simplify $\left(1-\sqrt{3}\right)^2$. [1]

(b) By writing 15° as $60^\circ - 45^\circ$ find the value of $\cos(15^\circ)$. [3]

The following diagram shows the triangle ABC where $\mathrm{AB} = 2$, $\mathrm{AC} = \sqrt{2}$ and $\mathrm{B\hat{A}C} = 15^\circ$.



(c) Find BC in the form $a + \sqrt{b}$ where $a, b \in \mathbb{Z}$. [4]

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22 – (MAT/12_HL_Summer_2016_Q9) – *Trigonometry*

Consider the equation $\frac{\sqrt{3}-1}{\sin x}+\frac{\sqrt{3}+1}{\cos x}=4\sqrt{2}$, $0<x<\frac{\pi}{2}$. Given that $\sin\left(\frac{\pi}{12}\right)=\frac{\sqrt{6}-\sqrt{2}}{4}$ and $\cos\left(\frac{\pi}{12}\right)=\frac{\sqrt{6}+\sqrt{2}}{4}$

(a) verify that $x=\frac{\pi}{12}$ is a solution to the equation; [3]

(b) hence find the other solution to the equation for $0<x<\frac{\pi}{2}$. [5]

23 – (MAT/10_HL_Winter_2016_Q13) – *Trigonometry*

(a) Find the value of $\sin\frac{\pi}{4}+\sin\frac{3\pi}{4}+\sin\frac{5\pi}{4}+\sin\frac{7\pi}{4}+\sin\frac{9\pi}{4}$. [2]

(b) Show that $\frac{1-\cos 2x}{2\sin x}\equiv\sin x, x\neq k\pi$ where $k\in\mathbb{Z}$. [2]

(c) Use the principle of mathematical induction to prove that
$\sin x+\sin 3x+\ldots+\sin(2n-1)x=\frac{1-\cos 2nx}{2\sin x}, n\in\mathbb{Z}^+, x\neq k\pi$ where $k\in\mathbb{Z}$. [9]

(d) Hence or otherwise solve the equation $\sin x+\sin 3x=\cos x$ in the interval $0<x<\pi$. [6]

24 – (MAT/11_HL_Summer_2017_Q3) – *Trigonometry*

[Maximum mark: 5]

Solve the equation $\sec^2 x + 2\tan x = 0$, $0 \le x \le 2\pi$.

25 - (MAT/12_HL_Summer_2017_Q11) - *Trigonometry, Complex Numbers*

(a) Solve $2\sin(x+60^\circ)=\cos(x+30^\circ)$, $0^\circ \le x \le 180^\circ$. [5]

(b) Show that $\sin 105^\circ + \cos 105^\circ = \frac{1}{\sqrt{2}}$. [3]

(c) Let $z = 1 - \cos 2\theta - \mathrm{i}\sin 2\theta$, $z \in \mathbb{C}$, $0 \le \theta \le \pi$.

(i) Find the modulus and argument of z in terms of θ. Express each answer in its simplest form.

(ii) Hence find the cube roots of z in modulus-argument form. [14]

26 – (MAT/11_HL_Summer_2018_Q8) – *Trigonometry*

[Maximum mark: 5]

Let $a = \sin b$, $0 < b < \frac{\pi}{2}$.

Find, in terms of b, the solutions of $\sin 2x = -a$, $0 \le x \le \pi$.

27 – (MAT/10_HL_Winter_2018_Q3) – *Trigonometry*

Consider the function $g(x) = 4\cos x + 1$, $a \le x \le \frac{\pi}{2}$ where $a < \frac{\pi}{2}$.

(a) For $a = -\frac{\pi}{2}$, sketch the graph of $y = g(x)$. Indicate clearly the maximum and minimum values of the function. [3]

(b) Write down the least value of a such that g has an inverse. [1]

(c) For the value of a found in part (b),

(i) write down the domain of g^{-1};

(ii) find an expression for $g^{-1}(x)$. [3]

28 – (MAT/11_HL_Summer_2019_Q4) – *Trigonometry*

The lengths of two of the sides in a triangle are $4\,\text{cm}$ and $5\,\text{cm}$. Let θ be the angle between the two given sides. The triangle has an area of $\frac{5\sqrt{15}}{2}\,\text{cm}^2$.

(a) Show that $\sin\theta = \frac{\sqrt{15}}{4}$. [1]

(b) Find the two possible values for the length of the third side. [6]

29 – (MAT/11_HL_Summer_2019_Q9) – *Trigonometry, Integration*

(a) Show that $(\sin x + \cos x)^2 = 1 + \sin 2x$. [2]

(b) Show that $\sec 2x + \tan 2x = \dfrac{\cos x + \sin x}{\cos x - \sin x}$. [4]

(c) Hence or otherwise find $\int_0^{\frac{\pi}{6}} (\sec 2x + \tan 2x)\,\mathrm{d}x$ in the form $\ln\left(a + \sqrt{b}\right)$ where $a, b \in \mathbb{Z}$. [9]

30 – (MAT/10_HL_Winter_2019_Q4) – *Trigonometry*

[Maximum mark: 7]

A and B are acute angles such that $\cos A = \frac{2}{3}$ and $\sin B = \frac{1}{3}$.

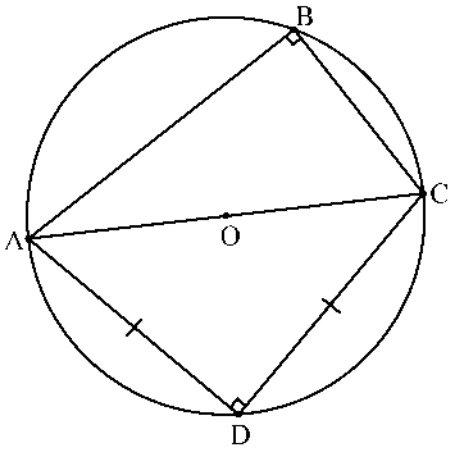
Show that $\cos(2A + B) = -\frac{2\sqrt{2}}{27} - \frac{4\sqrt{5}}{27}$.

31 – (MAT/10_HL_Winter_2019_Q9) – *Trigonometry*

[Maximum mark: 14]

(a) Given that $\cos 75^\circ = q$, show that $\cos 105^\circ = -q$. [1]

In the following diagram, the points A, B, C and D are on the circumference of a circle with centre O and radius r. [AC] is a diameter of the circle. $BC = r$. $AD = CD$ and $A\hat{B}C = A\hat{D}C = 90$.



(b) Show that $B\hat{A}D = 75^\circ$. [3]

(c) (i) By considering triangle ABD, show that $BD^2 = 5r^2 - 2r^2 q\sqrt{6}$.

(ii) By considering triangle CBD, find another expression for BD^2 in terms of r and q. [7]

(d) Use your answers to part (c) to show that $\cos 75 = \dfrac{1}{\sqrt{6}+\sqrt{2}}$. [3]

32 - (MAT/10_HL_Winter_2020_Q6) - *Quadratics, Trigonometry*

[Maximum mark: 4]

Consider the equation $ax^2 + bx + c = 0$, where $a \neq 0$. Given that the roots of this equation are $x = \sin\theta$ and $x = \cos\theta$, show that $b^2 = a^2 + 2ac$.

33 – (MAT/10_HL_Winter_2020_Q8) – *Trigonometry*

(a) Show that $\frac{\sin x \tan x}{1-\cos x} \equiv 1 + \frac{1}{\cos x}$, $x \neq 2n\pi$, $n \in \mathbb{Z}$. [3]

(b) Hence determine the range of values of k for which $\frac{\sin x \tan x}{1-\cos x} = k$ has no real solutions. [4]

34 – (MathAA/12_HL_Summer_2021_Q2) – *Trigonometry*

[Maximum mark: 7]

Solve the equation $2\cos^2 x + 5\sin x = 4$, $0 \le x \le 2\pi$.

35 – (MathAA/11_HL_Summer_2021_Q5) – *Trigonometry*

[Maximum mark: 8]

(a) Show that $\sin 2x + \cos 2x - 1 = 2\sin x(\cos x - \sin x)$. [2]

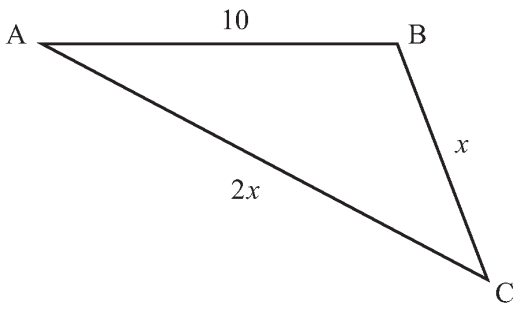
(b) Hence or otherwise, solve $\sin 2x + \cos 2x - 1 + \cos x - \sin x = 0$ for $0 < x < 2\pi$. [6]

36 – (MathAA/12_HL_Summer_2021_Q6) – *Trigonometry*

[Maximum mark: 7]

The following diagram shows triangle ABC, with $\mathrm{AB} = 10$, $\mathrm{BC} = x$ and $\mathrm{AC} = 2x$.

diagram not to scale



Given that $\cos \hat{\mathrm{C}} = \frac{3}{4}$, find the area of the triangle.

Give your answer in the form $\frac{p\sqrt{q}}{2}$ where $p, q \in \mathbb{Z}^+$.

37 – (MathAA/11_HL_Summer_2021_Q6) – *Trigonometry*

[Maximum mark: 4]

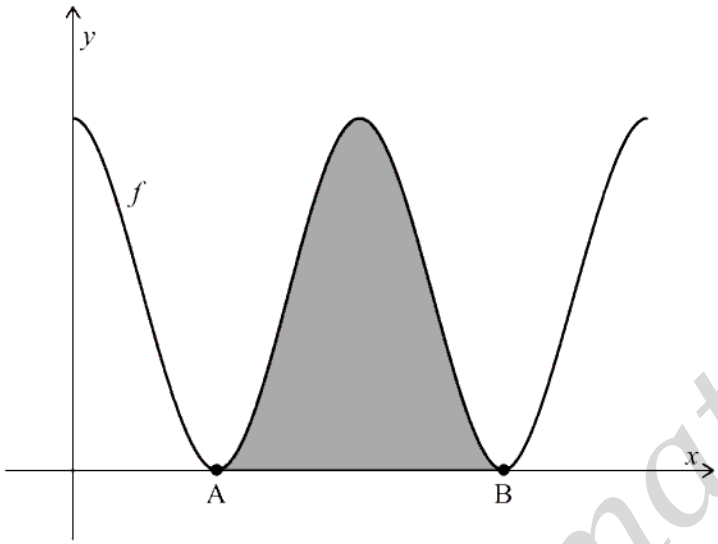
It is given that $\operatorname{cosec}\theta = \frac{3}{2}$, where $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Find the exact value of $\cot\ \theta$.

38 – (MathAA/12_HL_Summer_2021_Q10) – *Trigonometry*

[Maximum mark: 15]

Consider the function f defined by $f(x) = 6 + 6\cos x$, for $0 \le x \le 4\pi$.

The following diagram shows the graph of $y = f(x)$.



The graph of f touches the x-axis at points A and B, as shown. The shaded region is enclosed by the graph of $y = f(x)$ and the x-axis, between the points A and B.

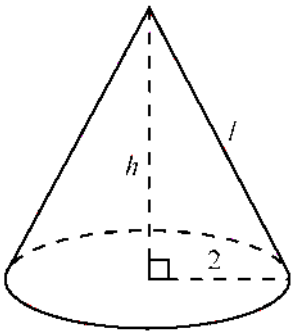
(a) Find the x-coordinates of A and B. [3]

(b) Show that the area of the shaded region is 12π. [5]

The right cone in the following diagram has a total surface area of 12π, equal to the shaded area in the previous diagram.

The cone has a base radius of 2, height h, and slant height l.

diagram not to scale



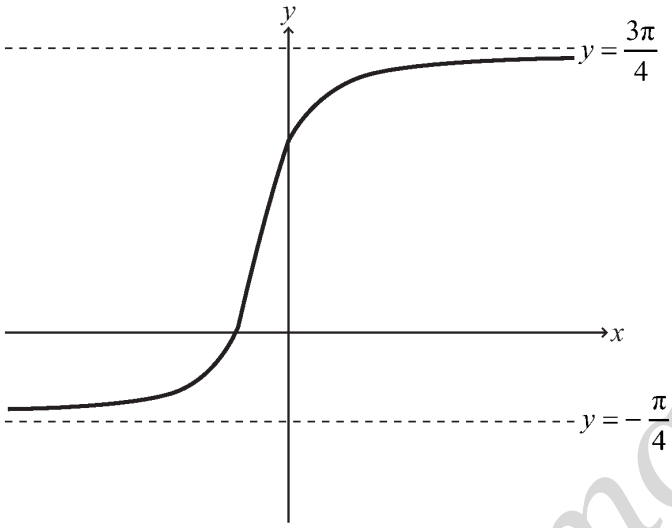
(c) Find the value of l. [3]

(d) Hence, find the volume of the cone. [4]

39 – (MathAA/12_HL_Summer_2021_Q12) – *Trigonometry, Mathematical Induction*

[Maximum mark: 19]

The following diagram shows the graph of $y = \arctan(2x+1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$, with asymptotes at $y = -\frac{\pi}{4}$ and $y = \frac{3\pi}{4}$.



(a) Describe a sequence of transformations that transforms the graph of $y = \arctan x$ to the graph of $y = \arctan(2x+1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$. [3]

(b) Show that $\arctan p + \arctan q \equiv \arctan\left(\frac{p+q}{1-pq}\right)$ where p, $q > 0$ and $pq < 1$. [4]

(c) Verify that $\arctan(2x+1) = \arctan\left(\frac{x}{x+1}\right) + \frac{\pi}{4}$ for $x \in \mathbb{R}$, $x > 0$. [3]

(d) Using mathematical induction and the result from part (b), prove that $\sum_{r=1}^{n} \arctan\left(\frac{1}{2r^2}\right) = \arctan\left(\frac{n}{n+1}\right)$ for $n \in \mathbb{Z}^+$. [9]

40 – (MathAA/10_HL_Winter_2021_Q6) – *Trigonometry*

[Maximum mark: 7]

(a) Show that $2x-3-\frac{6}{x-1}=\frac{2x^2-5x-3}{x-1}$, $x \in \mathbb{R}$, $x \neq 1$. [2]

(b) Hence or otherwise, solve the equation $2\sin 2\theta - 3 - \frac{6}{\sin 2\theta - 1} = 0$ for $0 \leq \theta \leq \pi$, $\theta \neq \frac{\pi}{4}$. [5]

41 – (MathAA/12_HL_Summer_2022_Q4) – *Trigonometry*

[Maximum mark: 5]

Find the least positive value of x for which $\cos\left(\frac{x}{2}+\frac{\pi}{3}\right)=\frac{1}{\sqrt{2}}$.

42 – (MathAA/10_HL_Winter_2022_Q8) – *Trigonometry, Algebra-functions - Roots*

[Maximum mark: 5]

Let $f(x) = \cos(x - k)$, where $0 \le x \le a$ and $a, k \in \mathbb{R}$.

(a) Consider the case where $k = \frac{\pi}{2}$.

By sketching a suitable graph, or otherwise, find the largest value of a for which the inverse function f^{-1} exists. [2]

(b) Find the largest value of a for which the inverse function f^{-1} exists in the case where $k = \pi$. [1]

(c) Find the largest value of a for which the inverse function f^{-1} exists in the case where $\pi < k < 2\pi$. Give your answer in terms of k. [2]

1 – (MAT/12_HL_Summer_2012_Q5) – *Matrices*

Three non-singular 2×2 matrices A, $\boldsymbol{B}$ and X satisfy $4\boldsymbol{A}-5\boldsymbol{BX}=\boldsymbol{B}$.

(a) Find X in terms of A and $\boldsymbol{B}$. *[2 marks]*

(b) Given that $\boldsymbol{A}=2\boldsymbol{B}$, find X. *[3 marks]*

2 – (MAT/10_HL_Winter_2012_Q6) – *Matrices*

Consider the following equations, where $a,\ b \in \mathbb{R}$:

$$x + 3y + (a-1)z = 1$$
$$2x + 2y + (a-2)z = 1$$
$$3x + y + (a-3)z = b.$$

(a) If each of these equations defines a plane, show that, for any value of a, the planes do not intersect at a unique point. *[3 marks]*

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(b) Find the value of b for which the intersection of the planes is a straight line. *[4 marks]*

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3 – (MAT/10_HL_Winter_2012_Q11) – *Matrices*

Consider the matrix $A = \begin{pmatrix} a & a-1 \\ b & b \end{pmatrix}$, $b \neq 0$.

(a) For $a = 2$ and $b = 1$, show that $(A^2 - 3A)^2 = I$. *[3 marks]*

(b) Find the value of a and the value of b in each of the following cases:

(i) $A\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$;

(ii) $A^{-1}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$. *[6 marks]*

(c) Consider the lines l_1 and l_2 defined by the equations $ax + (a-1)y = 0$ and $bx + by = 1$, respectively, where $b \neq 0$.

(i) Find the coordinates of the intersection point of the lines l_1 and l_2 in terms of a and b.

(ii) Given that the lines are perpendicular, find the coordinates of the point of intersection in terms of b. *[9 marks]*

4 – (MAT/12_HL_Summer_2013_Q2) – *Matrices*

Consider the matrices $\boldsymbol{A} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 3 & -5 \\ 2 & -3 \end{pmatrix}$.

(a) Find $\det \boldsymbol{A}$ and hence write down the matrix $\boldsymbol{A}^{-1}$. *[2 marks]*

(b) Find the matrix $\boldsymbol{A}^{-1}\boldsymbol{B}$. *[3 marks]*

5 – (MAT/11_HL_Summer_2013_Q3) – *Matrices*

[Maximum mark: 5]

Given $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 4 & -3 \\ 1 & 1 \end{pmatrix}$, find the matrix $\boldsymbol{X}$, such that $\boldsymbol{AXA}^{-1} = \boldsymbol{B}$.

6 – (MAT/11_HL_Summer_2013_Q6) – *Matrices*

[Maximum mark: 6]

The matrix A is such that $A^2 = I$, where I is the identity matrix. Use mathematical induction to prove that $(A+I)^n = 2^{n-1}(A+I)$, for all $n \in \mathbb{Z}^+$.

7 – (MAT/10_HL_Winter_2013_Q4) – *Matrices*

[Maximum mark: 5]

Consider the matrix $A = \begin{pmatrix} 1 & a(a+1) \\ 1 & b(b+1) \end{pmatrix}$, $a \neq b$. Given that A is singular, find the value of $a+b$.

8 – (MAT/12_HL_Summer_2014_Q3) – *Matrices*

(a) Show that the following system of equations has an infinite number of solutions. *[2]*

$$\begin{aligned} x + y + 2z &= -2 \\ 3x - y + 14z &= 6 \\ x + 2y &= -5 \end{aligned}$$

The system of equations represents three planes in space.

(b) Find the parametric equations of the line of intersection of the three planes. *[3]*

9 – (MAT/12_HL_Summer_2016_Q1) – *Matrices*

[Maximum mark: 6]

The following system of equations represents three planes in space.

$$x + 3y + z = -1$$
$$x + 2y - 2z = 15$$
$$2x + y - z = 6$$

Find the coordinates of the point of intersection of the three planes.

10 – (MAT/10_HL_Winter_2018_Q4) – *Matrices*

Consider the following system of equations where $a \in \mathbb{R}$.

$$\begin{aligned} 2x + 4y - z &= 10 \\ x + 2y + az &= 5 \\ 5x + 12y &= 2a. \end{aligned}$$

(a) Find the value of a for which the system of equations does not have a unique solution. [2]

(b) Find the solution of the system of equations when $a = 2$. [5]

1 – (MAT/12_HL_Summer_2012_Q2) – *Vectors - Lines - Planes*

[Maximum mark: 5]

Find the values of x for which the vectors $\begin{pmatrix} 1 \\ 2\cos x \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2\sin x \\ 1 \end{pmatrix}$ are perpendicular, $0 \le x \le \frac{\pi}{2}$.

2 – (MAT/10_HL_Winter_2012_Q9) – *Vectors - Lines - Planes*

Two boats, A and B, move so that at time t hours, their position vectors, in kilometres, are $\boldsymbol{r}_A = (9t)\boldsymbol{i} + (3-6t)\boldsymbol{j}$ and $\boldsymbol{r}_B = (7-4t)\boldsymbol{i} + (7t-6)\boldsymbol{j}$.

(a) Find the coordinates of the common point of the paths of the two boats. *[4 marks]*

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(b) Show that the boats do not collide. *[2 marks]*

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3 – (MAT/11_HL_Summer_2013_Q2) – *Vectors - Lines - Planes*

Consider the points $A(1, 2, 3)$, $B(1, 0, 5)$ and $C(2, -1, 4)$.

(a) Find $\overrightarrow{AB} \times \overrightarrow{AC}$. *[4 marks]*

(b) Hence find the area of the triangle ABC. *[2 marks]*

4 – (MAT/12_HL_Summer_2013_Q11) – *Vectors - Lines - Planes*

The vertices of a triangle ABC have coordinates given by $A(-1, 2, 3)$, $B(4, 1, 1)$ and $C(3, -2, 2)$.

(a) (i) Find the lengths of the sides of the triangle.

(ii) Find $\cos B\hat{A}C$. *[6 marks]*

(b) (i) Show that $\overrightarrow{BC} \times \overrightarrow{CA} = -7\boldsymbol{i} - 3\boldsymbol{j} - 16\boldsymbol{k}$.

(ii) Hence, show that the area of the triangle ABC is $\frac{1}{2}\sqrt{314}$. *[5 marks]*

(c) Find the Cartesian equation of the plane containing the triangle ABC. *[3 marks]*

(d) Find a vector equation of (AB). *[2 marks]*

The point D on (AB) is such that $\overrightarrow{OD}$ is perpendicular to $\overrightarrow{BC}$ where O is the origin.

(e) (i) Find the coordinates of D.

(ii) Show that D does not lie between A and B. *[5 marks]*

5 – (MAT/10_HL_Winter_2013_Q11) – *Vectors - Lines - Planes*

Consider the points $A(1, 0, 0)$, $B(2, 2, 2)$ and $C(0, 2, 1)$.

(a) Find the vector $\overrightarrow{CA} \times \overrightarrow{CB}$. *[4]*

(b) Find an exact value for the area of the triangle ABC. *[3]*

(c) Show that the Cartesian equation of the plane Π_1, containing the triangle ABC, is $2x + 3y - 4z = 2$. *[3]*

A second plane Π_2 is defined by the Cartesian equation $\Pi_2 : 4x - y - z = 4$. L_1 is the line of intersection of the planes Π_1 and Π_2.

(d) Find a vector equation for L_1. *[5]*

A third plane Π_3 is defined by the Cartesian equation $16x + \alpha y - 3z = \beta$.

(e) Find the value of α if all three planes contain L_1. *[3]*

(f) Find conditions on α and β if the plane Π_3 does **not** intersect with L_1. *[2]*

6 – (MAT/12_HL_Summer_2014_Q6) – *Vectors - Lines - Planes*

PQRS is a rhombus. Given that $\overrightarrow{PQ} = \boldsymbol{a}$ and $\overrightarrow{QR} = \boldsymbol{b}$,

(a) express the vectors $\overrightarrow{PR}$ and $\overrightarrow{QS}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$; *[2]*

(b) hence show that the diagonals in a rhombus intersect at right angles. *[4]*

7 – (MAT/11_HL_Summer_2014_Q12) – *Vectors - Lines - Planes*

(a) Show that the points $O(0,0,0)$, $A(6,0,0)$, $B\left(6,-\sqrt{24},\sqrt{12}\right)$, $C\left(0,-\sqrt{24},\sqrt{12}\right)$ form a square. *[3]*

(b) Find the coordinates of M, the mid-point of [OB]. *[1]*

(c) Show that an equation of the plane Π, containing the square OABC, is $y+\sqrt{2}z=0$. *[3]*

(d) Find a vector equation of the line L, through M, perpendicular to the plane Π. *[3]*

(e) Find the coordinates of D, the point of intersection of the line L with the plane whose equation is $y=0$. *[3]*

(f) Find the coordinates of E, the reflection of the point D in the plane Π. *[3]*

(g) (i) Find the angle $O\hat{D}A$.

(ii) State what this tells you about the solid OABCDE. *[6]*

8 – (MAT/12_HL_Summer_2014_Q12) – *Vectors - Lines - Planes*

Given the points $A(1, 0, 4)$, $B(2, 3, -1)$ and $C(0, 1, -2)$,

(a) find the vector equation of the line L_1 passing through the points A and B. *[2]*

The line L_2 has Cartesian equation $\frac{x-1}{3} = \frac{y+2}{1} = \frac{z-1}{-2}$.

(b) Show that L_1 and L_2 are skew lines. *[5]*

Consider the plane Π_1, parallel to both lines L_1 and L_2. Point C lies in the plane Π_1.

(c) Find the Cartesian equation of the plane Π_1. *[4]*

The line L_3 has vector equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} k \\ 1 \\ -1 \end{pmatrix}$.

The plane Π_2 has Cartesian equation $x + y = 12$.

The angle between the line L_3 and the plane Π_2 is 60°.

(d) (i) Find the value of k.

(ii) Find the point of intersection P of the line L_3 and the plane Π_2. *[7]*

9 – (MAT/10_HL_Winter_2014_Q3) – *Vectors - Lines - Planes*

[Maximum mark: 5]

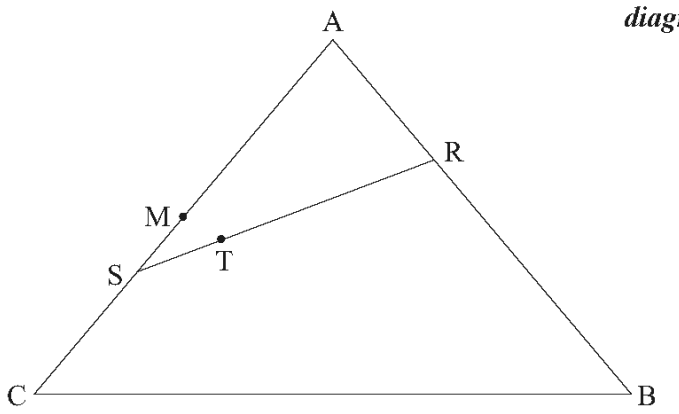
A point P, relative to an origin O, has position vector $\overrightarrow{OP} = \begin{pmatrix} 1+s \\ 3+2s \\ 1-s \end{pmatrix}$, $s \in \mathbb{R}$.

Find the minimum length of $\overrightarrow{OP}$.

10 – (MAT/10_HL_Winter_2014_Q12) – *Vectors - Lines - Planes*

The position vectors of the points A, B and C are $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively, relative to an origin O. The following diagram shows the triangle ABC and points M, R, S and T.

diagram not to scale



M is the midpoint of [AC].

R is a point on [AB] such that $\overrightarrow{AR} = \frac{1}{3}\overrightarrow{AB}$.

S is a point on [AC] such that $\overrightarrow{AS} = \frac{2}{3}\overrightarrow{AC}$.

T is a point on [RS] such that $\overrightarrow{RT} = \frac{2}{3}\overrightarrow{RS}$.

(a) (i) Express $\overrightarrow{AM}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

(ii) Hence show that $\overrightarrow{BM} = \frac{1}{2}\boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}\boldsymbol{c}$. *[4]*

(b) (i) Express $\overrightarrow{RA}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(ii) Show that $\overrightarrow{RT} = -\frac{2}{9}\boldsymbol{a} - \frac{2}{9}\boldsymbol{b} + \frac{4}{9}\boldsymbol{c}$. *[5]*

(c) Prove that T lies on [BM]. *[5]*

11 – (MAT/11_HL_Summer_2015_Q13) – *Vectors - Lines - Planes*

Two lines l_1 and l_2 are given respectively by the equations $\boldsymbol{r_1} = \overrightarrow{\mathrm{OA}} + \lambda \boldsymbol{v}$ and $\boldsymbol{r_2} = \overrightarrow{\mathrm{OB}} + \mu \boldsymbol{w}$ where $\overrightarrow{\mathrm{OA}} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$, $\boldsymbol{v} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, $\overrightarrow{\mathrm{OB}} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{w} = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ and O is the origin. Let P be a point on l_1 and let Q be a point on l_2.

(a) Find $\overrightarrow{\mathrm{PQ}}$, in terms of λ and μ. [2]

(b) Find the value of λ and the value of μ for which $\overrightarrow{\mathrm{PQ}}$ is perpendicular to the direction vectors of both l_1 and l_2. [5]

(c) Hence find the shortest distance between l_1 and l_2. [3]

(d) Find the Cartesian equation of the plane Π, which contains line l_1 and is parallel to the direction vector of line l_2. [5]

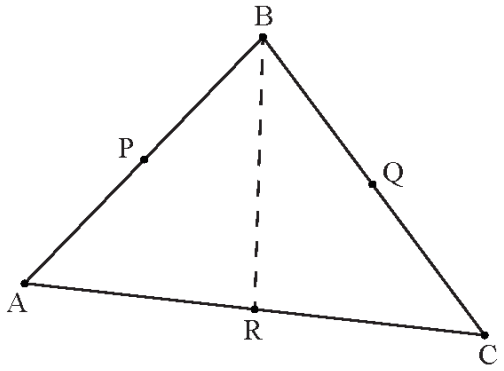
Let $\overrightarrow{\mathrm{OT}} = \overrightarrow{\mathrm{OB}} + \eta\,(\boldsymbol{v} \times \boldsymbol{w})$.

(e) Find the value of η for which the point T lies in the plane Π. [2]

(f) For this value of η, calculate $\left|\overrightarrow{\mathrm{BT}}\right|$. [2]

(g) State what you notice about your answers to (c) and (f), and give a geometrical interpretation of this result. [2]

12 – (MAT/10_HL_Winter_2015_Q13) – *Vectors - Lines - Planes*



Consider the triangle ABC. The points P, Q and R are the midpoints of the line segments $[\mathrm{AB}]$, $[\mathrm{BC}]$ and $[\mathrm{AC}]$ respectively.

Let $\overrightarrow{\mathrm{OA}} = \boldsymbol{a}$, $\overrightarrow{\mathrm{OB}} = \boldsymbol{b}$ and $\overrightarrow{\mathrm{OC}} = \boldsymbol{c}$.

(a) Find $\overrightarrow{\mathrm{BR}}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. [2]

(b) (i) Find a vector equation of the line that passes through B and R in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ and a parameter λ.

(ii) Find a vector equation of the line that passes through A and Q in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ and a parameter μ.

(iii) Hence show that $\overrightarrow{\mathrm{OG}} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$ given that G is the point where $[\mathrm{BR}]$ and $[\mathrm{AQ}]$ intersect. [9]

(c) Show that the line segment $[\mathrm{CP}]$ also includes the point G. [3]

The coordinates of the points A, B and C are $(1, 3, 1)$, $(3, 7, -5)$ and $(2, 2, 1)$ respectively.
A point X is such that $[\mathrm{GX}]$ is perpendicular to the plane ABC.

(d) Given that the tetrahedron ABCX has volume 12 units^3, find possible coordinates of X. [9]

13 – (MAT/11_HL_Summer_2016_Q8) – *Vectors - Lines - Planes*

[Maximum mark: 5]

O, A, B and C are distinct points such that $\overrightarrow{\mathrm{OA}} = \boldsymbol{a}$, $\overrightarrow{\mathrm{OB}} = \boldsymbol{b}$ and $\overrightarrow{\mathrm{OC}} = \boldsymbol{c}$.

It is given that $\boldsymbol{c}$ is perpendicular to $\overrightarrow{\mathrm{AB}}$ and $\boldsymbol{b}$ is perpendicular to $\overrightarrow{\mathrm{AC}}$.

Prove that $\boldsymbol{a}$ is perpendicular to $\overrightarrow{\mathrm{BC}}$.

14 – (MAT/12_HL_Summer_2016_Q10) – *Vectors - Lines - Planes*

A line L has equation $\frac{x-2}{p} = \frac{y-q}{2} = z - 1$ where $p, q \in \mathbb{R}$.

A plane Π has equation $x + y + 3z = 9$.

(a) Show that L is not perpendicular to Π. [3]

(b) Given that L lies in the plane Π, find the value of p and the value of q. [4]

Consider the different case where the acute angle between L and Π is θ where $\theta = \arcsin\left(\frac{1}{\sqrt{11}}\right)$.

(c) (i) Show that $p = -2$.

(ii) If L intersects Π at $z = -1$, find the value of q. [11]

15 – (MAT/11_HL_Summer_2016_Q11) – *Vectors - Lines - Planes*

Two planes have equations

$$\Pi_1 : 4x + y + z = 8 \text{ and } \Pi_2 : 4x + 3y - z = 0$$

(a) Find the cosine of the angle between the two planes in the form $\sqrt{\frac{p}{q}}$ where $p, q \in \mathbb{Z}$. [4]

Let L be the line of intersection of the two planes.

(b) (i) Show that L has direction $\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$.

(ii) Show that the point $\mathrm{A}(1, 0, 4)$ lies on both planes.

(iii) Write down a vector equation of L. [6]

B is the point on Π_1 with coordinates $(a, b, 1)$.

(c) Given the vector $\overrightarrow{\mathrm{AB}}$ is perpendicular to L find the value of a and the value of b. [5]

(d) Show that $\mathrm{AB} = 3\sqrt{2}$. [1]

The point P lies on L and $\mathrm{A\hat{B}P} = 45°$.

(e) Find the coordinates of the two possible positions of P. [5]

16 – (MAT/10_HL_Winter_2016_Q1) – *Vectors - Lines - Planes*

Find the coordinates of the point of intersection of the planes defined by the equations $x+y+z=3$, $x-y+z=5$ and $x+y+2z=6$.

17 - (MAT/10_HL_Winter_2016_Q4) - *Vectors - Lines - Planes*

Consider the vectors $\boldsymbol{a} = \boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k}$, $\boldsymbol{b} = -3\boldsymbol{j} + 2\boldsymbol{k}$.

(a) Find $\boldsymbol{a} \times \boldsymbol{b}$. [2]

(b) Hence find the Cartesian equation of the plane containing the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and passing through the point $(1, 0, -1)$. [3]

18 – (MAT/10_HL_Winter_2016_Q8) – *Vectors - Lines - Planes*

Consider the lines l_1 and l_2 defined by

$l_1 : \boldsymbol{r} = \begin{pmatrix} -3 \\ -2 \\ a \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$ and $l_2 : \dfrac{6-x}{3} = \dfrac{y-2}{4} = 1 - z$ where a is a constant.

Given that the lines l_1 and l_2 intersect at a point P,

(a) find the value of a; [4]

(b) determine the coordinates of the point of intersection P. [2]

19 – (MAT/11_HL_Summer_2017_Q5) – *Vectors - Lines - Planes*

ABCD is a parallelogram, where $\overrightarrow{AB} = -\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$ and $\overrightarrow{AD} = 4\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$.

(a) Find the area of the parallelogram ABCD. [3]

(b) By using a suitable scalar product of two vectors, determine whether $A\hat{B}C$ is acute or obtuse. [4]

20 – (MAT/10_HL_Winter_2017_Q2) – *Vectors - Lines - Planes*

[Maximum mark: 6]

The points A and B are given by $\mathrm{A}(0, 3, -6)$ and $\mathrm{B}(6, -5, 11)$.

The plane Π is defined by the equation $4x - 3y + 2z = 20$.

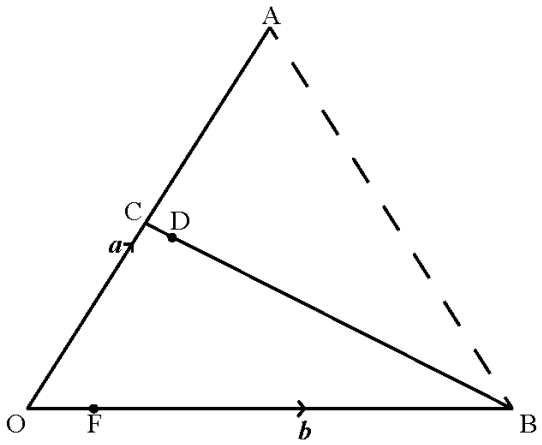
(a) Find a vector equation of the line L passing through the points A and B. [3]

(b) Find the coordinates of the point of intersection of the line L with the plane Π. [3]

21 – (MAT/10_HL_Winter_2017_Q9) – *Vectors - Lines - Planes*

[Maximum mark: 18]

In the following diagram, $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$. C is the midpoint of [OA] and $\overrightarrow{OF} = \frac{1}{6}\overrightarrow{FB}$.



(a) Find, in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

(i) $\overrightarrow{OF}$;

(ii) $\overrightarrow{AF}$. [3]

It is given also that $\overrightarrow{AD} = \lambda\overrightarrow{AF}$ and $\overrightarrow{CD} = \mu\overrightarrow{CB}$, where $\lambda, \mu \in \mathbb{R}$.

(b) Find an expression for

(i) $\overrightarrow{OD}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and λ;

(ii) $\overrightarrow{OD}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and μ. [4]

(c) Show that $\mu = \frac{1}{13}$, and find the value of λ. [4]

(d) Deduce an expression for $\overrightarrow{CD}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$ only. [2]

(e) Given that area $\Delta OAB = k(\text{area } \Delta CAD)$, find the value of k. [5]

22 - (MAT/12_HL_Summer_2018_Q1) - *Vectors - Lines - Planes*

[Maximum mark: 4]

The acute angle between the vectors $3\boldsymbol{i} - 4\boldsymbol{j} - 5\boldsymbol{k}$ and $5\boldsymbol{i} - 4\boldsymbol{j} + 3\boldsymbol{k}$ is denoted by θ.
Find $\cos\theta$.

23 – (MAT/12_HL_Summer_2018_Q9) – *Vectors - Lines - Planes*

The points $\mathrm{A}, \mathrm{B}, \mathrm{C}$ and D have position vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{d}$, relative to the origin O.
It is given that $\overrightarrow{\mathrm{AB}} = \overrightarrow{\mathrm{DC}}$.

(a) (i) Explain why ABCD is a parallelogram.

(ii) Using vector algebra, show that $\overrightarrow{\mathrm{AD}} = \overrightarrow{\mathrm{BC}}$. [4]

The position vectors $\overrightarrow{\mathrm{OA}}$, $\overrightarrow{\mathrm{OB}}$, $\overrightarrow{\mathrm{OC}}$ and $\overrightarrow{\mathrm{OD}}$ are given by

$$\begin{aligned}\boldsymbol{a} &= \boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}\\ \boldsymbol{b} &= 3\boldsymbol{i} - \boldsymbol{j} + p\boldsymbol{k}\\ \boldsymbol{c} &= q\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}\\ \boldsymbol{d} &= -\boldsymbol{i} + r\boldsymbol{j} - 2\boldsymbol{k}\end{aligned}$$

where p, q and r are constants.

(b) Show that $p = 1$, $q = 1$ and $r = 4$. [5]

(c) Find the area of the parallelogram ABCD. [4]

The point where the diagonals of ABCD intersect is denoted by M.

(d) Find the vector equation of the straight line passing through M and normal to the plane Π containing ABCD. [4]

(e) Find the Cartesian equation of Π. [3]

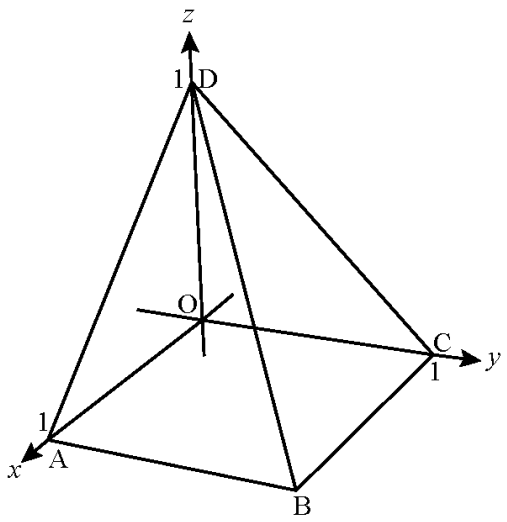
The plane Π cuts the x, y and z axes at X, Y and Z respectively.

(f) (i) Find the coordinates of X, Y and Z.

(ii) Find YZ. [4]

24 – (MAT/11_HL_Summer_2018_Q10) – *Vectors - Lines - Planes*

The following figure shows a square based pyramid with vertices at $O(0, 0, 0)$, $A(1, 0, 0)$, $B(1, 1, 0)$, $C(0, 1, 0)$ and $D(0, 0, 1)$.



(a) Find the Cartesian equation of the plane Π_1, passing through the points A, B and D. [3]

The Cartesian equation of the plane Π_2, passing through the points B, C and D, is $y + z = 1$.

(b) Find the angle between the faces ABD and BCD. [4]

The plane Π_3 passes through O and is normal to the line BD.

(c) Find the Cartesian equation of Π_3. [3]

Π_3 cuts AD and BD at the points P and Q respectively.

(d) Show that P is the midpoint of AD. [4]

(e) Find the area of the triangle OPQ. [5]

25 – (MAT/10_HL_Winter_2018_Q5) – *Vectors - Lines - Planes*

The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are defined by $\boldsymbol{a} = \begin{pmatrix} 1 \\ 1 \\ t \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 0 \\ -t \\ 4t \end{pmatrix}$, where $t \in \mathbb{R}$.

(a) Find and simplify an expression for $\boldsymbol{a} \bullet \boldsymbol{b}$ in terms of t. [2]

(b) Hence or otherwise, find the values of t for which the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is obtuse. [4]

26 – (MAT/10_HL_Winter_2018_Q9) – *Vectors - Lines - Planes*

Consider a triangle OAB such that O has coordinates $(0, 0, 0)$, A has coordinates $(0, 1, 2)$ and B has coordinates $(2b, 0, b-1)$ where $b < 0$.

(a) Find, in terms of b, a Cartesian equation of the plane Π containing this triangle. [5]

Let M be the midpoint of the line segment $[\mathrm{OB}]$.

(b) Find, in terms of b, the equation of the line L which passes through M and is perpendicular to the plane Π. [3]

(c) Show that L does not intersect the y-axis for any negative value of b. [7]

27 – (MAT/11_HL_Summer_2019_Q1) – *Vectors - Lines - Planes*

[Maximum mark: 4]

Let $\boldsymbol{a} = \begin{pmatrix} 2 \\ k \\ -1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} -3 \\ k+2 \\ k \end{pmatrix}$, $k \in \mathbb{R}$.

Given that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular, find the possible values of k.

28 – (MAT/12_HL_Summer_2019_Q2) – *Vectors - Lines - Planes*

Three points in three-dimensional space have coordinates $A(0, 0, 2)$, $B(0, 2, 0)$ and $C(3, 1, 0)$.

(a) Find the vector

(i) $\overrightarrow{AB}$;

(ii) $\overrightarrow{AC}$. [2]

(b) Hence or otherwise, find the area of the triangle ABC. [4]

29 – (MAT/11_HL_Summer_2019_Q11) – *Vectors - Lines - Planes*

Two distinct lines, l_1 and l_2, intersect at a point P. In addition to P, four distinct points are marked out on l_1 and three distinct points on l_2. A mathematician decides to join some of these eight points to form polygons.

(a) (i) Find how many sets of four points can be selected which can form the vertices of a quadrilateral.

(ii) Find how many sets of three points can be selected which can form the vertices of a triangle. [6]

The line l_1 has vector equation $\boldsymbol{r}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$ and the line l_2 has vector equation

$\boldsymbol{r}_2 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 6 \\ 2 \end{pmatrix}$, $\mu \in \mathbb{R}$.

The point P has coordinates $(4, 6, 4)$.

(b) Verify that P is the point of intersection of the two lines. [3]

The point A has coordinates $(3, 4, 3)$ and lies on l_1.

(c) Write down the value of λ corresponding to the point A. [1]

The point B has coordinates $(-1, 0, 2)$ and lies on l_2.

(d) Write down $\overrightarrow{\mathrm{PA}}$ and $\overrightarrow{\mathrm{PB}}$. [2]

Let C be the point on l_1 with coordinates $(1, 0, 1)$ and D be the point on l_2 with parameter $\mu = -2$.

(e) Find the area of the quadrilateral CDBA. [8]

30 – (MAT/10_HL_Winter_2019_Q3) – *Vectors - Lines - Planes*

[Maximum mark: 5]

Three planes have equations:

$$\begin{aligned} 2x - y + z &= 5 \\ x + 3y - z &= 4 \\ 3x - 5y + az &= b \end{aligned}$$, where $a, b \in \mathbb{R}$.

Find the set of values of a and b such that the three planes have no points of intersection.

31 – (MAT/10_HL_Winter_2019_Q8) – *Vectors - Lines - Planes*

[Maximum mark: 6]

A straight line, L_θ, has vector equation $\boldsymbol{r} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ \sin\theta \\ \cos\theta \end{pmatrix}$, $\lambda, \theta \in \mathbb{R}$.

The plane, Π_p, has equation $x = p$, $p \in \mathbb{R}$.

Show that the angle between L_θ and Π_p is independent of both θ and p.

32 – (MAT/10_HL_Winter_2019_Q11) – *Vectors - Lines - Planes, Graphs*

[Maximum mark: 17]

Points $A(0, 0, 10)$, $B(0, 10, 0)$, $C(10, 0, 0)$, $V(p, p, p)$ form the vertices of a tetrahedron.

(a) (i) Show that $\overrightarrow{AB} \times \overrightarrow{AV} = -10\begin{pmatrix} 10-2p \\ p \\ p \end{pmatrix}$ and find a similar expression for $\overrightarrow{AC} \times \overrightarrow{AV}$.

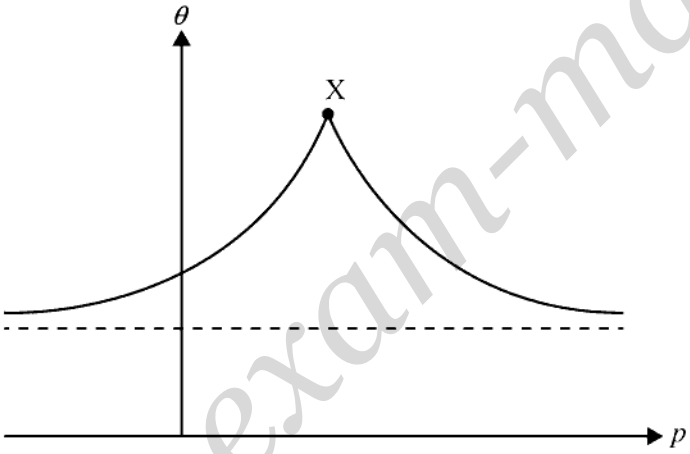
(ii) Hence, show that, if the angle between the faces ABV and ACV is θ, then $\cos\theta = \dfrac{p(3p-20)}{6p^2-40p+100}$. [8]

(b) Consider the case where the faces ABV and ACV are perpendicular.

(i) Find the two possible coordinates of V.

(ii) Comment on the positions of V in relation to the plane ABC. [4]

(c) The following diagram shows the graph of θ against p. The maximum point is shown by X.



(i) At X, find the value of p and the value of θ.

(ii) Find the equation of the horizontal asymptote of the graph. [5]

33 – (MathAA/12_HL_Summer_2021_Q5) – *Vectors - Lines - Planes*

[Maximum mark: 4]

Given any two non-zero vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, show that $|\boldsymbol{a} \times \boldsymbol{b}|^2 = |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2$.

34 – (MathAA/12_HL_Summer_2021_Q8) – *Vectors - Lines - Planes*

[Maximum mark: 8]

The lines l_1 and l_2 have the following vector equations where $\lambda, \mu \in \mathbb{R}$.

$$l_1 : \boldsymbol{r}_1 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

$$l_2 : \boldsymbol{r}_2 = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

(a) Show that l_1 and l_2 do not intersect. [3]

(b) Find the minimum distance between l_1 and l_2. [5]

35 – (MathAA/11_HL_Summer_2021_Q11) – *Vectors - Lines - Planes*

[Maximum mark: 19]

Consider the line L_1 defined by the Cartesian equation $\frac{x+1}{2} = y = 3 - z$.

(a) (i) Show that the point $(-1, 0, 3)$ lies on L_1.

(ii) Find a vector equation of L_1. [4]

Consider a second line L_2 defined by the vector equation $\boldsymbol{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix}$,

where $t \in \mathbb{R}$ and $a \in \mathbb{R}$.

(b) Find the possible values of a when the acute angle between L_1 and L_2 is $45°$. [8]

It is given that the lines L_1 and L_2 have a unique point of intersection, A, when $a \neq k$.

(c) Find the value of k, and find the coordinates of the point A in terms of a. [7]

36 – (MathAA/11_HL_Summer_2022_Q11) – *Vectors - Lines - Planes*

[Maximum mark: 15]

Consider the three planes

$$\Pi_1 : 2x - y + z = 4$$
$$\Pi_2 : x - 2y + 3z = 5$$
$$\Pi_3 : 9x + 3y - 2z = 32$$

(a) Show that the three planes do not intersect. [4]

(b) (i) Verify that the point $P(1, -2, 0)$ lies on both Π_1 and Π_2.

(ii) Find a vector equation of L, the line of intersection of Π_1 and Π_2. [5]

(c) Find the distance between L and Π_3. [6]

1 – (MAT/11_HL_Summer_2014_Q2) – *Statistics*

[Maximum mark: 4]

Four numbers are such that their mean is 13, their median is 14 and their mode is 15. Find the four numbers.

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2 – (MAT/11_HL_Summer_2015_Q2) – *Statistics*

A mathematics test is given to a class of 20 students. One student scores 0, but all the other students score 10.

(a) Find the mean score for the class. [2]

(b) Write down the median score. [1]

(c) Write down the number of students who scored

(i) above the mean score;

(ii) below the median score. [2]

3 – (MAT/12_HL_Summer_2017_Q7) – *Statistics, Probability*

(a) The random variable X has the Poisson distribution $\mathrm{Po}(m)$. Given that $\mathrm{P}(X > 0) = \frac{3}{4}$, find the value of m in the form $\ln a$ where a is an integer. [3]

(b) The random variable Y has the Poisson distribution $\mathrm{Po}(2m)$. Find $\mathrm{P}(Y > 1)$ in the form $\frac{b - \ln c}{c}$ where b and c are integers. [4]

4 – (MAT/11_HL_Summer_2017_Q10) – *Probability, Statistics*

The continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} k\sin\left(\dfrac{\pi x}{6}\right), & 0 \le x \le 6 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of k. [4]

(b) By considering the graph of f write down

(i) the mean of X;

(ii) the median of X;

(iii) the mode of X. [3]

(c) (i) Show that $P(0 \le X \le 2) = \dfrac{1}{4}$.

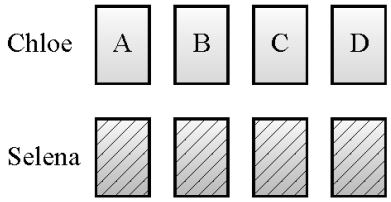
(ii) Hence state the interquartile range of X. [6]

(d) Calculate $P(X \le 4 \mid X \ge 3)$. [2]

5 – (MAT/10_HL_Winter_2017_Q10) – *Statistics*

[Maximum mark: 11]

Chloe and Selena play a game where each have four cards showing capital letters $\mathrm{A}, \mathrm{B}, \mathrm{C}$ and D.
Chloe lays her cards face up on the table in order $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$ as shown in the following diagram.



Selena shuffles her cards and lays them face down on the table. She then turns them over one by one to see if her card matches with Chloe's card directly above.
Chloe wins if **no** matches occur; otherwise Selena wins.

(a) Show that the probability that Chloe wins the game is $\frac{3}{8}$. [6]

Chloe and Selena repeat their game so that they play a total of 50 times.
Suppose the discrete random variable X represents the number of times Chloe wins.

(b) Determine

(i) the mean of X;

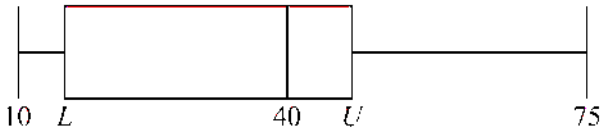
(ii) the variance of X. [5]

6 – (MathAA/11_HL_Summer_2021_Q3) – *Statistics*

[Maximum mark: 5]

A research student weighed lizard eggs in grams and recorded the results. The following box and whisker diagram shows a summary of the results where L and U are the lower and upper quartiles respectively.

diagram not to scale



The interquartile range is 20 grams and there are no outliers in the results.

(a) Find the minimum possible value of U. [3]

(b) Hence, find the minimum possible value of L. [2]

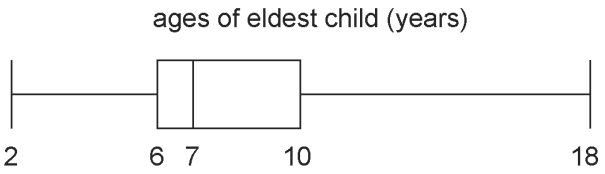
7 – (MathAA/11_HL_Summer_2022_Q2) – *Statistics*

[Maximum mark: 7]

A survey at a swimming pool is given to one adult in each family. The age of the adult, a years old, and of their eldest child, c years old, are recorded.

The ages of the eldest child are summarized in the following box and whisker diagram.

diagram not to scale



(a) Find the largest value of c that would not be considered an outlier. [3]

The regression line of a on c is $a = \frac{7}{4}c + 20$. The regression line of c on a is $c = \frac{1}{2}a - 9$.

(b) (i) One of the adults surveyed is 42 years old. Estimate the age of their eldest child.

(ii) Find the mean age of all the adults surveyed. [4]

1 – (MAT/12_HL_Summer_2012_Q3) – *Probability*

On a particular day, the probability that it rains is $\frac{2}{5}$. The probability that the "Tigers" soccer team wins on a day when it rains is $\frac{2}{7}$ and the probability that they win on a day when it does not rain is $\frac{4}{7}$.

(a) Draw a tree diagram to represent these events and their outcomes. *[1 mark]*

(b) What is the probability that the "Tigers" soccer team wins? *[2 marks]*

(c) Given that the "Tigers" soccer team won, what is the probability that it rained on that day? *[2 marks]*

2 – (MAT/12_HL_Summer_2012_Q11) – *Functions - Roots, Probability*

Consider the following functions:

$$f(x) = \frac{2x^2+3}{75},\ x \geq 0$$

$$g(x) = \frac{|3x-4|}{10},\ x \in \mathbb{R}.$$

(a) State the range of f and of g. *[2 marks]*

(b) Find an expression for the composite function $f \circ g(x)$ in the form $\frac{ax^2+bx+c}{3750}$, where a, b and $c \in \mathbb{Z}$. *[4 marks]*

(c) (i) Find an expression for the inverse function $f^{-1}(x)$.

(ii) State the domain and range of f^{-1}. *[4 marks]*

The domains of f and g are now restricted to $\{0, 1, 2, 3, 4\}$.

(d) By considering the values of f and g on this new domain, determine which of f and g could be used to find a probability distribution for a discrete random variable X, stating your reasons clearly. *[6 marks]*

(e) Using this probability distribution, calculate the mean of X. *[2 marks]*

3 – (MAT/10_HL_Winter_2012_Q5) – *Probability*

The continuous random variable X has probability density function given by

$$f(x)=\begin{cases} ae^{-x}, & 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) State the mode of X. *[1 mark]*

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(b) Determine the value of a. *[3 marks]*

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(c) Find $\mathrm{E}(X)$. *[4 marks]*

4 – (MAT/11_HL_Summer_2013_Q4) – *Probability*

[Maximum mark: 5]

The probability density function of the random variable X is defined as

$$f(x)=\begin{cases}\sin x, & 0\le x\le \frac{\pi}{2}\\ 0, & \text{otherwise.}\end{cases}$$

Find $\mathrm{E}(X)$.

5 – (MAT/12_HL_Summer_2013_Q4) – *Probability*

Tim and Caz buy a box of 16 chocolates of which 10 are milk and 6 are dark. Caz randomly takes a chocolate and eats it. Then Tim randomly takes a chocolate and eats it.

(a) Draw a tree diagram representing the possible outcomes, clearly labelling each branch with the correct probability. *[3 marks]*

(b) Find the probability that Tim and Caz eat the same type of chocolate. *[2 marks]*

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6 – (MAT/11_HL_Summer_2013_Q9) – *Probability*

[Maximum mark: 6]

Two events A and B are such that $\mathrm{P}(A \cup B) = 0.7$ and $\mathrm{P}(A \mid B') = 0.6$.

Find $\mathrm{P}(B)$.

7 – (MAT/11_HL_Summer_2013_Q13) – *Probability, Binomial Theorem*

On Saturday, Alfred and Beatrice play 6 different games against each other. In each game, one of the two wins. The probability that Alfred wins any one of these games is $\frac{2}{3}$.

(a) Show that the probability that Alfred wins exactly 4 of the games is $\frac{80}{243}$. *[3 marks]*

(b) (i) Explain why the total number of possible outcomes for the results of the 6 games is 64.

(ii) By expanding $(1+x)^6$ and choosing a suitable value for x, prove

$$64 = \binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}.$$

(iii) State the meaning of this equality in the context of the 6 games played. *[4 marks]*

(c) The following day Alfred and Beatrice play the 6 games again. Assume that the probability that Alfred wins any one of these games is still $\frac{2}{3}$.

(i) Find an expression for the probability Alfred wins 4 games on the first day and 2 on the second day. Give your answer in the form $\binom{6}{r}^2\left(\frac{2}{3}\right)^s\left(\frac{1}{3}\right)^t$ where the values of r, s and t are to be found.

(ii) Using your answer to (c)(i) and 6 similar expressions write down the probability that Alfred wins a total of 6 games over the two days as the sum of 7 probabilities.

(iii) Hence prove that $\binom{12}{6} = \binom{6}{0}^2 + \binom{6}{1}^2 + \binom{6}{2}^2 + \binom{6}{3}^2 + \binom{6}{4}^2 + \binom{6}{5}^2 + \binom{6}{6}^2$. *[9 marks]*

(d) Alfred and Beatrice play n games. Let A denote the number of games Alfred wins. The expected value of A can be written as $\mathrm{E}(A) = \sum_{r=0}^{n} r\binom{n}{r}\frac{a^r}{b^n}$.

(i) Find the values of a and b.

(ii) By differentiating the expansion of $(1+x)^n$, prove that the expected number of games Alfred wins is $\frac{2n}{3}$. *[6 marks]*

8 – (MAT/10_HL_Winter_2013_Q2) – *Probability*

The discrete random variable X has probability distribution:

x	0	1	2	3
$\mathrm{P}(X = x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	a

(a) Find the value of a. *[1]*

(b) Find $\mathrm{E}(X)$. *[2]*

(c) Find $\mathrm{Var}(X)$. *[3]*

9 – (MAT/12_HL_Summer_2014_Q1) – *Probability*

Events A and B are such that $\mathrm{P}(A)=\frac{2}{5}$, $\mathrm{P}(B)=\frac{11}{20}$ and $\mathrm{P}(A|B)=\frac{2}{11}$.

(a) Find $\mathrm{P}(A\cap B)$. *[2]*

(b) Find $\mathrm{P}(A\cup B)$. *[2]*

(c) State with a reason whether or not events A and B are independent. *[2]*

10 – (MAT/12_HL_Summer_2014_Q11) – *Probability*

(a) Mobile phone batteries are produced by two machines. Machine A produces 60% of the daily output and machine B produces 40%. It is found by testing that on average 2% of batteries produced by machine A are faulty and 1% of batteries produced by machine B are faulty.

(i) Draw a tree diagram clearly showing the respective probabilities.

(ii) A battery is selected at random. Find the probability that it is faulty.

(iii) A battery is selected at random and found to be faulty. Find the probability that it was produced by machine A. *[6]*

(b) In a pack of seven transistors, three are found to be defective. Three transistors are selected from the pack at random without replacement. The discrete random variable X represents the number of defective transistors selected.

(i) Find $\mathrm{P}(X = 2)$.

(ii) **Copy** and complete the following table:

x	0	1	2	3
$\mathrm{P}(X = x)$				

(iii) Determine $\mathrm{E}(X)$. *[6]*

11 – (MAT/10_HL_Winter_2014_Q4) – *Probability*

Events A and B are such that $\mathrm{P}(A) = 0.2$ and $\mathrm{P}(B) = 0.5$.

(a) Determine the value of $\mathrm{P}(A \cup B)$ when

(i) A and B are mutually exclusive;

(ii) A and B are independent. *[4]*

(b) Determine the range of possible values of $\mathrm{P}(A|B)$. *[3]*

12 – (MAT/10_HL_Winter_2014_Q9) – *Probability*

A continuous random variable T has probability density function f defined by

$$f(t) = \begin{cases} |2-t|, & 1 \le t \le 3 \\ 0, & \text{otherwise}. \end{cases}$$

(a) Sketch the graph of $y = f(t)$. *[2]*

(b) Find the interquartile range of T. *[4]*

13 – (MAT/12_HL_Summer_2015_Q1) – *Probability*

A and B are two events such that $\mathrm{P}(A) = 0.25$, $\mathrm{P}(B) = 0.6$ and $\mathrm{P}(A \cup B) = 0.7$.

(a) Find $\mathrm{P}(A \cap B)$. [2]

(b) Determine whether events A and B are independent. [2]

14 – (MAT/11_HL_Summer_2015_Q10) – *Probability*

[Maximum mark: 6]

A football team, Melchester Rovers are playing a tournament of five matches.
The probabilities that they win, draw or lose a match are $\frac{1}{2}$, $\frac{1}{6}$ and $\frac{1}{3}$ respectively.
These probabilities remain constant; the result of a match is independent of the results of other matches. At the end of the tournament their coach Roy loses his job if they lose three **consecutive** matches, otherwise he does not lose his job. Find the probability that Roy loses his job.

15 – (MAT/10_HL_Winter_2015_Q6) – *Probability*

A box contains four red balls and two white balls. Darren and Marty play a game by each taking it in turn to take a ball from the box, without replacement. The first player to take a white ball is the winner.

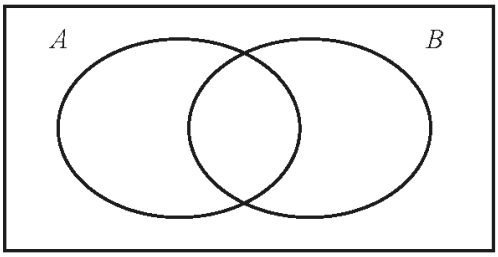
(a) Darren plays first, find the probability that he wins. [4]

The game is now changed so that the ball chosen is replaced after each turn.
Darren still plays first.

(b) Show that the probability of Darren winning has not changed. [3]

16 – (MAT/11_HL_Summer_2016_Q4) – *Probability*

(a) On the Venn diagram shade the region $A' \cap B'$. [1]



Two events A and B are such that $\mathrm{P}(A \cap B') = 0.2$ and $\mathrm{P}(A \cup B) = 0.9$.

(b) Find $\mathrm{P}(A' | B')$. [4]

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17 – (MAT/12_HL_Summer_2016_Q5) – *Probability*

A biased coin is tossed five times. The probability of obtaining a head in any one throw is p.

Let X be the number of heads obtained.

(a) Find, in terms of p, an expression for $\mathrm{P}(X = 4)$. [2]

(b) (i) Determine the value of p for which $\mathrm{P}(X = 4)$ is a maximum.

(ii) For this value of p, determine the expected number of heads. [6]

18 – (MAT/12_HL_Summer_2016_Q7) – *Probability*

A and B are independent events such that $\mathrm{P}(A) = \mathrm{P}(B) = p$, $p \neq 0$.

(a) Show that $\mathrm{P}(A \cup B) = 2p - p^2$. [2]

(b) Find $\mathrm{P}(A \mid A \cup B)$ in simplest form. [4]

19 – (MAT/10_HL_Winter_2016_Q2) – *Probability*

The faces of a fair six-sided die are numbered $1, 2, 2, 4, 4, 6$. Let X be the discrete random variable that models the score obtained when this die is rolled.

(a) Complete the probability distribution table for X. [2]

x				
$\mathrm{P}(X = x)$				

(b) Find the expected value of X. [2]

20 – (MAT/10_HL_Winter_2016_Q10) – *Probability*

Consider two events A and B defined in the same sample space.

(a) Show that $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(A' \cap B)$. [3]

(b) Given that $\mathrm{P}(A \cup B) = \frac{4}{9}$, $\mathrm{P}(B \mid A) = \frac{1}{3}$ and $\mathrm{P}(B \mid A') = \frac{1}{6}$,

(i) show that $\mathrm{P}(A) = \frac{1}{3}$;

(ii) hence find $\mathrm{P}(B)$. [6]

21 – (MAT/11_HL_Summer_2017_Q4) – *Probability*

[Maximum mark: 5]

Three girls and four boys are seated randomly on a straight bench. Find the probability that the girls sit together and the boys sit together.

22 – (MAT/12_HL_Summer_2017_Q7) – *Statistics, Probability*

(a) The random variable X has the Poisson distribution $\mathrm{Po}(m)$. Given that $\mathrm{P}(X > 0) = \frac{3}{4}$, find the value of m in the form $\ln a$ where a is an integer. [3]

(b) The random variable Y has the Poisson distribution $\mathrm{Po}(2m)$. Find $\mathrm{P}(Y > 1)$ in the form $\frac{b - \ln c}{c}$ where b and c are integers. [4]

23 – (MAT/11_HL_Summer_2017_Q10) – *Probability, Statistics*

The continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} k\sin\left(\dfrac{\pi x}{6}\right), & 0 \le x \le 6 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of k. [4]

(b) By considering the graph of f write down

(i) the mean of X;

(ii) the median of X;

(iii) the mode of X. [3]

(c) (i) Show that $P(0 \le X \le 2) = \dfrac{1}{4}$.

(ii) Hence state the interquartile range of X. [6]

(d) Calculate $P(X \le 4 \mid X \ge 3)$. [2]

24 – (MAT/11_HL_Summer_2018_Q3) – *Probability*

Two unbiased tetrahedral (four-sided) dice with faces labelled $1, 2, 3, 4$ are thrown and the scores recorded. Let the random variable T be the maximum of these two scores. The probability distribution of T is given in the following table.

t	1	2	3	4
$\mathrm{P}(T = t)$	$\frac{1}{16}$	a	b	$\frac{7}{16}$

(a) Find the value of a and the value of b. [3]

(b) Find the expected value of T. [2]

25 - (MAT/12_HL_Summer_2018_Q3) - *Probability*

The discrete random variable X has the following probability distribution, where p is a constant.

x	0	1	2	3	4
$\mathrm{P}(X = x)$	p	$0.5 - p$	0.25	0.125	p^3

(a) Find the value of p. [2]

(b) (i) Find μ, the expected value of X.

(ii) Find $\mathrm{P}(X > \mu)$. [4]

26 – (MAT/10_HL_Winter_2018_Q1) – *Probability*

Consider two events, A and B, such that $\mathrm{P}(A) = \mathrm{P}(A' \cap B) = 0.4$ and $\mathrm{P}(A \cap B) = 0.1$.

(a) By drawing a Venn diagram, or otherwise, find $\mathrm{P}(A \cup B)$. [3]

(b) Show that the events A and B are not independent. [3]

27 – (MAT/11_HL_Summer_2019_Q6) – *Probability*

Let X be a random variable which follows a normal distribution with mean μ. Given that $\mathrm{P}(X < \mu - 5) = 0.2$, find

(a) $\mathrm{P}(X > \mu + 5)$: [2]

(b) $\mathrm{P}(X < \mu + 5 \mid X > \mu - 5)$. [5]

28 – (MAT/11_HL_Summer_2019_Q10) – *Remainder & Factor Theorem, Probability*

The function $p(x)$ is defined by $p(x) = x^3 - 3x^2 + 8x - 24$ where $x \in \mathbb{R}$.

(a) Find the remainder when $p(x)$ is divided by

(i) $(x - 2)$

(ii) $(x - 3)$. [3]

(b) Prove that $p(x)$ has only one real zero. [4]

(c) Write down the transformation that will transform the graph of $y = p(x)$ onto the graph of $y = 8x^3 - 12x^2 + 16x - 24$. [2]

The random variable X follows a Poisson distribution with a mean of λ and $6\mathrm{P}(X = 3) = 3\mathrm{P}(X = 2) - 2\mathrm{P}(X = 1) + 3\mathrm{P}(X = 0)$.

(d) Find the value of λ. [6]

29 – (MAT/12_HL_Summer_2019_Q10) – *Integration, Probability*

The random variable X has probability density function f given by

$$f(x)=\begin{cases} k\left(\pi-\arcsin x\right) & 0\le x\le 1 \\ 0 & \text{otherwise} \end{cases}$$, where k is a positive constant.

(a) State the mode of X. [1]

(b) (i) Find $\int \arcsin x \, \mathrm{d}x$.

(ii) Hence show that $k=\dfrac{2}{2+\pi}$. [6]

(c) Given that $y=\left(\dfrac{x^2}{2}\right)\arcsin x-\left(\dfrac{1}{4}\right)\arcsin x+\left(\dfrac{x}{4}\right)\sqrt{1-x^2}$, show that

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=x\arcsin x$;

(ii) $\mathrm{E}\left(X\right)=\dfrac{3\pi}{4\left(\pi+2\right)}$. [9]

30 – (MAT/10_HL_Winter_2019_Q1) – *Probability*

[Maximum mark: 5]

The probability distribution of a discrete random variable, X, is given by the following table, where N and p are constants.

x	1	5	10	N
$\mathrm{P}(X = x)$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{5}$	p

(a) Find the value of p. [2]

(b) Given that $\mathrm{E}(X) = 10$, find the value of N. [3]

31 – (MAT/10_HL_Winter_2020_Q1) – *Probability*

[Maximum mark: 4]

A discrete random variable X has the probability distribution given by the following table.

x	0	1	2	3
$\mathrm{P}(X=x)$	p	$\frac{1}{4}$	$\frac{1}{6}$	q

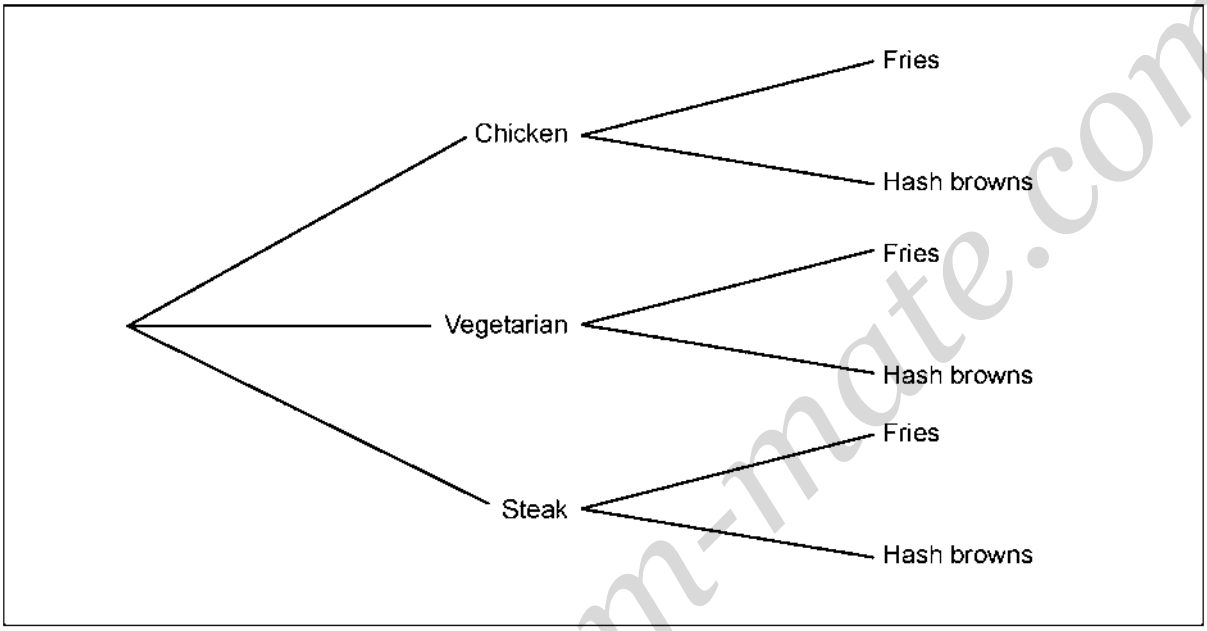
Given that $\mathrm{E}(X)=\frac{19}{12}$, determine the value of p and the value of q.

32 – (MAT/10_HL_Winter_2020_Q3) – *Probability*

At Nusaybah's Breakfast Diner, three types of omelette are available to order: chicken, vegetarian and steak. Each omelette is served with either a portion of fries or hash browns. It is known that 20% of customers choose a chicken omelette, 70% choose a vegetarian omelette and 10% choose a steak omelette.

It is also known that 65% of those ordering the chicken omelette, 70% of those ordering the vegetarian omelette and 60% of those ordering the steak omelette, order fries.

The following tree diagram represents the orders made by each customer.



(a) Complete the tree diagram by adding the respective probabilities to each branch. [2]

(b) Find the probability that a randomly selected customer orders fries. [2]

(c) Find the probability that a randomly selected customer orders fries, given that they do **not** order a chicken omelette. [3]

33 – (MathAA/11_HL_Summer_2021_Q10) – *Probability*

[Maximum mark: 16]

A biased four-sided die, A, is rolled. Let X be the score obtained when die A is rolled. The probability distribution for X is given in the following table.

x	1	2	3	4
$\mathrm{P}(X=x)$	p	p	p	$\frac{1}{2}p$

(a) Find the value of p. [2]

(b) Hence, find the value of $\mathrm{E}(X)$. [2]

A second biased four-sided die, B, is rolled. Let Y be the score obtained when die B is rolled. The probability distribution for Y is given in the following table.

y	1	2	3	4
$\mathrm{P}(Y=y)$	q	q	q	r

(c) (i) State the range of possible values of r.

(ii) Hence, find the range of possible values of q. [3]

(d) Hence, find the range of possible values for $\mathrm{E}(Y)$. [3]

Agnes and Barbara play a game using these dice. Agnes rolls die A once and Barbara rolls die B once. The probability that Agnes' score is less than Barbara's score is $\frac{1}{2}$.

(e) Find the value of $\mathrm{E}(Y)$. [6]

34 – (MathAA/10_HL_Winter_2021_Q4) – *Probability*

[Maximum mark: 5]

Box 1 contains 5 red balls and 2 white balls.
Box 2 contains 4 red balls and 3 white balls.

(a) A box is chosen at random and a ball is drawn. Find the probability that the ball is red. [3]

Let A be the event that "box 1 is chosen" and let R be the event that "a red ball is drawn".

(b) Determine whether events A and R are independent. [2]

35 – (MathAA/11_HL_Summer_2022_Q7) – *Probability*

[Maximum mark: 8]

The continuous random variable X has probability density function

$$f(x)=\begin{cases}\dfrac{k}{\sqrt{4-3x^2}}, & 0\le x\le 1\\ 0, & \text{otherwise.}\end{cases}$$

(a) Find the value of k. [4]

(b) Find $\mathrm{E}(X)$. [4]

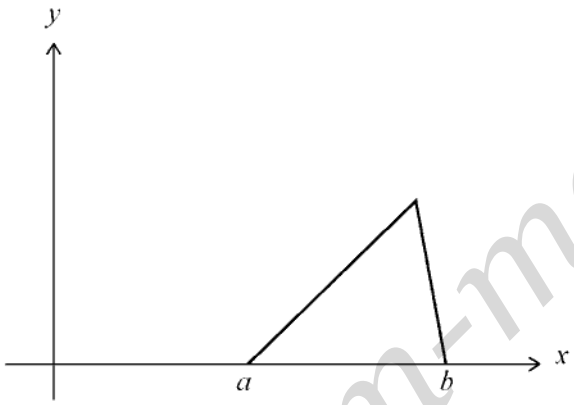
36 – (MathAA/12_HL_Summer_2022_Q8) – *Probability*

[Maximum mark: 6]

A continuous random variable X has the probability density function

$$f(x)=\begin{cases}\dfrac{2}{(b-a)(c-a)}(x-a), & a\le x\le c\\ \dfrac{2}{(b-a)(b-c)}(b-x), & c<x\le b\\ 0, & \text{otherwise}\end{cases}.$$

The following diagram shows the graph of $y=f(x)$ for $a\le x\le b$.



Given that $c\ge\dfrac{a+b}{2}$, find an expression for the median of X in terms of a, b and c.

37 – (MathAA/12_HL_Summer_2022_Q10) – *Probability*

[Maximum mark: 16]

A biased four-sided die with faces labelled 1, 2, 3 and 4 is rolled and the result recorded. Let X be the result obtained when the die is rolled. The probability distribution for X is given in the following table where p and q are constants.

x	1	2	3	4
$\mathrm{P}(X=x)$	p	0.3	q	0.1

For this probability distribution, it is known that $\mathrm{E}(X)=2$.

(a) Show that $p=0.4$ and $q=0.2$. [5]

(b) Find $\mathrm{P}(X>2)$. [2]

Nicky plays a game with this four-sided die. In this game she is allowed a maximum of five rolls. Her score is calculated by adding the results of each roll. Nicky wins the game if her score is at least ten.

After three rolls of the die, Nicky has a score of four.

(c) Assuming that rolls of the die are independent, find the probability that Nicky wins the game. [5]

David has two pairs of unbiased four-sided dice, a yellow pair and a red pair.
Both yellow dice have faces labelled 1, 2, 3 and 4. Let S represent the sum obtained by rolling the two yellow dice. The probability distribution for S is shown below.

s	2	3	4	5	6	7	8
$\mathrm{P}(S=s)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

The first red die has faces labelled 1, 2, 2 and 3. The second red die has faces labelled 1, a, a and b, where $a<b$ and $a, b \in \mathbb{Z}^+$. The probability distribution for the sum obtained by rolling the red pair is the same as the distribution for the sum obtained by rolling the yellow pair.

(d) Determine the value of b. [2]

(e) Find the value of a, providing evidence for your answer. [2]

38 – (MathAA/10_HL_Winter_2022_Q6) – *Probability*

[Maximum mark: 6]

Events A and B are such that $\mathrm{P}(A) = 0.3$ and $\mathrm{P}(B) = 0.8$.

(a) Determine the value of $\mathrm{P}(A \cap B)$ in the case where the events A and B are independent. [1]

(b) Determine the minimum possible value of $\mathrm{P}(A \cap B)$. [3]

(c) Determine the maximum possible value of $\mathrm{P}(A \cap B)$, justifying your answer. [2]

1 – (MAT/21_HL_Summer_2012_Q1) – *Differentiation*

[Maximum mark: 5]

Given that the graph of $y = x^3 - 6x^2 + kx - 4$ has exactly one point at which the gradient is zero, find the value of k.

2 – (MAT/22_HL_Summer_2012_Q6) – *Differentiation*

(a) Sketch the curve $y = \dfrac{\cos x}{\sqrt{x^2+1}}$, $-4 \le x \le 4$ showing clearly the coordinates of the x-intercepts, any maximum points and any minimum points. *[4 marks]*

(b) Write down the gradient of the curve at $x = 1$. *[1 mark]*

(c) Find the equation of the normal to the curve at $x = 1$. *[3 marks]*

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3 – (MAT/21_HL_Summer_2012_Q10) – *Differentiation*

[Maximum mark: 8]

A triangle is formed by the three lines $y = 10 - 2x$, $y = mx$ and $y = -\frac{1}{m}x$, where $m > \frac{1}{2}$.
Find the value of m for which the area of the triangle is a minimum.

4 – (MAT/21_HL_Summer_2012_Q11) – *Differentiation*

The function $f(x) = 3\sin x + 4\cos x$ is defined for $0 < x < 2\pi$.

(a) Write down the coordinates of the minimum point on the graph of f. *[1 mark]*

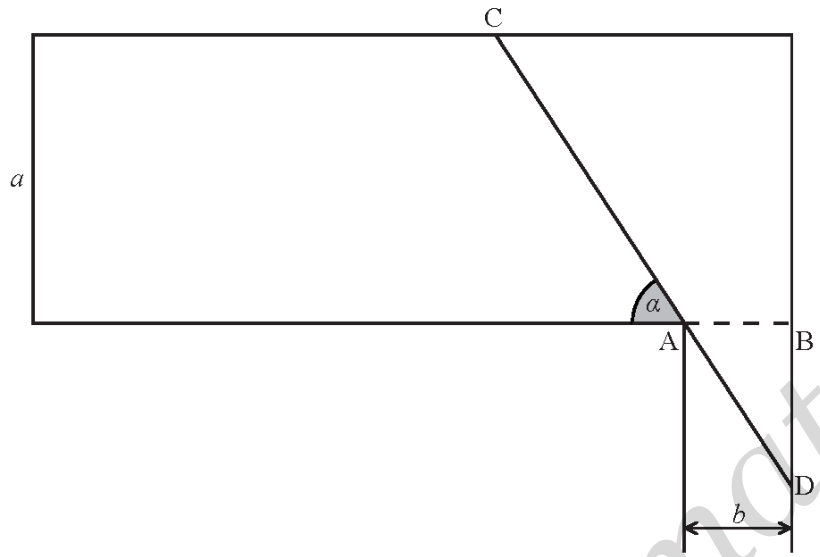
(b) The points $\mathrm{P}(p, 3)$ and $\mathrm{Q}(q, 3)$, $q > p$, lie on the graph of $y = f(x)$. Find p and q. *[2 marks]*

(c) Find the coordinates of the point, on $y = f(x)$, where the gradient of the graph is 3. *[4 marks]*

(d) Find the coordinates of the point of intersection of the normals to the graph at the points P and Q. *[7 marks]*

5 - (MAT/20_HL_Winter_2012_Q12) - *Differentiation*

The diagram shows the plan of an art gallery a metres wide. [AB] represents a doorway, leading to an exit corridor b metres wide. In order to remove a painting from the art gallery, CD (denoted by L) is measured for various values of α, as represented in the diagram.



(a) If α is the angle between [CD] and the wall, show that $L = \frac{a}{\sin\alpha} + \frac{b}{\cos\alpha}$, $0 < \alpha < \frac{\pi}{2}$. *[3 marks]*

(b) If $a = 5$ and $b = 1$, find the maximum length of a painting that can be removed through this doorway. *[4 marks]*

Let $a = 3k$ and $b = k$.

(c) Find $\frac{dL}{d\alpha}$. *[3 marks]*

(d) Find, in terms of k, the maximum length of a painting that can be removed from the gallery through this doorway. *[6 marks]*

(e) Find the minimum value of k if a painting 8 metres long is to be removed through this doorway. *[2 marks]*

6 – (MAT/22_HL_Summer_2013_Q12) – *Differentiation, Integration*

Consider the differential equation $y\frac{\mathrm{d}y}{\mathrm{d}x} = \cos 2x$.

(a) (i) Show that the function $y = \cos x + \sin x$ satisfies the differential equation.

(ii) Find the general solution of the differential equation. Express your solution in the form $y = f(x)$, involving a constant of integration.

(iii) For which value of the constant of integration does your solution coincide with the function given in part (i)? *[10 marks]*

(b) A different solution of the differential equation, satisfying $y = 2$ when $x = \frac{\pi}{4}$, defines a curve C.

(i) Determine the equation of C in the form $y = g(x)$, and state the range of the function g.

A region R in the xy plane is bounded by C, the x-axis and the vertical lines $x = 0$ and $x = \frac{\pi}{2}$.

(ii) Find the area of R.

(iii) Find the volume generated when that part of R above the line $y = 1$ is rotated about the x-axis through 2π radians. *[12 marks]*

7 – (MAT/21_HL_Summer_2013_Q13) – *Differentiation*

The function f has inverse f^{-1} and derivative $f'(x)$ for all $x \in \mathbb{R}$. For all functions with these properties you are given the result that for $a \in \mathbb{R}$ with $b = f(a)$ and $f'(a) \neq 0$

$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

(a) Verify that this is true for $f(x) = x^3 + 1$ at $x = 2$. *[6 marks]*

(b) Given that $g(x) = xe^{x^2}$, show that $g'(x) > 0$ for all values of x. *[3 marks]*

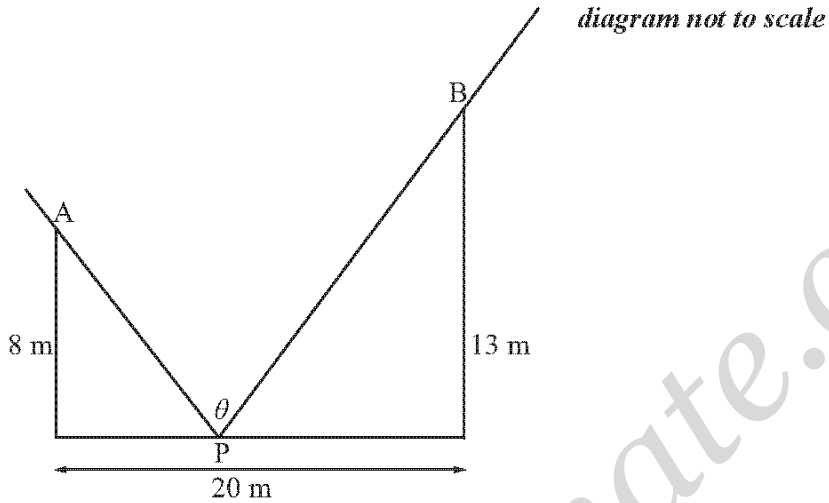
(c) Using the result given at the start of the question, find the value of the gradient function of $y = g^{-1}(x)$ at $x = 2$. *[4 marks]*

(d) (i) With f and g as defined in parts (a) and (b), solve $g \circ f(x) = 2$.

(ii) Let $h(x) = (g \circ f)^{-1}(x)$. Find $h'(2)$. *[6 marks]*

8 – (MAT/22_HL_Summer_2013_Q13) – *Differentiation*

A straight street of width 20 metres is bounded on its parallel sides by two vertical walls, one of height 13 metres, the other of height 8 metres. The intensity of light at point P at ground level on the street is proportional to the angle θ where $\theta = \mathrm{A\hat{P}B}$, as shown in the diagram.



(a) Find an expression for θ in terms of x, where x is the distance of P from the base of the wall of height 8m. *[2 marks]*

(b) (i) Calculate the value of θ when $x = 0$.

(ii) Calculate the value of θ when $x = 20$. *[2 marks]*

(c) Sketch the graph of θ, for $0 \le x \le 20$. *[2 marks]*

(d) Show that $\dfrac{\mathrm{d}\theta}{\mathrm{d}x} = \dfrac{5(744 - 64x - x^2)}{(x^2 + 64)(x^2 - 40x + 569)}$. *[6 marks]*

(e) Using the result in part (d), or otherwise, determine the value of x corresponding to the maximum light intensity at P. Give your answer to four significant figures. *[3 marks]*

(f) The point P moves across the street with speed $0.5\ \mathrm{m\,s^{-1}}$. Determine the rate of change of θ with respect to time when P is at the midpoint of the street. *[4 marks]*

9 – (MAT/22_HL_Summer_2014_Q9) – *Differentiation*

[Maximum mark: 5]

Sand is being poured to form a cone of height h cm and base radius r cm. The height remains equal to the base radius at all times. The height of the cone is increasing at a rate of $0.5\,\text{cm}\,\text{min}^{-1}$.

Find the rate at which sand is being poured, in $\text{cm}^3\,\text{min}^{-1}$, when the height is 4 cm.

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10 – (MAT/21_HL_Summer_2014_Q10) – *Functions - Roots, Differentiation*

Let $f(x) = \dfrac{e^{2x}+1}{e^{x}-2}$.

(a) Find the equations of the horizontal and vertical asymptotes of the curve $y = f(x)$. *[4]*

(b) (i) Find $f'(x)$.

(ii) Show that the curve has exactly one point where its tangent is horizontal.

(iii) Find the coordinates of this point. *[8]*

(c) Find the equation of L_1, the normal to the curve at the point where it crosses the y-axis. *[4]*

The line L_2 is parallel to L_1 and tangent to the curve $y = f(x)$.

(d) Find the equation of the line L_2. *[5]*

11 – (MAT/22_HL_Summer_2014_Q10) – *Differentiation*

Consider the curve with equation $\left(x^2+y^2\right)^2 = 4xy^2$.

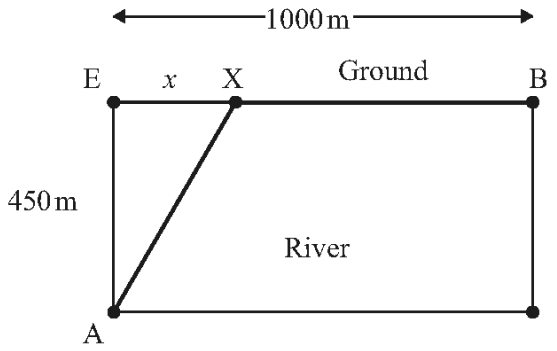
(a) Use implicit differentiation to find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$. *[5]*

(b) Find the equation of the normal to the curve at the point $(1, 1)$. *[3]*

12 – (MAT/22_HL_Summer_2014_Q12) – *Differentiation*

Engineers need to lay pipes to connect two cities A and B that are separated by a river of width 450 metres as shown in the following diagram. They plan to lay the pipes under the river from A to X and then under the ground from X to B. The cost of laying the pipes under the river is five times the cost of laying the pipes under the ground.

Let $\text{EX} = x$.



Let k be the cost, in dollars per metre, of laying the pipes under the ground.

(a) Show that the total cost C, in dollars, of laying the pipes from A to B is given by $C = 5k\sqrt{202500 + x^2} + (1000 - x)k$. *[2]*

(b) (i) Find $\frac{\mathrm{d}C}{\mathrm{d}x}$.

(ii) Hence find the value of x for which the total cost is a minimum, justifying that this value is a minimum. *[7]*

(c) Find the minimum total cost in terms of k. *[1]*

The angle at which the pipes are joined is $\text{A}\hat{\text{X}}\text{B} = \theta$.

(d) Find θ for the value of x calculated in (b). *[2]*

For safety reasons θ must be at least 120°.

Given this new requirement,

(e) (i) find the new value of x which minimises the total cost;

(ii) find the percentage increase in the minimum total cost. *[3]*

13 - (MAT/20_HL_Winter_2014_Q10) - *Differentiation*

Consider the triangle PQR where $\mathrm{Q\hat{P}R} = 30^\circ$, $\mathrm{PQ} = (x+2)\,\mathrm{cm}$ and $\mathrm{PR} = (5-x)^2\,\mathrm{cm}$, where $-2 < x < 5$.

(a) Show that the area, A cm^2, of the triangle is given by $A = \frac{1}{4}\left(x^3 - 8x^2 + 5x + 50\right)$. *[2]*

(b) (i) State $\frac{\mathrm{d}A}{\mathrm{d}x}$.

(ii) Verify that $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$ when $x = \frac{1}{3}$. *[3]*

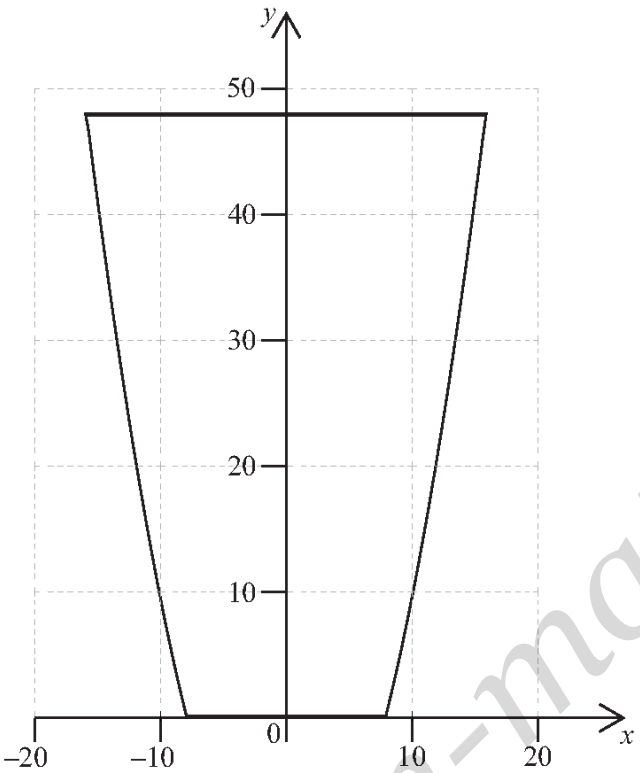
(c) (i) Find $\frac{\mathrm{d}^2A}{\mathrm{d}x^2}$ and hence justify that $x = \frac{1}{3}$ gives the maximum area of triangle PQR.

(ii) State the maximum area of triangle PQR.

(iii) Find QR when the area of triangle PQR is a maximum. *[7]*

14 – (MAT/20_HL_Winter_2014_Q13) – *Differentiation, Integration*

The vertical cross-section of a container is shown in the following diagram.



The curved sides of the cross-section are given by the equation $y = 0.25x^2 - 16$. The horizontal cross-sections are circular. The depth of the container is $48\,\text{cm}$.

(a) If the container is filled with water to a depth of h cm, show that the volume, $V\,\text{cm}^3$, of the water is given by $V = 4\pi\left(\frac{h^2}{2} + 16h\right)$. *[3]*

The container, initially full of water, begins leaking from a small hole at a rate given by $\frac{dV}{dt} = -\frac{250\sqrt{h}}{\pi(h+16)}$ where t is measured in seconds.

(b) (i) Show that $\frac{dh}{dt} = -\frac{250\sqrt{h}}{4\pi^2(h+16)^2}$.

(ii) State $\frac{dt}{dh}$ and hence show that $t = \frac{-4\pi^2}{250}\int\left(h^{\frac{3}{2}} + 32h^{\frac{1}{2}} + 256h^{-\frac{1}{2}}\right)dh$.

(iii) Find, correct to the nearest minute, the time taken for the container to become empty. (60 seconds = 1 minute) *[10]*

Once empty, water is pumped back into the container at a rate of $8.5\,\text{cm}^3\,\text{s}^{-1}$. At the same time, water continues leaking from the container at a rate of $\frac{250\sqrt{h}}{\pi(h+16)}\text{cm}^3\,\text{s}^{-1}$.

(c) Using an appropriate sketch graph, determine the depth at which the water ultimately stabilizes in the container. *[3]*

15 - (MAT/21_HL_Summer_2015_Q5) - *Differentiation*

[Maximum mark: 5]

A bicycle inner tube can be considered as a joined up cylinder of fixed length $200\,\text{cm}$ and radius $r\,\text{cm}$. The radius r increases as the inner tube is pumped up. Air is being pumped into the inner tube so that the volume of air in the tube increases at a constant rate of $30\,\text{cm}^3\,\text{s}^{-1}$. Find the rate at which the radius of the inner tube is increasing when $r = 2\,\text{cm}$.

16 – (MAT/21_HL_Summer_2015_Q6) – *Differentiation*

[Maximum mark: 4]

A function f is defined by $f(x) = x^3 + e^x + 1$, $x \in \mathbb{R}$. By considering $f'(x)$ determine whether f is a one-to-one or a many-to-one function.

17 – (MAT/21_HL_Summer_2015_Q9) – *Differentiation*

[Maximum mark: 7]

Find the equation of the normal to the curve $y = \dfrac{e^x \cos x \ln(x+e)}{\left(x^{17}+1\right)^5}$ at the point where $x = 0$.

In your answer give the value of the gradient, of the normal, to three decimal places.

18 – (MAT/22_HL_Summer_2015_Q11) – *Differentiation*

A curve is defined by $x^2 - 5xy + y^2 = 7$.

(a) Show that $\frac{dy}{dx} = \frac{5y - 2x}{2y - 5x}$. [3]

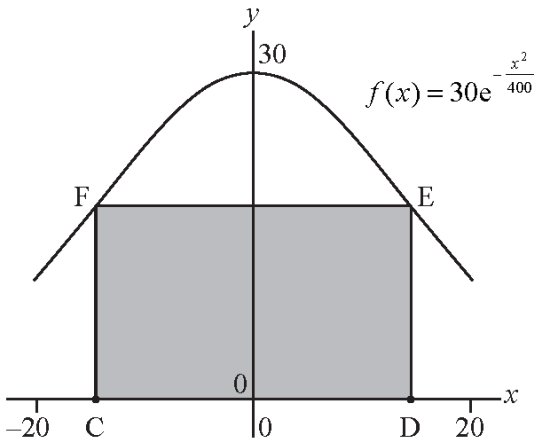
(b) Find the equation of the normal to the curve at the point $(6, 1)$. [4]

(c) Find the distance between the two points on the curve where each tangent is parallel to the line $y = x$. [8]

19 – (MAT/20_HL_Winter_2015_Q13) – *Differentiation*

The following diagram shows a vertical cross section of a building. The cross section of the roof of the building can be modelled by the curve $f(x) = 30e^{-\frac{x^2}{400}}$, where $-20 \le x \le 20$.

Ground level is represented by the x-axis.



(a) Find $f''(x)$. [4]

(b) Show that the gradient of the roof function is greatest when $x = -\sqrt{200}$. [3]

The cross section of the living space under the roof can be modelled by a rectangle CDEF with points $C(-a, 0)$ and $D(a, 0)$, where $0 < a \le 20$.

(c) Show that the maximum area A of the rectangle CDEF is $600\sqrt{2}e^{-\frac{1}{2}}$. [5]

(d) A function I is known as the Insulation Factor of CDEF. The function is defined as $I(a) = \frac{P(a)}{A(a)}$ where P = Perimeter and A = Area of the rectangle.

(i) Find an expression for P in terms of a.

(ii) Find the value of a which minimizes I.

(iii) Using the value of a found in part (ii) calculate the percentage of the cross sectional area under the whole roof that is not included in the cross section of the living space. [9]

20 – (MAT/22_HL_Summer_2016_Q7) – *Differentiation*

Consider the curve with equation $x^3 + y^3 = 4xy$.

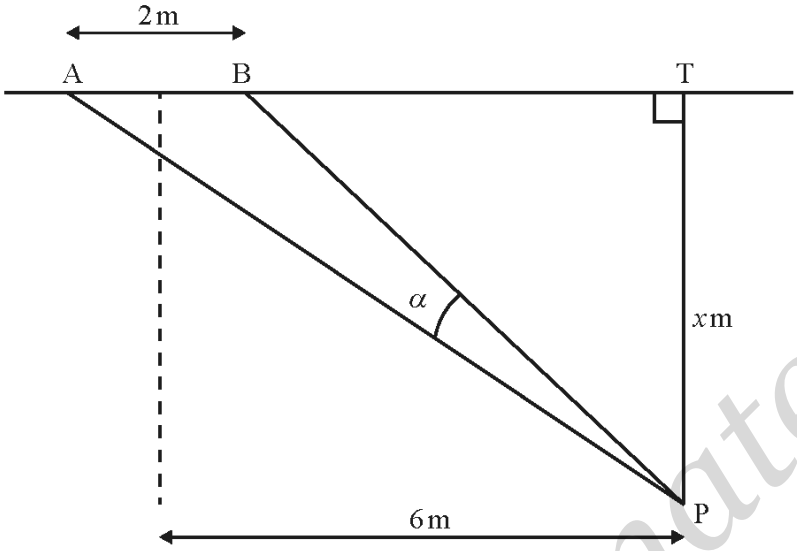
(a) Use implicit differentiation to show that $\frac{dy}{dx} = \frac{4y - 3x^2}{3y^2 - 4x}$. [3]

The tangent to this curve is parallel to the x-axis at the point where $x = k$, $k > 0$.

(b) Find the value of k. [5]

21 – (MAT/22_HL_Summer_2016_Q11) – *Differentiation*

Points A, B and T lie on a line on an indoor soccer field. The goal, $[\mathrm{AB}]$, is 2 metres wide. A player situated at point P kicks a ball at the goal. $[\mathrm{PT}]$ is perpendicular to (AB) and is 6 metres from a parallel line through the centre of $[\mathrm{AB}]$. Let PT be x metres and let $\alpha = \mathrm{A\hat{P}B}$ measured in degrees. Assume that the ball travels along the floor.



(a) Find the value of α when $x = 10$. [4]

(b) Show that $\tan\alpha = \dfrac{2x}{x^2 + 35}$. [4]

The maximum for $\tan\alpha$ gives the maximum for α.

(c) (i) Find $\dfrac{\mathrm{d}}{\mathrm{d}x}(\tan\alpha)$.

(ii) Hence or otherwise find the value of α such that $\dfrac{\mathrm{d}}{\mathrm{d}x}(\tan\alpha) = 0$.

(iii) Find $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}(\tan\alpha)$ and hence show that the value of α never exceeds 10°. [11]

(d) Find the set of values of x for which $\alpha \geq 7^\circ$. [3]

22 – (MAT/21_HL_Summer_2016_Q12) – *Differentiation*

Consider the curve, C defined by the equation $y^2 - 2xy = 5 - e^x$. The point A lies on C and has coordinates $(0, a)$, $a > 0$.

(a) Find the value of a. [2]

(b) Show that $\frac{dy}{dx} = \frac{2y - e^x}{2(y - x)}$. [4]

(c) Find the equation of the normal to C at the point A. [3]

(d) Find the coordinates of the second point at which the normal found in part (c) intersects C. [4]

(e) Given that $v = y^3$, $y > 0$, find $\frac{dv}{dx}$ at $x = 0$. [3]

23 – (MAT/22_HL_Summer_2016_Q12) – *Differentiation, Integration*

The functions f and g are defined by

$$f(x) = \frac{e^x + e^{-x}}{2}, x \in \mathbb{R}$$

$$g(x) = \frac{e^x - e^{-x}}{2}, x \in \mathbb{R}$$

(a) (i) Show that $\frac{1}{4f(x) - 2g(x)} = \frac{e^x}{e^{2x} + 3}$.

(ii) Use the substitution $u = e^x$ to find $\int_0^{\ln 3} \frac{1}{4f(x) - 2g(x)} dx$. Give your answer in the form $\frac{\pi\sqrt{a}}{b}$ where $a, b \in \mathbb{Z}^+$. [9]

Let $h(x) = nf(x) + g(x)$ where $n \in \mathbb{R}, n > 1$.

(b) (i) By forming a quadratic equation in e^x, solve the equation $h(x) = k$, where $k \in \mathbb{R}^+$.

(ii) Hence or otherwise show that the equation $h(x) = k$ has two real solutions provided that $k > \sqrt{n^2 - 1}$ and $k \in \mathbb{R}^+$. [8]

Let $t(x) = \frac{g(x)}{f(x)}$.

(c) (i) Show that $t'(x) = \frac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$ for $x \in \mathbb{R}$.

(ii) Hence show that $t'(x) > 0$ for $x \in \mathbb{R}$. [6]

24 – (MAT/20_HL_Winter_2016_Q6) – *Differentiation*

An earth satellite moves in a path that can be described by the curve $72.5x^2 + 71.5y^2 = 1$ where $x = x(t)$ and $y = y(t)$ are in thousands of kilometres and t is time in seconds.

Given that $\frac{dx}{dt} = 7.75 \times 10^{-5}$ when $x = 3.2 \times 10^{-3}$, find the possible values of $\frac{dy}{dt}$.

Give your answers in standard form.

25 – (MAT/20_HL_Winter_2016_Q10) – *Functions - Roots, Differentiation, Integration*

Let the function f be defined by $f(x) = \dfrac{2 - e^x}{2e^x - 1}$, $x \in D$.

(a) Determine D, the largest possible domain of f. [2]

(b) Show that the graph of f has three asymptotes and state their equations. [5]

(c) Show that $f'(x) = -\dfrac{3e^x}{\left(2e^x - 1\right)^2}$. [3]

(d) Use your answers from parts (b) and (c) to justify that f has an inverse and state its domain. [4]

(e) Find an expression for $f^{-1}(x)$. [4]

(f) Consider the region R enclosed by the graph of $y = f(x)$ and the axes.
Find the volume of the solid obtained when R is rotated through 2π about the y-axis. [4]

26 – (MAT/21_HL_Summer_2017_Q2) – *Differentiation*

The curve C is defined by equation $xy - \ln y = 1$, $y > 0$.

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. [4]

(b) Determine the equation of the tangent to C at the point $\left(\frac{2}{\mathrm{e}}, \mathrm{e}\right)$. [3]

27 – (MAT/22_HL_Summer_2017_Q2) – *Differentiation, Integration*

Consider the curve defined by the equation $4x^2 + y^2 = 7$.

(a) Find the equation of the normal to the curve at the point $\left(1, \sqrt{3}\right)$. [6]

(b) Find the volume of the solid formed when the region bounded by the curve, the x-axis for $x \geq 0$ and the y-axis for $y \geq 0$ is rotated through 2π about the x-axis. [3]

28 – (MAT/21_HL_Summer_2017_Q8) – *Differentiation*

A water trough which is 10 metres long has a uniform cross-section in the shape of a semicircle with radius 0.5 metres. It is partly filled with water as shown in the following diagram of the cross-section. The centre of the circle is O and the angle KOL is θ radians.

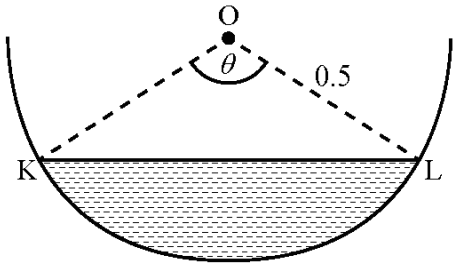


diagram not to scale

(a) Find an expression for the volume of water $V\left(m^3\right)$ in the trough in terms of θ. [3]

The volume of water is increasing at a constant rate of $0.0008\,\mathrm{m}^3\,\mathrm{s}^{-1}$.

(b) Calculate $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ when $\theta = \frac{\pi}{3}$. [4]

29 – (MAT/21_HL_Summer_2017_Q12) – *Functions - Roots, Graphs, Differentiation*

Consider $f(x) = -1 + \ln\left(\sqrt{x^2 - 1}\right)$.

(a) Find the largest possible domain D for f to be a function. [2]

The function f is defined by $f(x) = -1 + \ln\left(\sqrt{x^2 - 1}\right),\ x \in D$.

(b) Sketch the graph of $y = f(x)$ showing clearly the equations of asymptotes and the coordinates of any intercepts with the axes. [3]

(c) Explain why f is an even function. [1]

(d) Explain why the inverse function f^{-1} does not exist. [1]

The function g is defined by $g(x) = -1 + \ln\left(\sqrt{x^2 - 1}\right),\ x \in]1, \infty[$.

(e) Find the inverse function g^{-1} and state its domain. [4]

(f) Find $g'(x)$. [3]

(g) Hence, show that there are no solutions to

(i) $g'(x) = 0$;

(ii) $\left(g^{-1}\right)'(x) = 0$. [4]

30 – (MAT/20_HL_Winter_2017_Q10) – *Differentiation, Integration*

[Maximum mark: 17]

Consider the function $f(x) = \dfrac{\sqrt{x}}{\sin x}$, $0 < x < \pi$.

(a) (i) Show that the x-coordinate of the minimum point on the curve $y = f(x)$ satisfies the equation $\tan x = 2x$.

(ii) Determine the values of x for which $f(x)$ is a decreasing function. [7]

(b) Sketch the graph of $y = f(x)$ showing clearly the minimum point and any asymptotic behaviour. [3]

(c) Find the coordinates of the point on the graph of f where the normal to the graph is parallel to the line $y = -x$. [4]

Consider the region bounded by the curve $y = f(x)$, the x-axis and the lines $x = \dfrac{\pi}{6}$, $x = \dfrac{\pi}{3}$.

(d) This region is now rotated through 2π radians about the x-axis. Find the volume of revolution. [3]

31 – (MAT/20_HL_Winter_2017_Q11) – *Trigonometry, Differentiation*

[Maximum mark: 18]

Consider the function $f(x) = 2\sin^2 x + 7\sin 2x + \tan x - 9,\ 0 \le x < \frac{\pi}{2}$.

(a) (i) Determine an expression for $f'(x)$ in terms of x.

(ii) Sketch a graph of $y = f'(x)$ for $0 \le x < \frac{\pi}{2}$.

(iii) Find the x-coordinate(s) of the point(s) of inflexion of the graph of $y = f(x)$, labelling these clearly on the graph of $y = f'(x)$. [8]

(b) Let $u = \tan x$.

(i) Express $\sin x$ in terms of u.

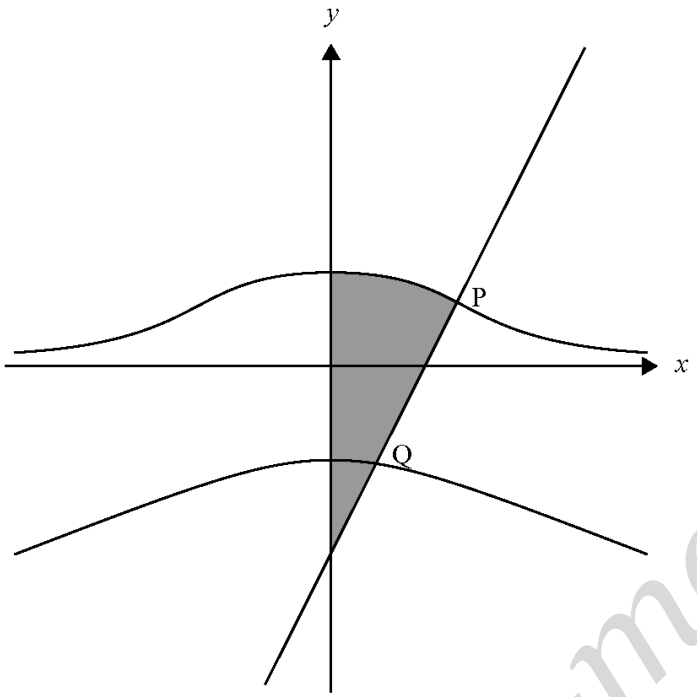
(ii) Express $\sin 2x$ in terms of u.

(iii) Hence show that $f(x) = 0$ can be expressed as $u^3 - 7u^2 + 15u - 9 = 0$. [7]

(c) Solve the equation $f(x) = 0$, giving your answers in the form $\arctan k$ where $k \in \mathbb{Z}$. [3]

32 – (MAT/21_HL_Summer_2018_Q9) – *Differentiation, Integration*

The following graph shows the two parts of the curve defined by the equation $x^2y = 5 - y^4$, and the normal to the curve at the point $\mathrm{P}(2, 1)$.



(a) Show that there are exactly two points on the curve where the gradient is zero. [7]

(b) Find the equation of the normal to the curve at the point P. [5]

(c) The normal at P cuts the curve again at the point Q. Find the x-coordinate of Q. [3]

(d) The shaded region is rotated by 2π about the y-axis. Find the volume of the solid formed. [7]

33 - (MAT/22_HL_Summer_2018_Q11) - *Differentiation*

A curve C is given by the implicit equation $x + y - \cos(xy) = 0$.

(a) Show that $\frac{dy}{dx} = -\left(\frac{1 + y\sin(xy)}{1 + x\sin(xy)}\right)$. [5]

(b) The curve $xy = -\frac{\pi}{2}$ intersects C at P and Q.

(i) Find the coordinates of P and Q.

(ii) Given that the gradients of the tangents to C at P and Q are m_1 and m_2 respectively, show that $m_1 \times m_2 = 1$. [7]

(c) Find the coordinates of the three points on C, nearest the origin, where the tangent is parallel to the line $y = -x$. [7]

34 – (MAT/20_HL_Winter_2018_Q5) – *Differentiation*

[Maximum mark: 5]

Differentiate from first principles the function $f(x) = 3x^3 - x$.

35 – (MAT/20_HL_Winter_2018_Q9) – *Functions - Roots, Differentiation*

The function f is defined by $f(x) = \dfrac{2\ln x + 1}{x - 3}$, $0 < x < 3$.

(a) Find $f'(x)$. [4]

(b) Hence, or otherwise, find the coordinates of the point of inflexion on the graph of $y = f(x)$. [4]

(c) Draw a set of axes showing x and y values between -3 and 3. On these axes

(i) sketch the graph of $y = f(x)$, showing clearly any axis intercepts and giving the equations of any asymptotes.

(ii) sketch the graph of $y = f^{-1}(x)$, showing clearly any axis intercepts and giving the equations of any asymptotes. [8]

(d) Hence, or otherwise, solve the inequality $f(x) > f^{-1}(x)$. [3]

36 – (MAT/21_HL_Summer_2019_Q1) – *Differentiation*

[Maximum mark: 4]

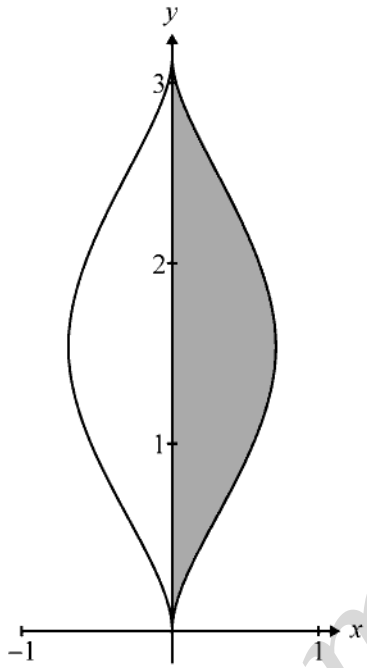
Let l be the tangent to the curve $y = xe^{2x}$ at the point $(1 , e^2)$.

Find the coordinates of the point where l meets the x-axis.

37 – (MAT/20_HL_Winter_2019_Q11) – *Differentiation, Integration*

[Maximum mark: 17]

The following diagram shows part of the graph of $2x^2 = \sin^3 y$ for $0 \le y \le \pi$.



(a) (i) Using implicit differentiation, find an expression for $\frac{dy}{dx}$.

(ii) Find the equation of the tangent to the curve at the point $\left(\frac{1}{4}, \frac{5\pi}{6}\right)$. [8]

The shaded region R is the area bounded by the curve, the y-axis and the lines $y = 0$ and $y = \pi$.

(b) Find the area of R. [3]

The region R is now rotated about the y-axis, through 2π radians, to form a solid.

(c) By writing $\sin^3 y$ as $(1 - \cos^2 y)\sin y$, show that the volume of the solid formed is $\frac{2\pi}{3}$. [6]

38 – (MAT/20_HL_Winter_2020_Q8) – *Differentiation*

A small bead is free to move along a smooth wire in the shape of the curve $y = \frac{10}{3 - 2e^{0.5x}}$ $(x \geq 0)$.

(a) Find an expression for $\frac{dy}{dx}$. [3]

At the point on the curve where $x = 4$, it is given that $\frac{dy}{dt} = -0.1 \text{ m s}^{-1}$.

(b) Find the value of $\frac{dx}{dt}$ at this exact same instant. [3]

39 – (MathAA/21_HL_Summer_2021_Q9) – *Differentiation, Kinematics, Relative Velocity*

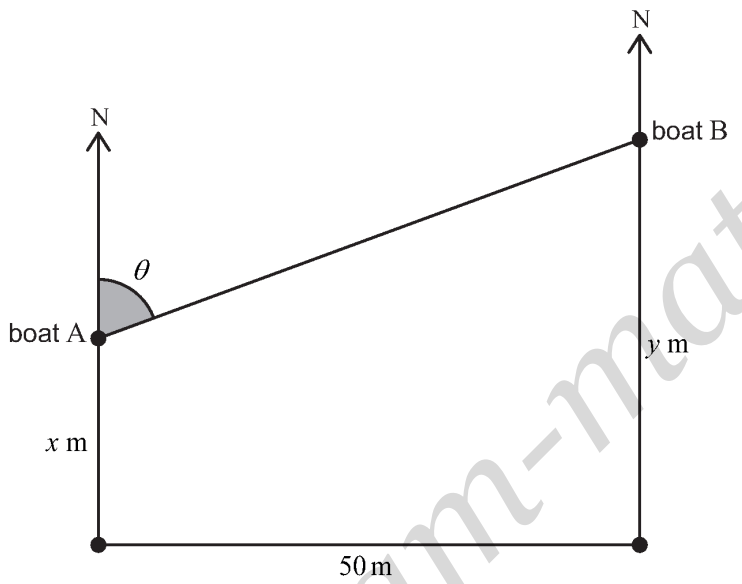
[Maximum mark: 7]

Two boats A and B travel due north.

Initially, boat B is positioned 50 metres due east of boat A.

The distances travelled by boat A and boat B, after t seconds, are x metres and y metres respectively. The angle θ is the radian measure of the bearing of boat B from boat A.
This information is shown on the following diagram.

diagram not to scale



(a) Show that $y = x + 50\cot\theta$. [1]

At time T, the following conditions are true.

Boat B has travelled 10 metres further than boat A.
Boat B is travelling at double the speed of boat A.
The rate of change of the angle θ is -0.1 radians per second.

(b) Find the speed of boat A at time T. [6]

40 – (MathAA/21_HL_Summer_2021_Q10) – *Exponentials - Logarithms, Differentiation*

[Maximum mark: 15]

Consider the function f defined by $f(x) = 90\,\mathrm{e}^{-0.5x}$ for $x \in \mathbb{R}^+$.

The graph of f and the line $y = x$ intersect at point P.

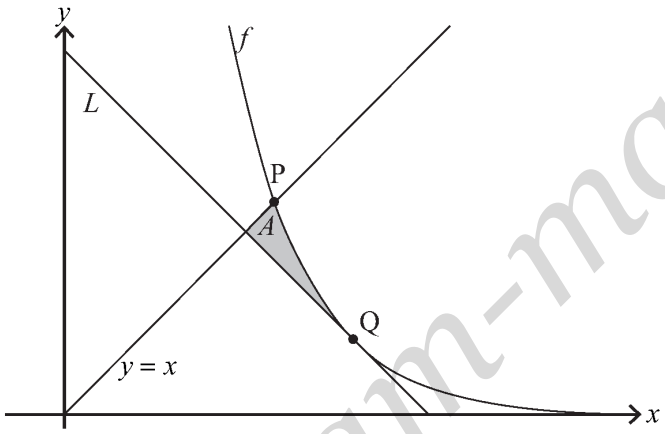
(a) Find the x-coordinate of P. [2]

The line L has a gradient of -1 and is a tangent to the graph of f at the point Q.

(b) Find the exact coordinates of Q. [4]

(c) Show that the equation of L is $y = -x + 2\ln 45 + 2$. [2]

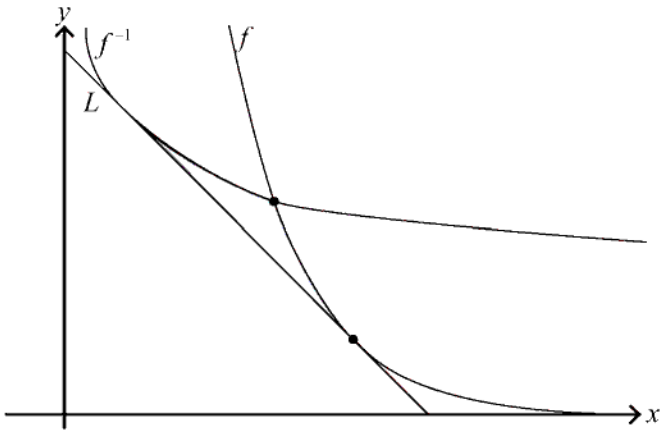
The shaded region A is enclosed by the graph of f and the lines $y = x$ and L.



(d) (i) Find the x-coordinate of the point where L intersects the line $y = x$.

(ii) Hence, find the area of A. [5]

The line L is tangent to the graphs of both f and the inverse function f^{-1}.



(e) Find the shaded area enclosed by the graphs of f and f^{-1} and the line L. [2]

41 – (MathAA/22_HL_Summer_2021_Q11) – *Differentiation*

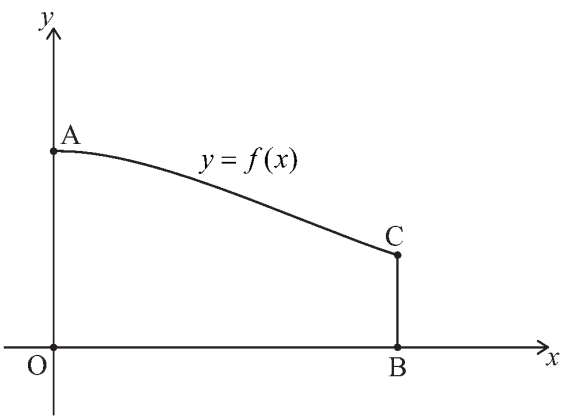
[Maximum mark: 18]

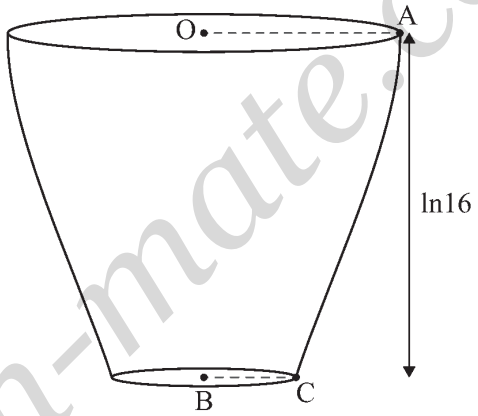
A function f is defined by $f(x)=\dfrac{k\mathrm{e}^{\frac{x}{2}}}{1+\mathrm{e}^{x}}$ where $x \in \mathbb{R}$, $x \geq 0$ and $k \in \mathbb{R}^{+}$.

The region enclosed by the graph of $y=f(x)$, the x-axis, the y-axis and the line $x=\ln 16$ is rotated $360°$ about the x-axis to form a solid of revolution.

(a) Show that the volume of the solid formed is $\dfrac{15k^2\pi}{34}$ cubic units. [6]

Pedro wants to make a small bowl with a volume of $300\,\mathrm{cm}^3$ based on the result from part (a). Pedro's design is shown in the following diagrams.





The vertical height of the bowl, BO, is measured along the x-axis. The radius of the bowl's top is OA and the radius of the bowl's base is BC. All lengths are measured in cm.

(b) Find the value of k that satisfies the requirements of Pedro's design. [2]

(c) Find

(i) OA;

(ii) BC. [4]

For design purposes, Pedro investigates how the cross-sectional radius of the bowl changes.

(d) (i) By sketching the graph of a suitable derivative of f, find where the cross-sectional radius of the bowl is decreasing most rapidly.

(ii) State the cross-sectional radius of the bowl at this point. [6]

42 – (MathAA/22_HL_Summer_2021_Q12) – *Algebra-functions - Roots, Differentiation*

[Maximum mark: 21]

A function f is defined by $f(x) = \arcsin\left(\frac{x^2-1}{x^2+1}\right)$, $x \in \mathbb{R}$.

(a) Show that f is an even function. [1]

(b) By considering limits, show that the graph of $y = f(x)$ has a horizontal asymptote and state its equation. [2]

(c) (i) Show that $f'(x) = \frac{2x}{\sqrt{x^2}\left(x^2+1\right)}$ for $x \in \mathbb{R}$, $x \neq 0$.

(ii) By using the expression for $f'(x)$ and the result $\sqrt{x^2} = |x|$, show that f is decreasing for $x < 0$. [9]

A function g is defined by $g(x) = \arcsin\left(\frac{x^2-1}{x^2+1}\right)$, $x \in \mathbb{R}$, $x \geq 0$.

(d) Find an expression for $g^{-1}(x)$, justifying your answer. [5]

(e) State the domain of g^{-1}. [1]

(f) Sketch the graph of $y = g^{-1}(x)$, clearly indicating any asymptotes with their equations and stating the values of any axes intercepts. [3]

43 – (MathAA/20_HL_Winter_2021_Q8) – *Differentiation*

[Maximum mark: 8]

Consider the curve C given by $y - x - xy\ln(xy)$ where $x > 0$, $y > 0$.

(a) Show that $\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right)\left(1 + \ln(xy)\right) = 1$. [3]

(b) Hence find the equation of the tangent to C at the point where $x = 1$. [5]

44 – (MathAA/21_HL_Summer_2022_Q10) – *Algebra-functions - Roots, Differentiation, Integration*

[Maximum mark: 20]

Consider the function $f(x)=\sqrt{x^2-1}$, where $1 \le x \le 2$.

(a) Sketch the curve $y=f(x)$, clearly indicating the coordinates of the endpoints. [2]

(b) (i) Show that the inverse function of f is given by $f^{-1}(x)=\sqrt{x^2+1}$.

(ii) State the domain and range of f^{-1}. [5]

The curve $y=f(x)$ is rotated 2π about the y-axis to form a solid of revolution that is used to model a water container.

(c) (i) Show that the volume, $V\,\mathrm{m}^3$, of water in the container when it is filled to a height of h metres is given by $V=\pi\left(\frac{1}{3}h^3+h\right)$.

(ii) Hence, determine the maximum volume of the container. [5]

At $t=0$, the container is empty. Water is then added to the container at a constant rate of $0.4\,\mathrm{m}^3\,\mathrm{s}^{-1}$.

(d) Find the time it takes to fill the container to its maximum volume. [2]

(e) Find the rate of change of the height of the water when the container is filled to half its maximum volume. [6]

45 – (MathAA/22_HL_Summer_2022_Q10) – *Trigonometry, Differentiation*

[Maximum mark: 15]

A scientist conducted a nine-week experiment on two plants, A and B, of the same species. He wanted to determine the effect of using a new plant fertilizer. Plant A was given fertilizer regularly, while Plant B was not.

The scientist found that the height of Plant A, h_A cm, at time t weeks can be modelled by the function $h_A(t) = \sin(2t + 6) + 9t + 27$, where $0 \le t \le 9$.

The scientist found that the height of Plant B, h_B cm, at time t weeks can be modelled by the function $h_B(t) = 8t + 32$, where $0 < t < 9$.

(a) Use the scientist's models to find the initial height of

(i) Plant B;

(ii) Plant A correct to three significant figures. [3]

(b) Find the values of t when $h_A(t) = h_B(t)$. [3]

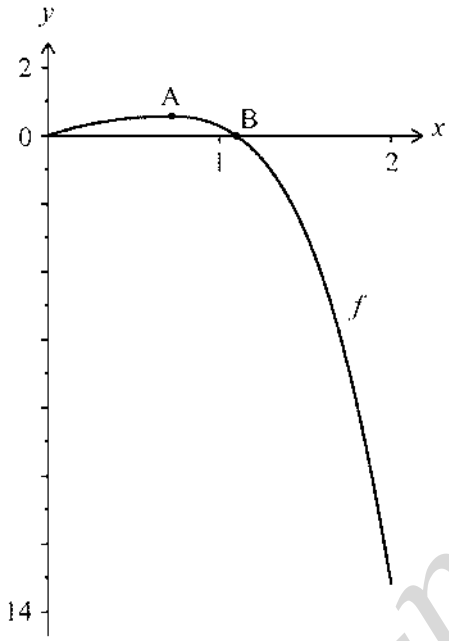
(c) For $t > 6$, prove that Plant A was always taller than Plant B. [3]

(d) For $0 \le t \le 9$, find the total amount of time when the rate of growth of Plant B was greater than the rate of growth of Plant A. [6]

46 – (MathAA/20_HL_Winter_2022_Q2) – *Differentiation, Integration*

[Maximum mark: 6]

The function f is defined as $f(x) = \ln(xe^x + 1) - x^4$, for $0 < x < 2$. The graph of f is shown in the following diagram.



The graph of f has a local maximum at point A. The graph intersects the x-axis at the origin and at point B.

(a) Find the coordinates of A. [2]

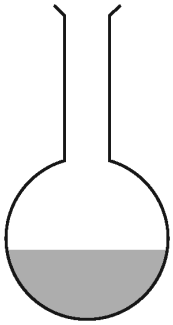
(b) Find the x-coordinate of B. [1]

(c) Find the total area enclosed by the graph of f, the x-axis and the line $x = 2$. [3]

47 – (MathAA/20_HL_Winter_2022_Q8) – *Differentiation*

[Maximum mark: 6]

The following diagram shows liquid in a round-bottomed glass flask, which is made of a sphere and a cylindrical neck.



Initially, the flask is empty. Liquid is poured into the flask at a rate of $2\,\text{cm}^3\text{s}^{-1}$. You may assume that the liquid does not reach the cylindrical neck.

The volume $V\,\text{cm}^3$ and the height $h\,\text{cm}$ of the liquid in the flask satisfy the equation $V = 5\pi h^2 - \frac{1}{3}\pi h^3$.

Find the rate of change of the height of the liquid in the flask at the instant when the volume of the liquid is $200\,\text{cm}^3$.

48 – (MathAA/20_HL_Winter_2022_Q11) – *Algebra-functions - Roots, Differentiation*

[Maximum mark: 15]

The function f is defined by $f(x) = e^{-2x}(3x - 4)$, where $x \in \mathbb{R}$.

(a) Find $f'(x)$. [3]

(b) Hence or otherwise, find the coordinates of the point on the graph of $y = f(x)$ where the tangent is parallel to the line $y = x$. [3]

The region enclosed by the curve $y = f(x)$, the x-axis and the y-axis is rotated through 2π radians about the x-axis to form a solid of revolution.

(c) Find the volume of this solid. [4]

Consider a function g such that $g(0) = 1$ and $g'(0) = 2$.

(d) Find the value of

(i) $(f \circ g)(0)$;

(ii) $(f \circ g)'(0)$. [5]

1 – (MAT/11_HL_Summer_2012_Q6) – *Integration*

The graph below shows the two curves $y=\frac{1}{x}$ and $y=\frac{k}{x}$, where $k>1$.

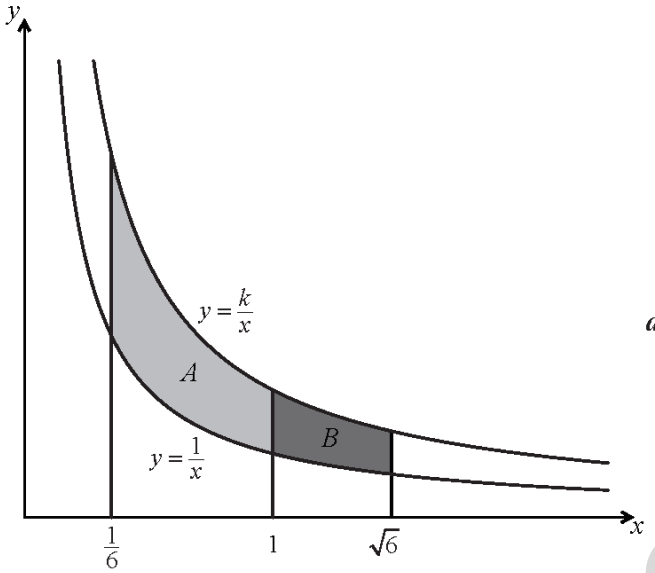


diagram not to scale

(a) Find the area of region A in terms of k. *[3 marks]*

(b) Find the area of region B in terms of k. *[2 marks]*

(c) Find the ratio of the area of region A to the area of region B. *[3 marks]*

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2 – (MAT/12_HL_Summer_2012_Q10) – *Functions - Roots, Graphs, Integration*

The function f is defined on the domain $\left[0, \frac{3\pi}{2}\right]$ by $f(x) = e^{-x}\cos x$.

(a) State the two zeros of f. *[1 mark]*

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(b) Sketch the graph of f. *[1 mark]*

(c) The region bounded by the graph, the x-axis and the y-axis is denoted by A and the region bounded by the graph and the x-axis is denoted by B. Show that the ratio of the area of A to the area of B is

$$\frac{e^{\pi}\left(e^{\frac{\pi}{2}}+1\right)}{e^{\pi}+1}.$$

[7 marks]

3 – (MAT/11_HL_Summer_2012_Q12) – *Differentiation, Integration*

Let $f(x)=\sqrt{\dfrac{x}{1-x}}$, $0<x<1$.

(a) Show that $f'(x)=\dfrac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}$ and deduce that f is an increasing function. *[5 marks]*

(b) Show that the curve $y=f(x)$ has one point of inflexion, and find its coordinates. *[6 marks]*

(c) Use the substitution $x=\sin^2\theta$ to show that $\int f(x)\,\mathrm{d}x=\arcsin\sqrt{x}-\sqrt{x-x^2}+c$. *[11 marks]*

4 – (MAT/12_HL_Summer_2013_Q1) – *Integration*

[Maximum mark: 6]

Find the exact value of $\int_1^2 \left((x-2)^2 + \frac{1}{x} + \sin \pi x \right) \mathrm{d}x$.

5 – (MAT/11_HL_Summer_2013_Q10) – *Integration*

(a) Find all values of x for $0.1 \le x \le 1$ such that $\sin(\pi x^{-1}) = 0$. *[2 marks]*

(b) Find $\int_{\frac{1}{n+1}}^{\frac{1}{n}} \pi x^{-2} \sin(\pi x^{-1})\,\mathrm{d}x$, showing that it takes different integer values when n is even and when n is odd. *[3 marks]*

(c) Evaluate $\int_{0.1}^{1} \left| \pi x^{-2} \sin(\pi x^{-1}) \right| \mathrm{d}x$. *[2 marks]*

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6 – (MAT/11_HL_Summer_2013_Q12) – *Quadratics, Graphs, Integration*

(a) Express $4x^2 - 4x + 5$ in the form $a(x-h)^2 + k$ where $a, h, k \in \mathbb{Q}$. *[2 marks]*

(b) The graph of $y = x^2$ is transformed onto the graph of $y = 4x^2 - 4x + 5$. Describe a sequence of transformations that does this, making the order of transformations clear. *[3 marks]*

The function f is defined by $f(x) = \dfrac{1}{4x^2 - 4x + 5}$.

(c) Sketch the graph of $y = f(x)$. *[2 marks]*

(d) Find the range of f. *[2 marks]*

(e) By using a suitable substitution show that $\int f(x)\,\mathrm{d}x = \dfrac{1}{4}\int \dfrac{1}{u^2+1}\,\mathrm{d}u$. *[3 marks]*

(f) Prove that $\displaystyle\int_1^{3.5} \frac{1}{4x^2 - 4x + 5}\,\mathrm{d}x = \frac{\pi}{16}$. *[7 marks]*

7 – (MAT/10_HL_Winter_2013_Q12) – *Complex Numbers, Trigonometry, Integration*

Consider the complex number $z = \cos\theta + \mathrm{i}\sin\theta$.

(a) Use De Moivre's theorem to show that $z^n + z^{-n} = 2\cos n\theta$, $n \in \mathbb{Z}^+$. *[2]*

(b) Expand $\left(z + z^{-1}\right)^4$. *[1]*

(c) Hence show that $\cos^4\theta = p\cos 4\theta + q\cos 2\theta + r$, where p, q and r are constants to be determined. *[4]*

(d) Show that $\cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$. *[3]*

(e) Hence find the value of $\int_0^{\frac{\pi}{2}} \cos^6\theta \,\mathrm{d}\theta$. *[3]*

The region S is bounded by the curve $y = \sin x\cos^2 x$ and the x-axis between $x = 0$ and $x = \frac{\pi}{2}$.

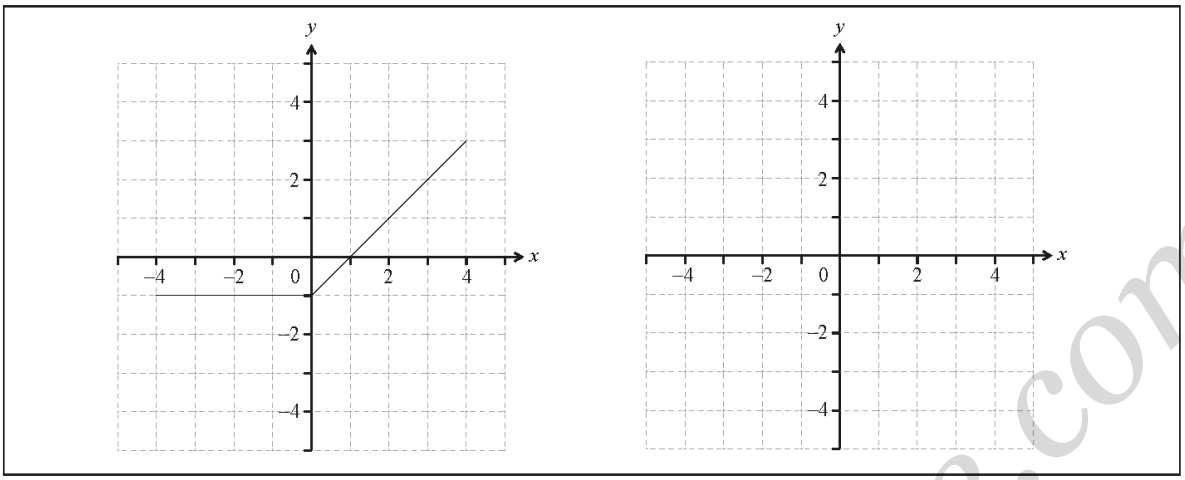
(f) S is rotated through 2π radians about the x-axis. Find the value of the volume generated. *[4]*

(g) (i) Write down an expression for the constant term in the expansion of $\left(z + z^{-1}\right)^{2k}$, $k \in \mathbb{Z}^+$.

(ii) Hence determine an expression for $\int_0^{\frac{\pi}{2}} \cos^{2k}\theta \,\mathrm{d}\theta$ in terms of k. *[3]*

8 – (MAT/11_HL_Summer_2014_Q6) – *Integration, Graphs*

The first set of axes below shows the graph of $y = f(x)$ for $-4 \le x \le 4$.



Let $g(x) = \int_{-4}^{x} f(t)\,\mathrm{d}t$ for $-4 \le x \le 4$.

(a) State the value of x at which $g(x)$ is a minimum. *[1]*

(b) On the second set of axes, sketch the graph of $y = g(x)$. *[5]*

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9 – (MAT/12_HL_Summer_2014_Q10) – *Integration*

[Maximum mark: 7]

Use the substitution $x = a\sec\theta$ to show that $\int_{a\sqrt{2}}^{2a} \frac{\mathrm{d}x}{x^3\sqrt{x^2-a^2}} = \frac{1}{24a^3}\left(3\sqrt{3}+\pi-6\right)$.

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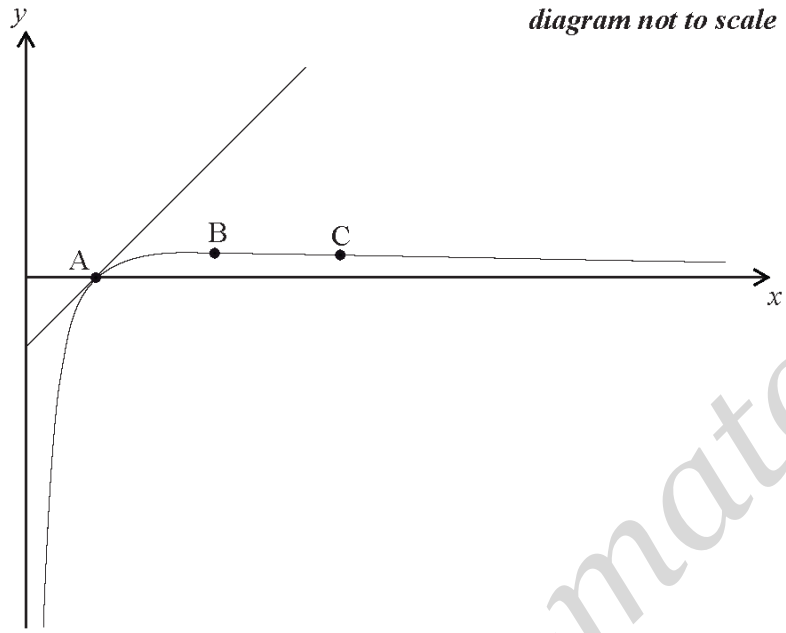
10 – (MAT/11_HL_Summer_2014_Q11) – *Differentiation, Integration*

Consider the function $f(x)=\dfrac{\ln x}{x}$, $x>0$.

The sketch below shows the graph of $y=f(x)$ and its tangent at a point A.



(a) Show that $f'(x)=\dfrac{1-\ln x}{x^2}$. *[2]*

(b) Find the coordinates of B, at which the curve reaches its maximum value. *[3]*

(c) Find the coordinates of C, the point of inflexion on the curve. *[5]*

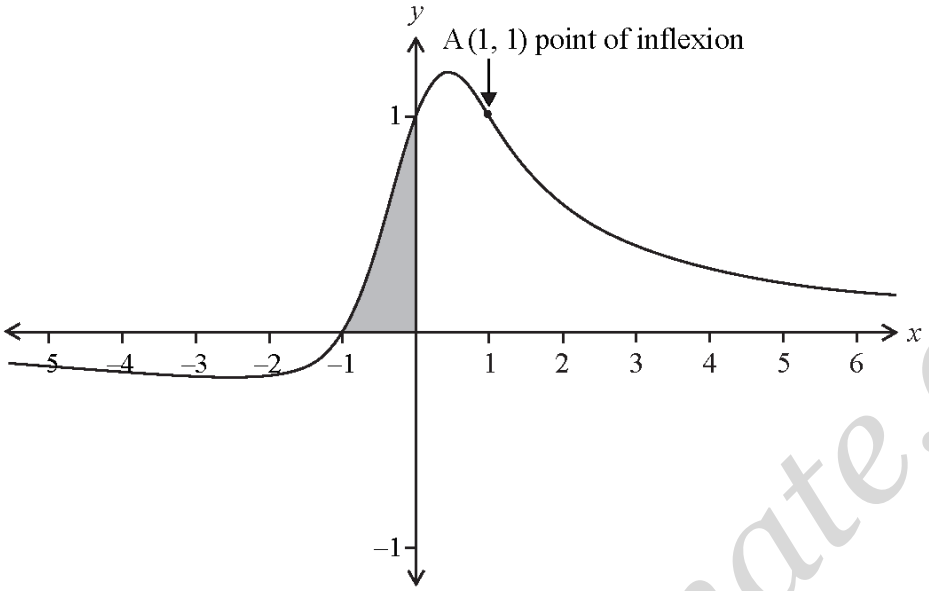
The graph of $y=f(x)$ crosses the x-axis at the point A.

(d) Find the equation of the tangent to the graph of f at the point A. *[4]*

(e) Find the area enclosed by the curve $y=f(x)$, the tangent at A, and the line $x=\mathrm{e}$. *[7]*

11 – (MAT/12_HL_Summer_2014_Q13) – *Differentiation, Integration*

The graph of the function $f(x)=\dfrac{x+1}{x^2+1}$ is shown below.



(a) Find $f'(x)$. *[2]*

(b) Hence find the x-coordinates of the points where the gradient of the graph of f is zero. *[1]*

(c) Find $f''(x)$ expressing your answer in the form $\dfrac{p(x)}{(x^2+1)^3}$, where $p(x)$ is a polynomial of degree 3. *[3]*

The point $(1, 1)$ is a point of inflexion. There are two other points of inflexion.

(d) Find the x-coordinates of the other two points of inflexion. *[4]*

(e) Find the area of the shaded region. Express your answer in the form $\dfrac{\pi}{a}-\ln\sqrt{b}$, where a and b are integers. *[6]*

12 – (MAT/10_HL_Winter_2014_Q6) – *Integration*

[Maximum mark: 6]

By using the substitution $u = 1 + \sqrt{x}$, find $\int \frac{\sqrt{x}}{1+\sqrt{x}}\,\mathrm{d}x$.

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13 – (MAT/10_HL_Winter_2014_Q11) – *Functions - Roots, Differentiation, Integration*

The function f is defined as $f(x) = \mathrm{e}^{3x+1}$, $x \in \mathbb{R}$.

(a) (i) Find $f^{-1}(x)$.

(ii) State the domain of f^{-1}. *[4]*

The function g is defined as $g(x) = \ln x$, $x \in \mathbb{R}^+$.
The graph of $y = g(x)$ and the graph of $y = f^{-1}(x)$ intersect at the point P.

(b) Find the coordinates of P. *[5]*

The graph of $y = g(x)$ intersects the x-axis at the point Q.

(c) Show that the equation of the tangent T to the graph of $y = g(x)$ at the point Q is $y = x - 1$. *[3]*

A region R is bounded by the graphs of $y = g(x)$, the tangent T and the line $x = \mathrm{e}$.

(d) Find the area of the region R. *[5]*

(e) (i) Show that $g(x) \le x - 1$, $x \in \mathbb{R}^+$.

(ii) By replacing x with $\frac{1}{x}$ in part (e)(i), show that $\frac{x-1}{x} \le g(x)$, $x \in \mathbb{R}^+$. *[6]*

14 – (MAT/10_HL_Winter_2014_Q13) – *Complex Numbers, Trigonometry, Integration*

(a) (i) Show that $(1+\mathrm{i}\tan\theta)^n+(1-\mathrm{i}\tan\theta)^n=\dfrac{2\cos n\theta}{\cos^n\theta}$, $\cos\theta\neq 0$.

(ii) Hence verify that $\mathrm{i}\tan\dfrac{3\pi}{8}$ is a root of the equation $(1+z)^4+(1-z)^4=0$, $z\in\mathbb{C}$.

(iii) State another root of the equation $(1+z)^4+(1-z)^4=0$, $z\in\mathbb{C}$. *[10]*

(b) (i) Use the double angle identity $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}$ to show that $\tan\dfrac{\pi}{8}=\sqrt{2}-1$.

(ii) Show that $\cos 4x=8\cos^4 x-8\cos^2 x+1$.

(iii) Hence find the value of $\displaystyle\int_0^{\frac{\pi}{8}}\frac{2\cos 4x}{\cos^2 x}\,\mathrm{d}x$. *[13]*

15 – (MAT/11_HL_Summer_2015_Q3) – *Integration*

(a) Find $\int (1+\tan^2 x)\,\mathrm{d}x$. [2]

(b) Find $\int \sin^2 x\,\mathrm{d}x$. [3]

16 – (MAT/12_HL_Summer_2015_Q5) – *Integration*

[Maximum mark: 6]

Show that $\int_1^2 x^3 \ln x \, dx = 4\ln 2 - \frac{15}{16}$.

17 – (MAT/11_HL_Summer_2015_Q6) – *Functions - Roots, Integration*

A function f is defined by $f(x)=\dfrac{3x-2}{2x-1}$, $x\in\mathbb{R}$, $x\neq\dfrac{1}{2}$.

(a) Find an expression for $f^{-1}(x)$. [4]

(b) Given that $f(x)$ can be written in the form $f(x)=A+\dfrac{B}{2x-1}$, find the values of the constants A and B. [2]

(c) Hence, write down $\int\dfrac{3x-2}{2x-1}\,\mathrm{d}x$. [1]

18 – (MAT/11_HL_Summer_2015_Q8) – *Integration*

[Maximum mark: 7]

By using the substitution $u = e^x + 3$, find $\int \frac{e^x}{e^{2x} + 6e^x + 13} dx$.

19 – (MAT/12_HL_Summer_2015_Q8) – *Integration*

By using the substitution $t = \tan x$, find $\int \frac{\mathrm{d}x}{1+\sin^2 x}$.

Express your answer in the form $m \arctan(n \tan x) + c$, where m, n are constants to be determined.

20 – (MAT/12_HL_Summer_2015_Q11) – *Functions - Roots, Differentiation, Integration*

Consider the functions $f(x) = \tan x,\ 0 \le x < \frac{\pi}{2}$ and $g(x) = \frac{x+1}{x-1},\ x \in \mathbb{R},\ x \ne 1$.

(a) Find an expression for $g \circ f(x)$, stating its domain. [2]

(b) Hence show that $g \circ f(x) = \frac{\sin x + \cos x}{\sin x - \cos x}$. [2]

(c) Let $y = g \circ f(x)$, find an exact value for $\frac{dy}{dx}$ at the point on the graph of $y = g \circ f(x)$ where $x = \frac{\pi}{6}$, expressing your answer in the form $a + b\sqrt{3},\ a,\ b \in \mathbb{Z}$. [6]

(d) Show that the area bounded by the graph of $y = g \circ f(x)$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{6}$ is $\ln\left(1 + \sqrt{3}\right)$. [6]

21 – (MAT/10_HL_Winter_2015_Q2) – *Integration*

[Maximum mark: 4]

Using integration by parts find $\int x \sin x \, dx$.

22 – (MAT/10_HL_Winter_2015_Q5) – *Integration*

[Maximum mark: 4]

Use the substitution $u = \ln x$ to find the value of $\int_{e}^{e^2} \frac{dx}{x \ln x}$.

23 – (MAT/10_HL_Winter_2015_Q12) – *Differentiation, Integration*

Consider the function defined by $f(x) = x\sqrt{1-x^2}$ on the domain $-1 \le x \le 1$.

(a) Show that f is an odd function. [2]

(b) Find $f'(x)$. [3]

(c) Hence find the x-coordinates of any local maximum or minimum points. [3]

(d) Find the range of f. [3]

(e) Sketch the graph of $y = f(x)$ indicating clearly the coordinates of the x-intercepts and any local maximum or minimum points. [3]

(f) Find the area of the region enclosed by the graph of $y = f(x)$ and the x-axis for $x \ge 0$. [4]

(g) Show that $\int_{-1}^{1} \left| x\sqrt{1-x^2} \right| \mathrm{d}x > \left| \int_{-1}^{1} x\sqrt{1-x^2}\,\mathrm{d}x \right|$. [2]

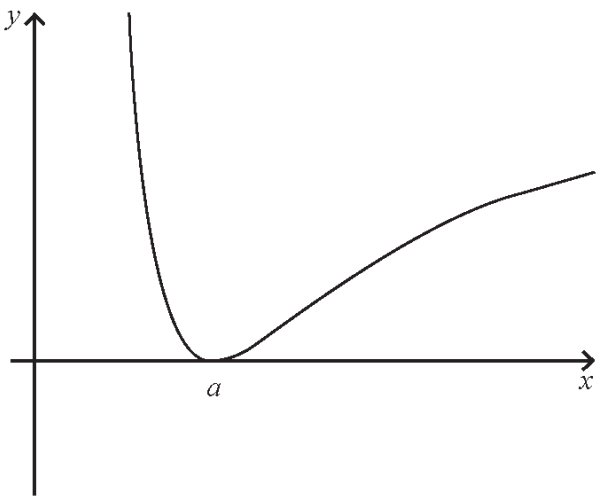
24 – (MAT/12_HL_Summer_2016_Q3) – *Trigonometry, Integration*

(a) Show that $\cot\alpha = \tan\left(\frac{\pi}{2} - \alpha\right)$ for $0 < \alpha < \frac{\pi}{2}$. [1]

(b) Hence find $\int_{\tan\alpha}^{\cot\alpha} \frac{1}{1+x^2}\,dx$, $0 < \alpha < \frac{\pi}{2}$. [4]

25 – (MAT/11_HL_Summer_2016_Q13) – *Integration*

The following diagram shows the graph of $y = \frac{(\ln x)^2}{x}$, $x > 0$.



(a) Given that the curve passes through the point $(a, 0)$, state the value of a. [1]

The region R is enclosed by the curve, the x-axis and the line $x = \mathrm{e}$.

(b) Use the substitution $u = \ln x$ to find the area of the region R. [5]

Let $I_n = \int_1^{\mathrm{e}} \frac{(\ln x)^n}{x^2} \mathrm{d}x$, $n \in \mathbb{N}$.

(c) (i) Find the value of I_0.

(ii) Prove that $I_n = -\frac{1}{\mathrm{e}} + nI_{n-1}$, $n \in \mathbb{Z}^+$.

(iii) Hence find the value of I_1. [7]

(d) Find the volume of the solid formed when the region R is rotated through 2π about the x-axis. [5]

26 – (MAT/12_HL_Summer_2017_Q6) – *Integration*

(a) Using the substitution $x = \tan\theta$ show that $\int_0^1 \frac{1}{\left(x^2+1\right)^2}\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \cos^2\theta\,\mathrm{d}\theta$. [4]

(b) Hence find the value of $\int_0^1 \frac{1}{\left(x^2+1\right)^2}\,\mathrm{d}x$. [3]

27 – (MAT/11_HL_Summer_2017_Q9) – *Integration*

[Maximum mark: 5]

Find $\int \arcsin x \, \mathrm{d}x$.

28 – (MAT/12_HL_Summer_2017_Q9) – *Functions - Roots, Differentiation, Integration*

Consider the function f defined by $f(x) = x^2 - a^2,\ x \in \mathbb{R}$ where a is a positive constant.

(a) Showing any x and y intercepts, any maximum or minimum points and any asymptotes, sketch the following curves on separate axes.

(i) $y = f(x)$;

(ii) $y = \dfrac{1}{f(x)}$;

(iii) $y = \left|\dfrac{1}{f(x)}\right|$. [8]

(b) Find $\int f(x)\cos x\,\mathrm{d}x$. [5]

The function g is defined by $g(x) = x\sqrt{f(x)}$ for $|x| > a$.

(c) By finding $g'(x)$ explain why g is an increasing function. [4]

29 – (MAT/11_HL_Summer_2017_Q11) – *Functions - Roots, Graphs, Integration*

(a) (i) Express x^2+3x+2 in the form $(x+h)^2+k$.

(ii) Factorize x^2+3x+2. [2]

Consider the function $f(x)=\dfrac{1}{x^2+3x+2}$, $x\in\mathbb{R}$, $x\neq-2$, $x\neq-1$.

(b) Sketch the graph of $f(x)$, indicating on it the equations of the asymptotes, the coordinates of the y-intercept and the local maximum. [5]

(c) Show that $\dfrac{1}{x+1}-\dfrac{1}{x+2}=\dfrac{1}{x^2+3x+2}$. [1]

(d) Hence find the value of p if $\int_0^1 f(x)\,\mathrm{d}x=\ln(p)$. [4]

(e) Sketch the graph of $y=f(|x|)$. [2]

(f) Determine the area of the region enclosed between the graph of $y=f(|x|)$, the x-axis and the lines with equations $x=-1$ and $x=1$. [3]

30 – (MAT/11_HL_Summer_2018_Q4) – *Integration*

Given that $\int_{-2}^{2} f(x)\mathrm{d}x = 10$ and $\int_{0}^{2} f(x)\mathrm{d}x = 12$, find

(a) $\int_{-2}^{0} \left(f(x)+2\right)\mathrm{d}x$; [4]

(b) $\int_{-2}^{0} f(x+2)\,\mathrm{d}x$. [2]

31 – (MAT/12_HL_Summer_2018_Q6) – *Differentiation, Integration*

Consider the functions f, g, defined for $x \in \mathbb{R}$, given by $f(x) = \mathrm{e}^{-x}\sin x$ and $g(x) = \mathrm{e}^{-x}\cos x$.

(a) Find

(i) $f'(x)$;

(ii) $g'(x)$. [3]

(b) Hence, or otherwise, find $\int_0^{\pi} \mathrm{e}^{-x}\sin x \,\mathrm{d}x$. [4]

32 – (MAT/11_HL_Summer_2018_Q7) – *Differentiation, Integration*

Let $y = \arccos\left(\frac{x}{2}\right)$.

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. [2]

(b) Find $\int_0^1 \arccos\left(\frac{x}{2}\right)\mathrm{d}x$. [7]

33 – (MAT/12_HL_Summer_2018_Q8) – *Integration*

(a) Use the substitution $u = x^{\frac{1}{2}}$ to find $\int \frac{\mathrm{d}x}{x^{\frac{3}{2}} + x^{\frac{1}{2}}}$. [4]

(b) Hence find the value of $\frac{1}{2}\int_{1}^{9} \frac{\mathrm{d}x}{x^{\frac{3}{2}} + x^{\frac{1}{2}}}$, expressing your answer in the form $\arctan q$, where $q \in \mathbb{Q}$. [3]

34 – (MAT/12_HL_Summer_2018_Q11) – *Exponentials - Logarithms, Integration*

(a) Show that $\log_{r^2} x = \frac{1}{2}\log_r x$ where $r, x \in \mathbb{R}^+$. [2]

It is given that $\log_2 y + \log_4 x + \log_4 2x = 0$.

(b) Express y in terms of x. Give your answer in the form $y = px^q$, where p, q are constants. [5]

The region R, is bounded by the graph of the function found in part (b), the x-axis, and the lines $x = 1$ and $x = \alpha$ where $\alpha > 1$. The area of R is $\sqrt{2}$.

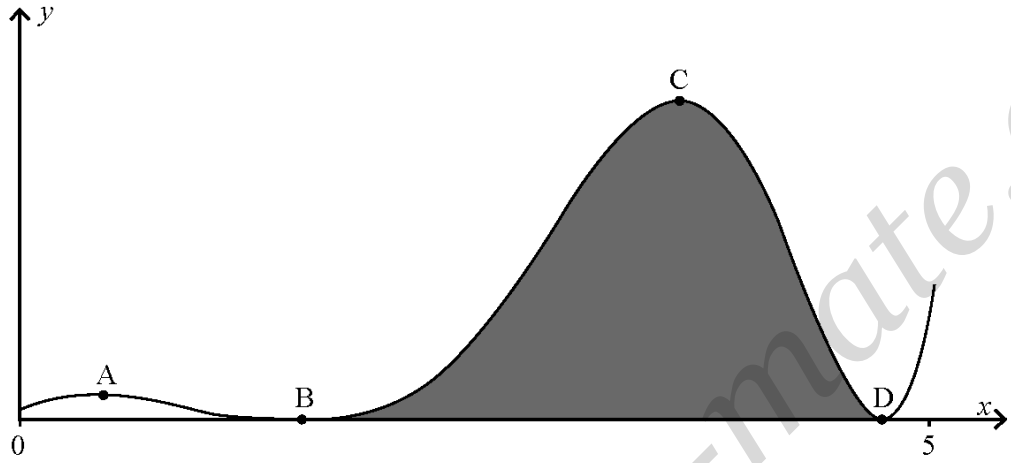
(c) Find the value of α. [5]

35 – (MAT/10_HL_Winter_2018_Q10) – *Integration*

(a) Use integration by parts to show that $\int e^x \cos 2x \, dx = \frac{2e^x}{5}\sin 2x + \frac{e^x}{5}\cos 2x + c,\ c \in \mathbb{R}$. [5]

(b) Hence, show that $\int e^x \cos^2 x \, dx = \frac{e^x}{5}\sin 2x + \frac{e^x}{10}\cos 2x + \frac{e^x}{2} + c,\ c \in \mathbb{R}$. [3]

The function f is defined by $f(x) = e^x \cos^2 x$, where $0 \le x \le 5$. The curve $y = f(x)$ is shown on the following graph which has local maximum points at A and C and touches the x-axis at B and D.



(c) Find the x-coordinates of A and of C, giving your answers in the form $a + \arctan b$, where $a, b \in \mathbb{R}$. [6]

(d) Find the area enclosed by the curve and the x-axis between B and D, as shaded on the diagram. [5]

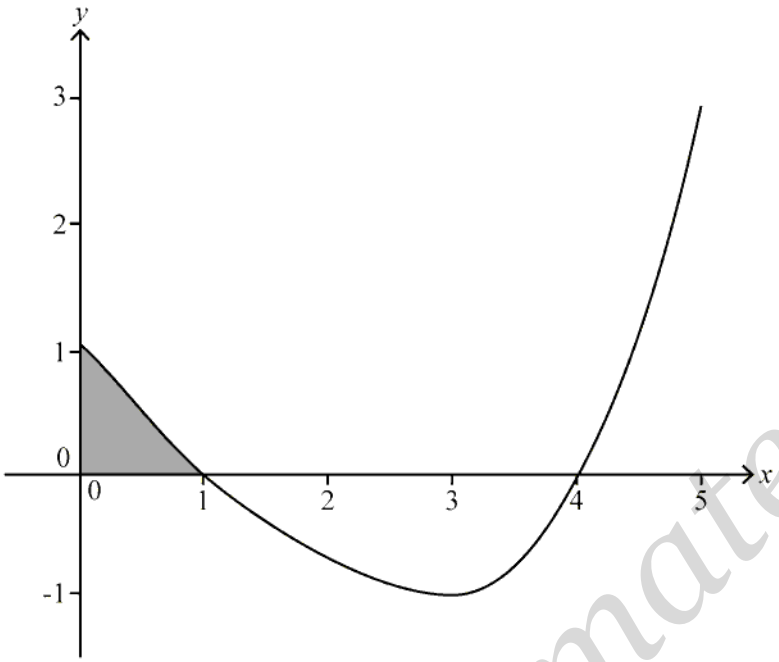
36 – (MAT/12_HL_Summer_2019_Q4) – *Integration*

[Maximum mark: 5]

Using the substitution $u = \sin x$, find $\int \frac{\cos^3 x \, dx}{\sqrt{\sin x}}$.

37 – (MAT/11_HL_Summer_2019_Q8) – *Integration*

The graph of $y = f'(x)$, $0 \le x \le 5$ is shown in the following diagram. The curve intercepts the x-axis at $(1, 0)$ and $(4, 0)$ and has a local minimum at $(3, -1)$.



(a) Write down the x-coordinate of the point of inflexion on the graph of $y = f(x)$. [1]

The shaded area enclosed by the curve $y = f'(x)$, the x-axis and the y-axis is 0.5.
Given that $f(0) = 3$,

(b) find the value of $f(1)$. [3]

The area enclosed by the curve $y = f'(x)$ and the x-axis between $x = 1$ and $x = 4$ is 2.5.

(c) Find the value of $f(4)$. [2]

(d) Sketch the curve $y = f(x)$, $0 \le x \le 5$ indicating clearly the coordinates of the maximum and minimum points and any intercepts with the coordinate axes. [3]

38 – (MAT/11_HL_Summer_2019_Q9) – *Trigonometry, Integration*

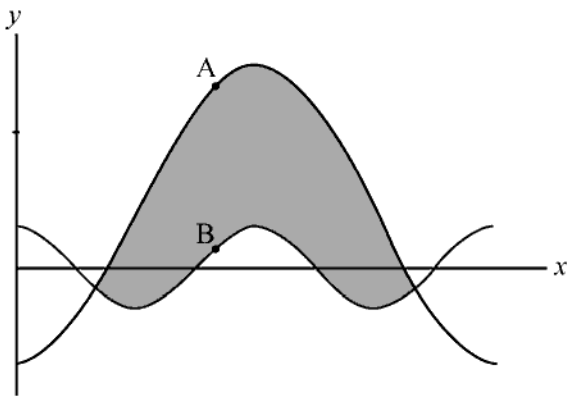
(a) Show that $(\sin x + \cos x)^2 = 1 + \sin 2x$. [2]

(b) Show that $\sec 2x + \tan 2x = \dfrac{\cos x + \sin x}{\cos x - \sin x}$. [4]

(c) Hence or otherwise find $\int_0^{\frac{\pi}{6}} (\sec 2x + \tan 2x)\,\mathrm{d}x$ in the form $\ln\left(a + \sqrt{b}\right)$ where $a, b \in \mathbb{Z}$. [9]

39 – (MAT/12_HL_Summer_2019_Q9) – *Differentiation, Integration*

Consider the functions f and g defined on the domain $0 < x < 2\pi$ by
$f(x) = 3\cos 2x$ and $g(x) = 4 - 11\cos x$.
The following diagram shows the graphs of $y = f(x)$ and $y = g(x)$.



(a) Find the x-coordinates of the points of intersection of the two graphs. [6]

(b) Find the exact area of the shaded region, giving your answer in the form $p\pi + q\sqrt{3}$, where $p, q \in \mathbb{Q}$. [5]

At the points A and B on the diagram, the gradients of the two graphs are equal.

(c) Determine the y-coordinate of A on the graph of g. [6]

40 – (MAT/12_HL_Summer_2019_Q10) – *Integration, Probability*

The random variable X has probability density function f given by

$$f(x)=\begin{cases} k\left(\pi-\arcsin x\right) & 0\le x\le 1 \\ 0 & \text{otherwise} \end{cases}$$, where k is a positive constant.

(a) State the mode of X. [1]

(b) (i) Find $\int \arcsin x \, \mathrm{d}x$.

(ii) Hence show that $k=\dfrac{2}{2+\pi}$. [6]

(c) Given that $y=\left(\dfrac{x^2}{2}\right)\arcsin x-\left(\dfrac{1}{4}\right)\arcsin x+\left(\dfrac{x}{4}\right)\sqrt{1-x^2}$, show that

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=x\arcsin x$;

(ii) $\mathrm{E}\left(X\right)=\dfrac{3\pi}{4\left(\pi+2\right)}$. [9]

41 – (MAT/10_HL_Winter_2019_Q2) – *Integration*

[Maximum mark: 6]

Given that $\int_{0}^{\ln k} e^{2x}\,dx = 12$, find the value of k.

42 – (MAT/10_HL_Winter_2019_Q7) – *Integration*

[Maximum mark: 7]

(a) Write $2x - x^2$ in the form $a(x-h)^2 + k$, where $a, h, k \in \mathbb{R}$. [2]

(b) Hence, find the value of $\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{\sqrt{2x - x^2}}\,dx$. [5]

43 – (MAT/10_HL_Winter_2019_Q10) – *Differentiation, Integration*

[Maximum mark: 19]

Consider $f(x) = \frac{2x-4}{x^2-1}$, $-1 < x < 1$.

(a) (i) Find $f'(x)$.

(ii) Show that, if $f'(x) = 0$, then $x = 2 - \sqrt{3}$. [5]

(b) For the graph of $y = f(x)$,

(i) find the coordinates of the y-intercept;

(ii) show that there are no x-intercepts;

(iii) sketch the graph, showing clearly any asymptotic behaviour. [5]

(c) Show that $\frac{3}{x+1} - \frac{1}{x-1} = \frac{2x-4}{x^2-1}$. [2]

(d) The area enclosed by the graph of $y = f(x)$ and the line $y = 4$ can be expressed as $\ln v$. Find the value of v. [7]

44 – (MAT/10_HL_Winter_2020_Q9) – *Integration*

[Maximum mark: 8]

By using the substitution $x = \tan u$, find the value of $\int_0^1 \frac{x^2}{\left(1+x^2\right)^3}\,\mathrm{d}x$.

45 – (MAT/10_HL_Winter_2020_Q12) – *Graphs, Integration*

Consider the function defined by $f(x) = \dfrac{kx-5}{x-k}$, where $x \in \mathbb{R} \setminus \{k\}$ and $k^2 \neq 5$.

(a) State the equation of the vertical asymptote on the graph of $y = f(x)$. [1]

(b) State the equation of the horizontal asymptote on the graph of $y = f(x)$. [1]

(c) Use an algebraic method to determine whether f is a self-inverse function. [4]

Consider the case where $k = 3$.

(d) Sketch the graph of $y = f(x)$, stating clearly the equations of any asymptotes and the coordinates of any points of intersections with the coordinate axes. [3]

(e) The region bounded by the x-axis, the curve $y = f(x)$, and the lines $x = 5$ and $x = 7$ is rotated through 2π about the x-axis. Find the volume of the solid generated, giving your answer in the form $\pi(a + b\ln 2)$, where $a, b \in \mathbb{Z}$. [6]

46 – (MathAA/12_HL_Summer_2021_Q9) – *Integration*

[Maximum mark: 7]

By using the substitution $u = \sin x$, find $\int \frac{\sin x \cos x}{\sin^2 x - \sin x - 2} \mathrm{d}x$.

47 – (MathAA/10_HL_Winter_2021_Q1) – *Integration*

[Maximum mark: 4]

Given that $\frac{dy}{dx} = \cos\left(x - \frac{\pi}{4}\right)$ and $y = 2$ when $x = \frac{3\pi}{4}$, find y in terms of x.

48 – (MathAA/11_HL_Summer_2022_Q1) – *Integration*

[Maximum mark: 5]

Find the value of $\int_1^9 \left(\frac{3\sqrt{x}-5}{\sqrt{x}} \right) \mathrm{d}x$.

49 – (MathAA/11_HL_Summer_2022_Q12) – *Sequences - Series, Differentiation, Integration*

[Maximum mark: 21]

The function f is defined by $f(x) = e^x \sin x$, where $x \in \mathbb{R}$.

(a) Find the Maclaurin series for $f(x)$ up to and including the x^3 term. [4]

(b) Hence, find an approximate value for $\int_0^1 e^{x^2} \sin(x^2)\,dx$. [4]

The function g is defined by $g(x) = e^x \cos x$, where $x \in \mathbb{R}$.

(c) (i) Show that $g(x)$ satisfies the equation $g''(x) = 2(g'(x) - g(x))$.

(ii) Hence, deduce that $g^{(4)}(x) = 2(g'''(x) - g''(x))$. [5]

(d) Using the result from part (c), find the Maclaurin series for $g(x)$ up to and including the x^4 term. [5]

(e) Hence, or otherwise, determine the value of $\lim_{x \to 0} \frac{e^x \cos x - 1 - x}{x^3}$. [3]

50 – (MathAA/12_HL_Summer_2022_Q7) – *Integration*

[Maximum mark: 6]

By using the substitution $u = \sec x$ or otherwise, find an expression for $\int_0^{\frac{\pi}{3}} \sec^n x \tan x \, \mathrm{d}x$ in terms of n, where n is a non-zero real number.

51 - (MathAA/10_HL_Winter_2022_Q4) - *Integration*

[Maximum mark: 5]

The derivative of the function f is given by $f'(x) = \frac{6x}{x^2+1}$.

The graph of $y = f(x)$ passes through the point $(1, 5)$. Find an expression for $f(x)$.

1 – (MAT/11_HL_Summer_2009_Q13) – *Complex Numbers, Differential Equations*

Part A *[Maximum mark: 9]*

If z is a non-zero complex number, we define $L(z)$ by the equation

$$L(z) = \ln|z| + \mathrm{i}\arg(z),\ 0 \le \arg(z) < 2\pi.$$

(a) Show that when z is a positive real number, $L(z) = \ln z$. *[2 marks]*

(b) Use the equation to calculate

(i) $L(-1)$;

(ii) $L(1-\mathrm{i})$;

(iii) $L(-1+\mathrm{i})$. *[5 marks]*

(c) Hence show that the property $L(z_1 z_2) = L(z_1) + L(z_2)$ does not hold for all values of z_1 and z_2. *[2 marks]*

Part B *[Maximum mark: 14]*

Let f be a function with domain $\mathbb{R}$ that satisfies the conditions,

$$f(x+y) = f(x)f(y), \text{ for all } x \text{ and } y \text{ and } f(0) \ne 0.$$

(a) Show that $f(0) = 1$. *[3 marks]*

(b) Prove that $f(x) \ne 0$, for all $x \in \mathbb{R}$. *[3 marks]*

(c) Assuming that $f'(x)$ exists for all $x \in \mathbb{R}$, use the definition of derivative to show that $f(x)$ satisfies the differential equation $f'(x) = k\,f(x)$, where $k = f'(0)$. *[4 marks]*

(d) Solve the differential equation to find an expression for $f(x)$. *[4 marks]*

2 – (MAT/10_HL_Winter_2009_Q8) – *Differential Equations*

A certain population can be modelled by the differential equation $\frac{dy}{dt} = k y \cos kt$, where y is the population at time t hours and k is a positive constant.

(a) Given that $y = y_0$ when $t = 0$, express y in terms of k, t and y_0. *[5 marks]*

(b) Find the ratio of the minimum size of the population to the maximum size of the population. *[2 marks]*

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3 – (MAT/12_HL_Summer_2010_Q14) – *Differential Equations, Trigonometry*

[Maximum mark: 15]

Throughout this question x satisfies $0 \le x < \frac{\pi}{2}$.

(a) Solve the differential equation $\sec^2 x \frac{dy}{dx} = -y^2$, where $y = 1$ when $x = 0$.

Give your answer in the form $y = f(x)$. *[7 marks]*

(b) (i) Prove that $1 \le \sec x \le 1 + \tan x$.

(ii) Deduce that $\frac{\pi}{4} \le \int_0^{\frac{\pi}{4}} \sec x \, dx \le \frac{\pi}{4} + \frac{1}{2}\ln 2$. *[8 marks]*

4 – (MAT/10_HL_Winter_2011_Q13) – *Differential Equations*

The curve C with equation $y = f(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{\ln y}(x+2),\ y > 1,$$

and $y = \mathrm{e}$ when $x = 2$.

(a) Find the equation of the tangent to C at the point $(2, \mathrm{e})$. *[3 marks]*

(b) Find $f(x)$. *[11 marks]*

(c) Determine the largest possible domain of f. *[6 marks]*

(d) Show that the equation $f(x) = f'(x)$ has no solution. *[4 marks]*

5 – (MathAA/10_HL_Winter_2021_Q8) – *Differential Equations*

[Maximum mark: 7]

Solve the differential equation $\frac{dy}{dx} = \frac{\ln 2x}{x^2} - \frac{2y}{x}$, $x > 0$, given that $y = 4$ at $x = \frac{1}{2}$.

Give your answer in the form $y = f(x)$.

6 – (MathAA/10_HL_Winter_2022_Q9) – *Differential Equations*

[Maximum mark: 10]

Consider the homogeneous differential equation $\frac{dy}{dx} = \frac{y^2 - 2x^2}{xy}$, where $x, y \neq 0$.

It is given that $y = 2$ when $x = 1$.

(a) By using the substitution $y = vx$, solve the differential equation. Give your answer in the form $y^2 = f(x)$. [8]

The points of zero gradient on the curve $y^2 = f(x)$ lie on two straight lines of the form $y = mx$ where $m \in \mathbb{Z}$.

(b) Find the values of m. [2]

1 – (MAT/11_HL_Summer_2014_Q8) – *Kinematics*

[Maximum mark: 6]

A body is moving in a straight line. When it is s metres from a fixed point O on the line its velocity, v, is given by $v = -\frac{1}{s^2}$, $s > 0$.

Find the acceleration of the body when it is 50 cm from O.

2 – (MAT/12_HL_Summer_2017_Q4) – *Kinematics*

A particle moves along a straight line. Its displacement, s metres, at time t seconds is given by $s = t + \cos 2t$, $t \geq 0$. The first two times when the particle is at rest are denoted by t_1 and t_2, where $t_1 < t_2$.

(a) Find t_1 and t_2. [5]

(b) Find the displacement of the particle when $t = t_1$. [2]

3 – (MAT/10_HL_Winter_2017_Q5) – *Kinematics*

[Maximum mark: 5]

A particle moves in a straight line such that at time t seconds $(t \geq 0)$, its velocity v, in ms^{-1}, is given by $v = 10te^{-2t}$. Find the exact distance travelled by the particle in the first half-second.

4 – (MathAA/12_HL_Summer_2021_Q11) – *Kinematics, Relative Velocity*

[Maximum mark: 20]

The acceleration, $a\,\mathrm{ms}^{-2}$, of a particle moving in a horizontal line at time t seconds, $t \geq 0$, is given by $a = -(1+v)$ where $v\,\mathrm{ms}^{-1}$ is the particle's velocity and $v > -1$.

At $t = 0$, the particle is at a fixed origin O and has initial velocity $v_0\,\mathrm{ms}^{-1}$.

(a) By solving an appropriate differential equation, show that the particle's velocity at time t is given by $v(t) = (1 + v_0)\mathrm{e}^{-t} - 1$. [6]

(b) Initially at O, the particle moves in the positive direction until it reaches its maximum displacement from O. The particle then returns to O.

Let s metres represent the particle's displacement from O and $s_{\max}$ its maximum displacement from O.

(i) Show that the time T taken for the particle to reach $s_{\max}$ satisfies the equation $\mathrm{e}^{T} = 1 + v_0$.

(ii) By solving an appropriate differential equation and using the result from part (b) (i), find an expression for $s_{\max}$ in terms of v_0. [7]

Let $v(T-k)$ represent the particle's velocity k seconds before it reaches $s_{\max}$, where

$$v(T-k) = (1 + v_0)\mathrm{e}^{-(T-k)} - 1.$$

(c) By using the result to part (b) (i), show that $v(T-k) = \mathrm{e}^{k} - 1$. [2]

Similarly, let $v(T+k)$ represent the particle's velocity k seconds after it reaches $s_{\max}$.

(d) Deduce a similar expression for $v(T+k)$ in terms of k. [2]

(e) Hence, show that $v(T-k) + v(T+k) \geq 0$. [3]

1 - (MathAA/10_HL_Winter_2022_Q2) - *Geometry*

[Maximum mark: 7]

Consider a circle with a diameter AB, where A has coordinates $(1, 4, 0)$ and B has coordinates $(-3, 2, -4)$.

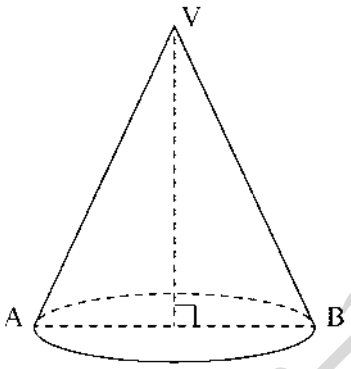
(a) Find

(i) the coordinates of the centre of the circle;

(ii) the radius of the circle. [4]

The circle forms the base of a right cone whose vertex V has coordinates $(-1, -1, 0)$.

diagram not to scale

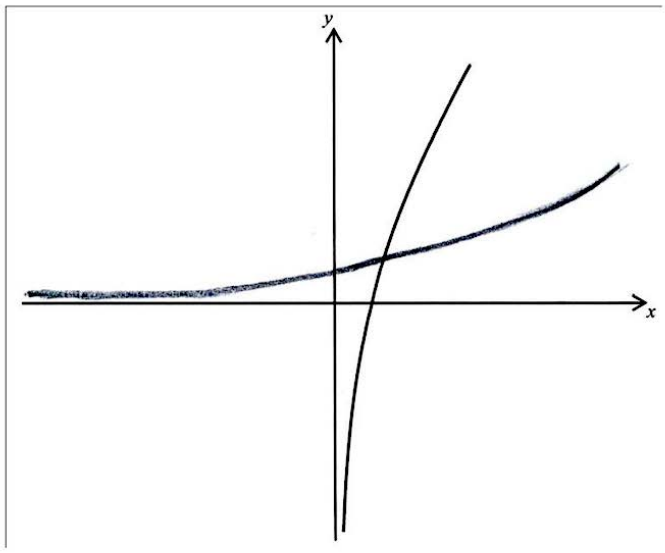


(b) Find the exact volume of the cone. [3]

ANSWERS

1 – (MAT/11_HL_Summer_2012_Q4) – *Graphs, Functions - Roots*

(a)



Note: Award ***A1*** for correct asymptote with correct behaviour and ***A1*** for shape.

(b) intersect on $y = x$

$x + \ln x = x \Rightarrow \ln x = 0$

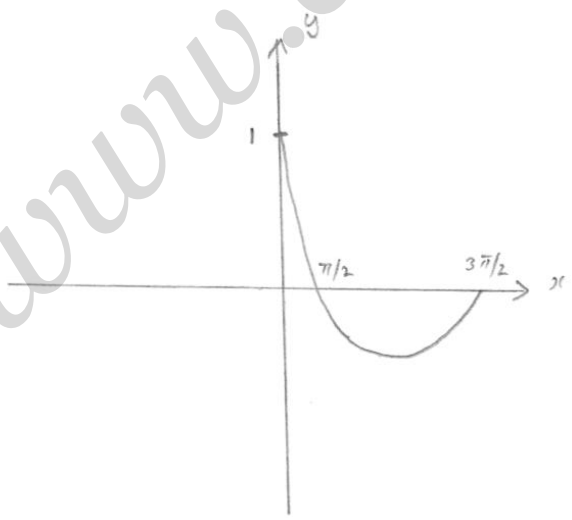
intersect at $(1, 1)$

2 – (MAT/12_HL_Summer_2012_Q10) – *Functions - Roots, Graphs, Integration*

(a) $e^{-x}\cos x = 0$

$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$

(b)



(c) attempt at integration by parts

EITHER

$$I=\int e^{-x}\cos x\,dx=-e^{-x}\cos x\,dx-\int e^{-x}\sin x\,dx$$

$$\Rightarrow I=-e^{-x}\cos x\,dx-\left[-e^{-x}\sin x+\int e^{-x}\cos x\,dx\right]$$

$$\Rightarrow I=\frac{e^{-x}}{2}(\sin x-\cos x)+C$$

Note: Do not penalize absence of C.

OR

$$I=\int e^{-x}\cos x\,dx=e^{-x}\sin x+\int e^{-x}\sin x\,dx$$

$$\Rightarrow I=e^{-x}\sin x-e^{-x}\cos x-\int e^{-x}\cos x\,dx$$

$$\Rightarrow I=\frac{e^{-x}}{2}(\sin x-\cos x)+C$$

Note: Do not penalize absence of C.

THEN

$$\int_0^{\frac{\pi}{2}} e^{-x}\cos x\,dx=\left[\frac{e^{-x}}{2}(\sin x-\cos x)\right]_0^{\frac{\pi}{2}}=\frac{e^{-\frac{\pi}{2}}}{2}+\frac{1}{2}$$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{-x}\cos x\,dx=\left[\frac{e^{-x}}{2}(\sin x-\cos x)\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}=-\frac{e^{-\frac{3\pi}{2}}}{2}-\frac{e^{-\frac{\pi}{2}}}{2}$$

ratio of $A:B$ is $\dfrac{\frac{e^{-\frac{\pi}{2}}}{2}+\frac{1}{2}}{\frac{e^{-\frac{3\pi}{2}}}{2}+\frac{e^{-\frac{\pi}{2}}}{2}}$

$$=\frac{e^{\frac{3\pi}{2}}\left(e^{-\frac{\pi}{2}}+1\right)}{e^{\frac{3\pi}{2}}\left(e^{-\frac{3\pi}{2}}+e^{-\frac{\pi}{2}}\right)}$$

$$=\frac{e^{\pi}\left(e^{\frac{\pi}{2}}+1\right)}{e^{\pi}+1}$$

3 – (MAT/12_HL_Summer_2012_Q11) – *Functions - Roots, Probability*

(a) $f(x) \geq \frac{1}{25}$

$g(x) \in \mathbb{R}, g(x) \geq 0$

(b) $f \circ g(x) = \dfrac{2\left(\frac{3x-4}{10}\right)^2 + 3}{75}$

$= \dfrac{\frac{2(9x^2 - 24x + 16)}{100} + 3}{75}$

$= \dfrac{9x^2 - 24x + 166}{3750}$

(c) (i) **METHOD 1**

$y = \dfrac{2x^2 + 3}{75}$

$x^2 = \dfrac{75y - 3}{2}$

$x = \sqrt{\dfrac{75y - 3}{2}}$

$\Rightarrow f^{-1}(x) = \sqrt{\dfrac{75x - 3}{2}}$

Note:	Accept ± in line 3 for the ***(A1)*** but not in line 4 for the ***A1***. Award the ***A1*** only if written in the form $f^{-1}(x) =$.

METHOD 2

$y = \dfrac{2x^2 + 3}{75}$

$x = \dfrac{2y^2 + 3}{75}$

$y = \sqrt{\dfrac{75x - 3}{2}}$

$\Rightarrow f^{-1}(x) = \sqrt{\dfrac{75x - 3}{2}}$

Note:	Accept ± in line 3 for the ***(A1)*** but not in line 4 for the ***A1***. Award the ***A1*** only if written in the form $f^{-1}(x) =$.

(ii) domain: $x \geq \frac{1}{25}$; range: $f^{-1}(x) \geq 0$

(d) probabilities from $f(x)$:

X	0	1	2	3	4
$P(X=x)$	$\frac{3}{75}$	$\frac{5}{75}$	$\frac{11}{75}$	$\frac{21}{75}$	$\frac{35}{75}$

Note: Award ***A1*** for one error, ***A0*** otherwise.

probabilities from $g(x)$:

X	0	1	2	3	4
$P(X=x)$	$\frac{4}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{5}{10}$	$\frac{8}{10}$

Note: Award ***A1*** for one error, ***A0*** otherwise.

only in the case of $f(x)$ does $\sum P(X=x)=1$, hence only $f(x)$ can be used as a probability mass function

(e)

$$E(x)=\sum x\cdot \mathrm{P}(X=x)$$

$$=\frac{5}{75}+\frac{22}{75}+\frac{63}{75}+\frac{140}{75}=\frac{230}{75}\left(=\frac{46}{15}\right)$$

4 – (MAT/10_HL_Winter_2012_Q3) – *Functions - Roots, Graphs*

(a) **METHOD 1**

$f(x)=(x+1)(x-1)(x-2)$
$=x^3-2x^2-x+2$
$a=-2$, $b=-1$ and $c=2$

METHOD 2

from the graph or using $f(0)=2$
$c=2$
setting up linear equations using $f(1)=0$ and $f(-1)=0$ (or $f(2)=0$)
obtain $a=-2$, $b=-1$

(b) (i) $(1, 0)$, $(3, 0)$ and $(4, 0)$

(ii) $g(0)$ occurs at $3f(-2)$
$=-36$

5 - (MAT/10_HL_Winter_2012_Q12) - *Functions - Roots, Mathematical Induction*

(a) $(f \circ f)(x) = f\left(\frac{x}{2-x}\right) = \frac{\frac{x}{2-x}}{2-\frac{x}{2-x}}$

$(f \circ f)(x) = \frac{x}{4-3x}$

(b) $P(n): \underbrace{(f \circ f \circ \ldots \circ f)}_{n \text{ times}}(x) = F_n(x)$

$P(1):\ f(x) = F_1(x)$

$LHS = f(x) = \frac{x}{2-x}$ and $RHS = F_1(x) = \frac{x}{2^1-(2^1-1)x} = \frac{x}{2-x}$

$\therefore P(1)$ true

assume that $P(k)$ is true, *i.e.*, $\underbrace{(f \circ f \circ \ldots \circ f)}_{\text{k times}}(x) = F_k(x)$

consider $P(k+1)$

EITHER

$\underbrace{(f \circ f \circ \ldots \circ f)}_{\text{k+1 times}}(x) = \left(f \circ \underbrace{f \circ f \circ \ldots \circ f}_{\text{k times}}\right)(x) = f\left(F_k(x)\right)$

$= f\left(\frac{x}{2^k-(2^k-1)x}\right) = \frac{\frac{x}{2^k-(2^k-1)x}}{2-\frac{x}{2^k-(2^k-1)x}}$

$= \frac{x}{2\left(2^k-(2^k-1)x\right)-x} = \frac{x}{2^{k+1}-(2^{k+1}-2)x-x}$

OR

$\underbrace{(f \circ f \circ \ldots \circ f)}_{\text{k+1 times}}(x) = \left(f \circ \underbrace{f \circ f \circ \ldots \circ f}_{\text{k times}}\right)(x) = F_k(f(x))$

$= F_k\left(\frac{x}{2-x}\right) = \frac{\frac{x}{2-x}}{2^k-(2^k-1)\frac{x}{2-x}}$

$= \frac{x}{2^{k+1}-2^k x-2^k x+x}$

THEN

$= \frac{x}{2^{k+1}-(2^{k+1}-1)x} = F_{k+1}(x)$

$P(k)$ true implies $P(k+1)$ true, $P(1)$ true so $P(n)$ true for all $n \in \mathbb{Z}^+$

(c) **METHOD 1**

$$x = \frac{y}{2^n - (2^n - 1)y} \Rightarrow 2^n x - (2^n - 1)xy = y$$

$$\Rightarrow 2^n x = \left((2^n - 1)x + 1\right)y \Rightarrow y = \frac{2^n x}{(2^n - 1)x + 1}$$

$$F_n^{-1}(x) = \frac{2^n x}{(2^n - 1)x + 1}$$

$$F_n^{-1}(x) = \frac{x}{\frac{2^n - 1}{2^n}x + \frac{1}{2^n}}$$

$$F_n^{-1}(x) = \frac{x}{(1 - 2^{-n})x + 2^{-n}}$$

$$F_n^{-1}(x) = \frac{x}{2^{-n} - (2^{-n} - 1)x}$$

METHOD 2

attempt $F_{-n}\left(F_n(x)\right)$

$$= F_{-n}\left(\frac{x}{2^n - (2^n - 1)x}\right) = \frac{\frac{x}{2^n - (2^n - 1)x}}{2^{-n} - (2^{-n} - 1)\frac{x}{2^n - (2^n - 1)x}}$$

$$= \frac{x}{2^{-n}(2^n - (2^n - 1)x) - (2^{-n} - 1)x}$$

Note: Award ***A1*** marks for numerators and denominators.

$$= \frac{x}{1} = x$$

METHOD 3

attempt $F_n\left(F_{-n}(x)\right)$

$$= F_n\left(\frac{x}{2^{-n} - (2^{-n} - 1)x}\right) = \frac{\frac{x}{2^{-n} - (2^{-n} - 1)x}}{2^n - (2^n - 1)\frac{x}{2^{-n} - (2^{-n} - 1)x}}$$

$$= \frac{x}{2^n\left(2^{-n} - (2^{-n} - 1)x\right) - (2^n - 1)x}$$

Note: Award ***A1*** marks for numerators and denominators.

$$= \frac{x}{1} = x$$

(d) (i) $F_n(0)=0,\ F_n(1)=1$

(ii) **METHOD 1**

$2^n-(2^n-1)x-1=(2^n-1)(1-x)$

>0 if $0<x<1$ and $n\in\mathbb{Z}^+$

so $2^n-(2^n-1)x>1$ and $F_n(x)=\dfrac{x}{2^n-(2^n-1)x}<\dfrac{x}{1}\ (<x)$

$F_n(x)=\dfrac{x}{2^n-(2^n-1)x}<x$ for $0<x<1$ and $n\in\mathbb{Z}^+$

METHOD 2

$\dfrac{x}{2^n-(2^n-1)x}<x \Leftrightarrow 2^n-(2^n-1)x>1$

$\Leftrightarrow (2^n-1)x<2^n-1$

$\Leftrightarrow x<\dfrac{2^n-1}{2^n-1}=1$ true in the interval $]0,1[$

(iii) $B_n=2\left(A_n-\dfrac{1}{2}\right)\ (=2A_n-1)$

6 – (MAT/12_HL_Summer_2013_Q12) – *Functions - Roots, Graphs*

(a) by division or otherwise

$f(x)=2-\dfrac{5}{x+2}$

(b) $f'(x)=\dfrac{5}{(x+2)^2}$

>0 as $(x+2)^2>0$ (on D)

Note: Do not penalise candidates who use the original form of the function to compute its derivative.

(c) $S=\left[-3,\dfrac{3}{2}\right]$

(d) (i) **EITHER**

rearrange $y = f(x)$ to make x the subject
obtain one-line equation, *e.g.* $2x - 1 = xy + 2y$

$$x = \frac{2y+1}{2-y}$$

OR

interchange x and y
obtain one-line equation, *e.g.* $2y - 1 = xy + 2x$

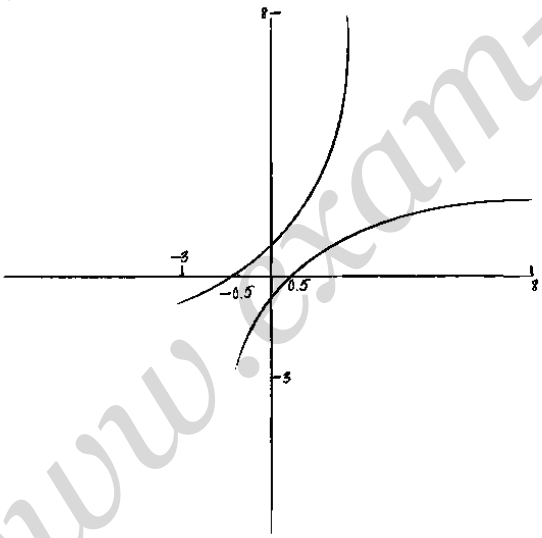
$$y = \frac{2x+1}{2-x}$$

THEN

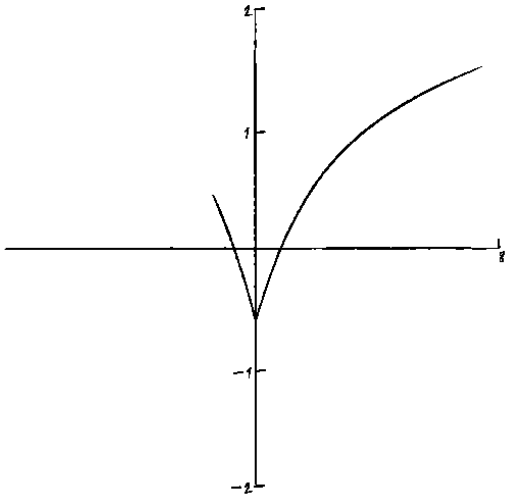
$$f^{-1}(x) = \frac{2x+1}{2-x}$$

Note:	Accept $\frac{5}{2-x} - 2$

(ii), (iii)



(e) (i)



Note:	***A1*** for correct sketch $x > 0$, ***A1*** for symmetry, ***A1*** for correct domain (from -1 to $+8$).

Note:	Candidates can **FT** on their sketch in (d)(ii).

(ii) attempt to solve $f(x) = -\frac{1}{4}$

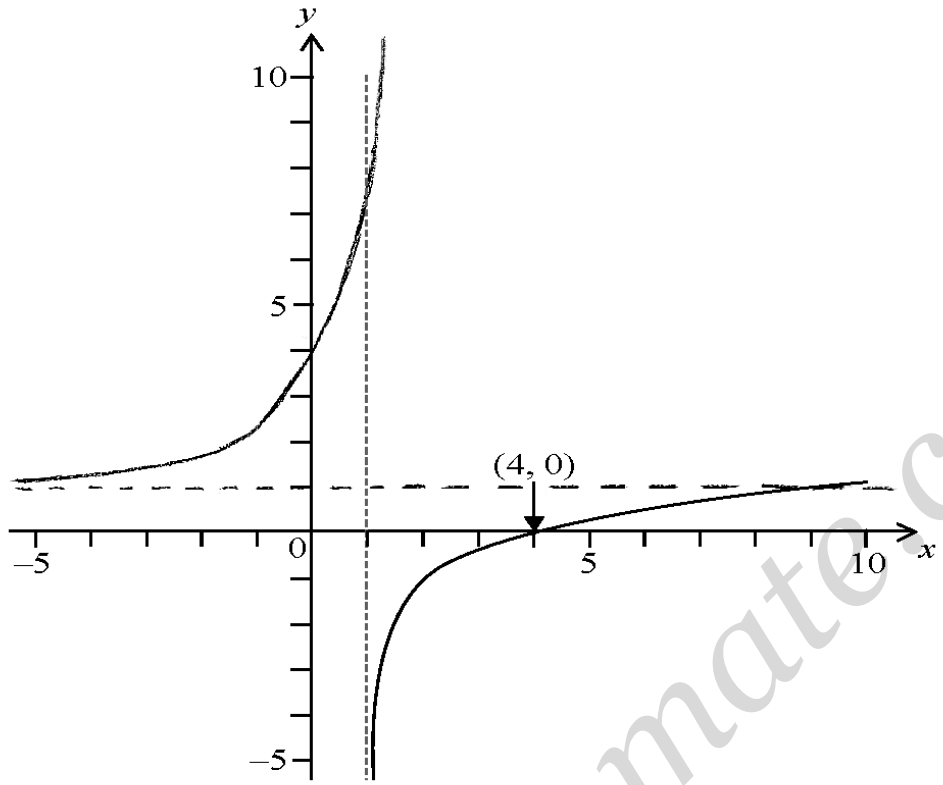
obtain $x = \frac{2}{9}$

use of symmetry or valid algebraic approach

obtain $x = -\frac{2}{9}$

7 – (MAT/10_HL_Winter_2013_Q3) – *Functions - Roots, Graphs*

(a)



shape with y-axis intercept $(0, 4)$

Note: Accept curve with an asymptote at $x = 1$ suggested.

correct asymptote $y = 1$

(b) range is $f^{-1}(x) > 1$ (or $]1, \infty[$)

Note: Also accept $]1, 10]$ or $]1, 10[$.

Note: Do not allow follow through from incorrect asymptote in (a).

(c) $(4, 0) \Rightarrow \ln(4a + b) = 0$

$\Rightarrow 4a + b = 1$

asymptote at $x = 1 \Rightarrow a + b = 0$

$\Rightarrow a = \frac{1}{3}, \ b = -\frac{1}{3}$

8 - (MAT/11_HL_Summer_2014_Q4) - *Functions - Roots*

$$r_1 + r_2 + r_3 = \frac{-48}{5}$$
$$r_1 r_2 r_3 = \frac{a-2}{5}$$
$$\frac{-48}{5} + \frac{a-2}{5} = 0$$
$$a = 50$$

9 - (MAT/12_HL_Summer_2014_Q8) - *Functions - Roots*

(a) $1-2(2)=-3$ and $\frac{3}{4}(2-2)^2-3=-3$

both answers are the same, hence f is continuous (at $x=2$)

Note: ***R1*** may be awarded for justification using a graph or referring to limits. Do not award ***A0R1***.

(b) reflection in the y-axis

$$f(-x)=\begin{cases} 1+2x, & x \geq -2 \\ \frac{3}{4}(x+2)^2-3, & x < -2 \end{cases}$$

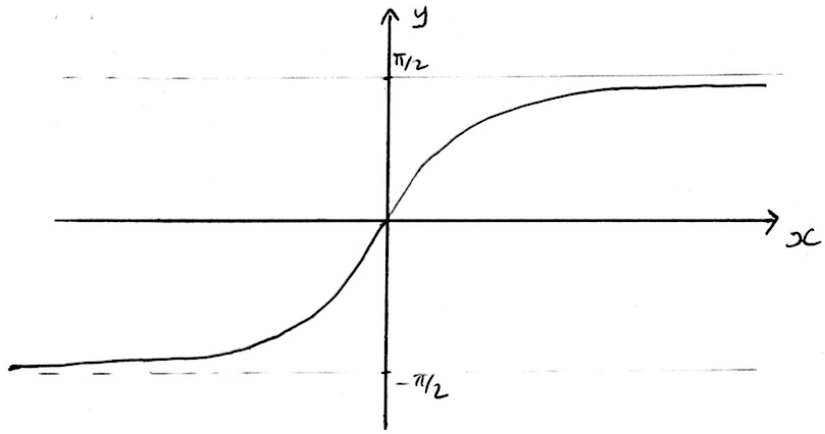
Note: Award ***M1*** for evidence of reflecting a graph in y-axis.

translation $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$$g(x)=\begin{cases} 2x-3, & x \geq 0 \\ \frac{3}{4}x^2-3, & x < 0 \end{cases}$$

10 – (MAT/12_HL_Summer_2014_Q14) – *Functions - Roots*

(a)



Note: ***A1*** for correct shape, ***A1*** for asymptotic behaviour at $y = \pm\frac{\pi}{2}$.

(b) $h \circ g(x) = \arctan\left(\frac{1}{x}\right)$

domain of $h \circ g$ is equal to the domain of g : $x \in \circ$, $x \neq 0$

(c) (i) $f(x) = \arctan(x) + \arctan\left(\frac{1}{x}\right)$

$$f'(x) = \frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}} \times -\frac{1}{x^2}$$

$$f'(x) = \frac{1}{1+x^2} + \frac{-\frac{1}{x^2}}{\frac{x^2+1}{x^2}}$$

$$= \frac{1}{1+x^2} - \frac{1}{1+x^2}$$

$$= 0$$

(ii) **METHOD 1**

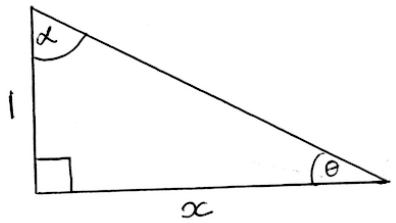
f is a constant

when $x > 0$

$f(1) = \frac{\pi}{4} + \frac{\pi}{4}$

$= \frac{\pi}{2}$

METHOD 2



from diagram

$\theta = \arctan\frac{1}{x}$

$\alpha = \arctan x$

$\theta + \alpha = \frac{\pi}{2}$

hence $f(x) = \frac{\pi}{2}$

METHOD 3

$\tan(f(x)) = \tan\left(\arctan(x) + \arctan\left(\frac{1}{x}\right)\right)$

$= \dfrac{x + \frac{1}{x}}{1 - x\left(\frac{1}{x}\right)}$

denominator $= 0$, so $f(x) = \frac{\pi}{2}$ (for $x > 0$)

(d) (i) Nigel is correct. ***A1***

METHOD 1

$\arctan(x)$ is an odd function and $\frac{1}{x}$ is an odd function

composition of two odd functions is an odd function and sum of two odd functions is an odd function ***R1***

METHOD 2

$$f(-x)=\arctan(-x)+\arctan\left(-\frac{1}{x}\right)=-\arctan(x)-\arctan\left(\frac{1}{x}\right)=-f(x)$$

therefore f is an odd function. ***R1***

(ii) $f(x)=-\frac{\pi}{2}$ ***A1***

11 – (MAT/10_HL_Winter_2014_Q1) – *Functions - Roots, Graphs*

(a) $g(x)=\frac{1}{x+3}+1$

Note: Award ***A1*** for $x+3$ in the denominator and ***A1*** for the "+1".

(b) $x=-3$
$y=1$

12 – (MAT/10_HL_Winter_2014_Q11) – *Functions - Roots, Differentiation, Integration*

(a) (i) $x = e^{3y+1}$

Note: The ***M1*** is for switching variables and can be awarded at any stage. Further marks do not rely on this mark being awarded.

taking the natural logarithm of both sides and attempting to transpose

$\left(f^{-1}(x)\right) = \frac{1}{3}(\ln x - 1)$

(ii) $x \in \mathbb{R}^+$ or equivalent, for example $x > 0$.

(b) $\ln x = \frac{1}{3}(\ln x - 1) \Rightarrow \ln x - \frac{1}{3}\ln x = -\frac{1}{3}$ (or equivalent)

$\ln x = -\frac{1}{2}$ (or equivalent)

$x = e^{-\frac{1}{2}}$

coordinates of P are $\left(e^{-\frac{1}{2}}, -\frac{1}{2}\right)$

(c) coordinates of Q are $(1, 0)$ seen anywhere

$\frac{dy}{dx} = \frac{1}{x}$

at Q, $\frac{dy}{dx} = 1$

$y = x - 1$

(d) let the required area be A

$$A = \int_1^e x - 1\,\mathrm{d}x - \int_1^e \ln x\,\mathrm{d}x$$

Note: The ***M1*** is for a difference of integrals. Condone absence of limits here.

attempting to use integration by parts to find $\int \ln x\,\mathrm{d}x$

$$= \left[\frac{x^2}{2} - x\right]_1^{\mathrm{e}} - [x\ln x - x]_1^{\mathrm{e}}$$

Note: Award ***A1*** for $\frac{x^2}{2} - x$ and ***A1*** for $x\ln x - x$.

Note: The second ***M1*** and second ***A1*** are independent of the first ***M1*** and the first ***A1***.

$$= \frac{\mathrm{e}^2}{2} - \mathrm{e} - \frac{1}{2}\left(= \frac{\mathrm{e}^2 - 2\mathrm{e} - 1}{2}\right)$$

(e) (i) **METHOD 1**

consider for example $h(x) = x - 1 - \ln x$

$h(1) = 0$ and $h'(x) = 1 - \frac{1}{x}$

as $h'(x) \geq 0$ for $x \geq 1$, then $h(x) \geq 0$ for $x \geq 1$

as $h'(x) \leq 0$ for $0 < x \leq 1$, then $h(x) \geq 0$ for $0 < x \leq 1$

so $g(x) \leq x - 1,\ x \in \mathbb{R}^+$

METHOD 2

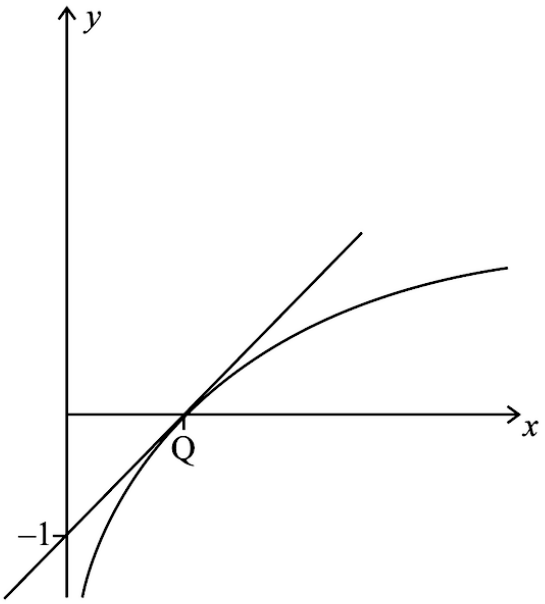
$g''(x) = -\frac{1}{x^2}$

$g''(x) < 0$ (concave down) for $x \in \mathbb{R}^+$

the graph of $y = g(x)$ is below its tangent ($y = x - 1$ at $x = 1$)

so $g(x) \leq x - 1,\ x \in \mathbb{R}^+$

METHOD 3



clear correct graphs of $y = x - 1$ and $\ln x$ for $x > 0$
statement to the effect that the graph of $\ln x$ is below the graph of its tangent at $x = 1$

(ii) replacing x by $\frac{1}{x}$ to obtain $\ln\left(\frac{1}{x}\right) \le \frac{1}{x} - 1\left(= \frac{1-x}{x}\right)$

$-\ln x \le \frac{1}{x} - 1\left(= \frac{1-x}{x}\right)$

$\ln x \ge 1 - \frac{1}{x}\left(= \frac{x-1}{x}\right)$

so $\frac{x-1}{x} \le g(x),\ x \in \mathbb{R}^+$

13 - (MAT/11_HL_Summer_2015_Q5) - *Functions - Roots*

(a) **EITHER**

$f(-x) = f(x)$

$\Rightarrow ax^2 - bx + c = ax^2 + bx + c \Rightarrow 2bx = 0, (\forall x \in \mathbb{R})$

OR

y-axis is eqn of symmetry

so $\dfrac{-b}{2a} = 0$

THEN

$\Rightarrow b = 0$

(b) $g(-x) = -g(x) \Rightarrow p\sin(-x) - qx + r = -p\sin x - qx - r$

$\Rightarrow -p\sin x - qx + r = -p\sin x - qx - r$

Note: ***M1*** is for knowing properties of sin.

$\Rightarrow 2r = 0 \Rightarrow r = 0$

Note: In (a) and (b) allow substitution of a particular value of x

(c) $h(-x) = h(x) = -h(x) \Rightarrow 2h(x) = 0 \Rightarrow h(x) = 0, (\forall x)$

14 - (MAT/11_HL_Summer_2015_Q6) - *Functions - Roots, Integration*

(a) $f: x \to y = \dfrac{3x-2}{2x-1}$ $\quad f^{-1}: y \to x$

$y = \dfrac{3x-2}{2x-1} \Rightarrow 3x-2 = 2xy - y$

$\Rightarrow 3x - 2xy = -y + 2$

$x(3-2y) = 2-y$

$x = \dfrac{2-y}{3-2y}$

$(f^{-1}(y) = \dfrac{2-y}{3-2y})$

$f^{-1}(x) = \dfrac{2-x}{3-2x} \qquad \left(x \neq \dfrac{3}{2}\right)$

Note: x and y might be interchanged earlier.

Note: First ***M1*** is for interchange of variables second ***M1*** for manipulation

Note: Final answer must be a function of x

(b) $\dfrac{3x-2}{2x-1} = A + \dfrac{B}{2x-1} \Rightarrow 3x-2 = A(2x-1)+B$

equating coefficients $3 = 2A$ and $-2 = -A + B$

$A = \dfrac{3}{2}$ and $B = -\dfrac{1}{2}$

Note: Could also be done by division or substitution of values.

(c) $\int f(x)\,\mathrm{d}x = \dfrac{3}{2}x - \dfrac{1}{4}\ln\left|2x-1\right| + c$

15 - (MAT/11_HL_Summer_2015_Q7) - *Functions - Roots*

(a) (i) $\left(-\frac{a_{n-1}}{a_n}=\right)-\frac{1}{2}$

(ii) $\left((-1)^n\frac{a_0}{a_n}=\right)-\frac{36}{2}=(-18)$

Note: First ***A1*** is for the negative sign.

(b) **METHOD 1**

if λ satisfies $p(\lambda)=0$ then $q(\lambda-4)=0$

so the roots of $q(x)$ are each 4 less than the roots of $p(x)$

so sum of roots is $-\frac{1}{2}-4\times 5=-20.5$

METHOD 2

$p(x+4)=2x^5+2\times 5\times 4x^4\ldots+x^4\ldots=2x^5+41x^4\ldots$

so sum of roots is $-\frac{41}{2}=-20.5$

16 – (MAT/11_HL_Summer_2015_Q9) – *Functions - Roots*

(a) $g \circ f(x) = g\left(f(x)\right)$

$= g\left(2x + \frac{\pi}{5}\right)$

$= 3\sin\left(2x + \frac{\pi}{5}\right) + 4$

(b) since $-1 \le \sin\theta \le +1$, range is $[1, 7]$

(c) $3\sin\left(2x + \frac{\pi}{5}\right) + 4 = 7 \Rightarrow 2x + \frac{\pi}{5} = \frac{\pi}{2} + 2n\pi \Rightarrow x = \frac{3\pi}{20} + n\pi$

so next biggest value is $\frac{23\pi}{20}$

Note: Allow use of period.

(d) **Note:** Transformations can be in any order but see notes below.

stretch scale factor 3 parallel to y axis (vertically)
vertical translation of 4 up

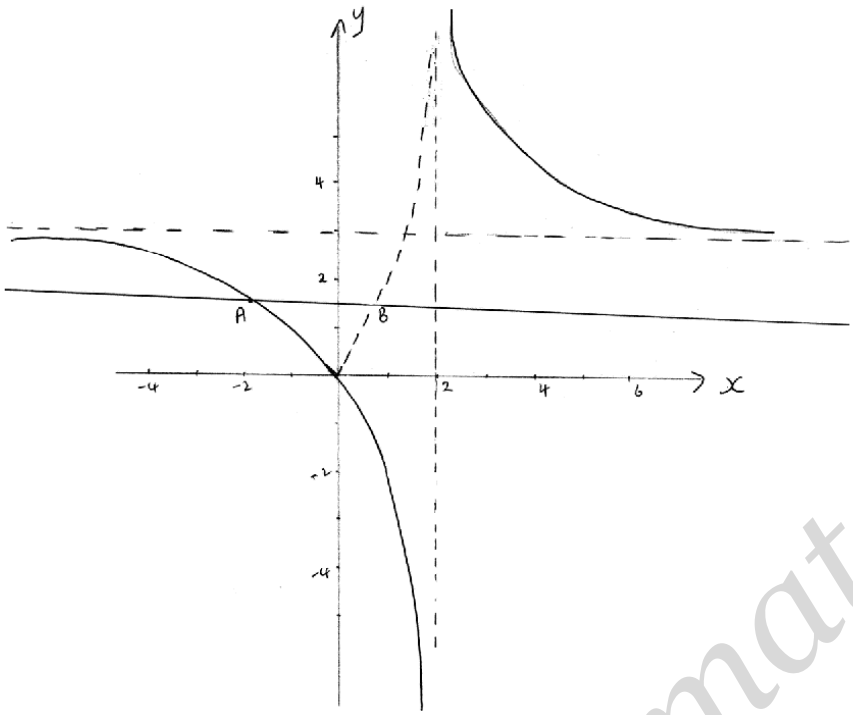
Note: Vertical translation is $\frac{4}{3}$ up if it occurs before stretch parallel to y axis.

stretch scale factor $\frac{1}{2}$ parallel to x axis (horizontally)

horizontal translation of $\frac{\pi}{10}$ to the left

17 – (MAT/12_HL_Summer_2015_Q10) – *Functions - Roots*

(a)



Note: In the diagram, points marked A and B refer to part (d) and do not need to be seen in part (a).

shape of curve

Note: This mark can only be awarded if there appear to be both horizontal and vertical asymptotes.

intersection at $(0, 0)$
horizontal asymptote at $y = 3$
vertical asymptote at $x = 2$

(b) $y = \dfrac{3x}{x-2}$

$xy - 2y = 3x$
$xy - 3x = 2y$
$x = \dfrac{2y}{y-3}$
$\left(f^{-1}(x)\right) = \dfrac{2x}{x-3}$

(c) **METHOD 1**

attempt to solve $\frac{2x}{x-3}=\frac{3x}{x-2}$

$2x(x-2)=3x(x-3)$

$x\left[2(x-2)-3(x-3)\right]=0$

$x(5-x)=0$

$x=0$ or $x=5$

METHOD 2

$x=\frac{3x}{x-2}$ or $x=\frac{2x}{x-3}$

$x=0$ or $x=5$

(d) **METHOD 1**

at $\mathrm{A}:\frac{3x}{x-2}=\frac{3}{2}$ AND at $\mathrm{B}:\frac{3x}{x-2}=-\frac{3}{2}$

$6x=3x-6$

$x=-2$

$6x=6-3x$

$x=\frac{2}{3}$

solution is $-2<x<\frac{2}{3}$

METHOD 2

$\left(\frac{3x}{x-2}\right)^2<\left(\frac{3}{2}\right)^2$

$9x^2<\frac{9}{4}(x-2)^2$

$3x^2+4x-4<0$

$(3x-2)(x+2)<0$

$x=-2$

$x=\frac{2}{3}$

solution is $-2<x<\frac{2}{3}$

(e) $-2 < x < 2$

18 – (MAT/12_HL_Summer_2015_Q11) – *Functions - Roots, Differentiation, Integration*

(a) $g \circ f(x) = \dfrac{\tan x + 1}{\tan x - 1}$

$x \neq \dfrac{\pi}{4}$, $0 \leq x < \dfrac{\pi}{2}$

(b) $\dfrac{\tan x + 1}{\tan x - 1} = \dfrac{\dfrac{\sin x}{\cos x} + 1}{\dfrac{\sin x}{\cos x} - 1}$

$= \dfrac{\sin x + \cos x}{\sin x - \cos x}$

(c) **METHOD 1**

$$\frac{dy}{dx} = \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$\frac{dy}{dx} = \frac{\left(2\sin x\cos x - \cos^2 x - \sin^2 x\right) - \left(2\sin x\cos x + \cos^2 x + \sin^2 x\right)}{\cos^2 x + \sin^2 x - 2\sin x\cos x}$$

$$= \frac{-2}{1 - \sin 2x}$$

Substitute $\dfrac{\pi}{6}$ into any formula for $\dfrac{dy}{dx}$

$$\frac{-2}{1 - \sin\frac{\pi}{3}}$$

$$= \frac{-2}{1 - \frac{\sqrt{3}}{2}}$$

$$= \frac{-4}{2 - \sqrt{3}}$$

$$= \frac{-4}{2 - \sqrt{3}}\left(\frac{2 + \sqrt{3}}{2 + \sqrt{3}}\right)$$

$$= \frac{-8 - 4\sqrt{3}}{1} = -8 - 4\sqrt{3}$$

METHOD 2

$$\frac{dy}{dx} = \frac{(\tan x - 1)\sec^2 x - (\tan x + 1)\sec^2 x}{(\tan x - 1)^2}$$

$$= \frac{-2\sec^2 x}{(\tan x - 1)^2}$$

$$= \frac{-2\sec^2 \frac{\pi}{6}}{\left(\tan\frac{\pi}{6} - 1\right)^2} = \frac{-2\left(\frac{4}{3}\right)}{\left(\frac{1}{\sqrt{3}} - 1\right)^2} = \frac{-8}{\left(1 - \sqrt{3}\right)^2}$$

Note: Award ***M1*** for substitution of $\frac{\pi}{6}$.

$$\frac{-8}{\left(1 - \sqrt{3}\right)^2} = \frac{-8}{\left(4 - 2\sqrt{3}\right)}\frac{\left(4 + 2\sqrt{3}\right)}{\left(4 + 2\sqrt{3}\right)} = -8 - 4\sqrt{3}$$

(d) Area $= \left| \int_0^{\frac{\pi}{6}} \frac{\sin x + \cos x}{\sin x - \cos x} \, dx \right|$

$= \left| \left[\ln \left| \sin x - \cos x \right| \right]_0^{\frac{\pi}{6}} \right|$

Note: Condone absence of limits and absence of modulus signs at this stage.

$= \left| \ln \left| \sin \frac{\pi}{6} - \cos \frac{\pi}{6} \right| - \ln \left| \sin 0 - \cos 0 \right| \right|$

$= \left| \ln \left| \frac{1}{2} - \frac{\sqrt{3}}{2} \right| - 0 \right|$

$= \left| \ln \left(\frac{\sqrt{3} - 1}{2} \right) \right|$

$= - \ln \left(\frac{\sqrt{3} - 1}{2} \right) = \ln \left(\frac{2}{\sqrt{3} - 1} \right)$

$= \ln \left(\frac{2}{\sqrt{3} - 1} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1} \right)$

$= \ln \left(\sqrt{3} + 1 \right)$

19 – (MAT/12_HL_Summer_2015_Q12) – *Functions - Roots*

(a) (i)–(iii) given the three roots $\alpha,\ \beta,\ \gamma$, we have

$$x^3+px^2+qx+c=(x-\alpha)(x-\beta)(x-\gamma)$$
$$=\left(x^2-(\alpha+\beta)x+\alpha\beta\right)(x-\gamma)$$
$$=x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma$$

comparing coefficients:

$$p=-(\alpha+\beta+\gamma)$$
$$q=(\alpha\beta+\beta\gamma+\gamma\alpha)$$
$$c=-\alpha\beta\gamma$$

(b) **METHOD 1**

i) Given $-\alpha-\beta-\gamma=-6$

And $\alpha\beta+\beta\gamma+\gamma\alpha=18$

Let the three roots be α,β,γ .

So $\beta-\alpha=\gamma-\beta$

or $2\beta=\alpha+\gamma$

Attempt to solve simultaneous equations:

$$\beta+2\beta=6$$
$$\beta=2$$

ii) $\alpha+\gamma=4$

$$2\alpha+2\gamma+\alpha\gamma=18$$
$$\Rightarrow \gamma^2-4\gamma+10=0$$
$$\Rightarrow \gamma=\frac{4\pm\mathrm{i}\sqrt{24}}{2}$$

Therefore $c=-\alpha\beta\gamma=-\left(\frac{4+\mathrm{i}\sqrt{24}}{2}\right)\left(\frac{4-\mathrm{i}\sqrt{24}}{2}\right)2=-20$

METHOD 2

(i) let the three roots be α, $\alpha - d$, $\alpha + d$
adding roots
to give $3\alpha = 6$
$\alpha = 2$

(ii) α is a root, so $2^3 - 6 \times 2^2 + 18 \times 2 + c = 0$

$8 - 24 + 36 + c = 0$
$c = -20$

METHOD 3

(i) let the three roots be α, $\alpha - d$, $\alpha + d$
adding roots
to give $3\alpha = 6$
$\alpha = 2$

(ii) $q = 18 = 2(2 - d) + (2 - d)(2 + d) + 2(2 + d)$

$d^2 = -6 \Rightarrow d = \sqrt{6}\,\mathrm{i}$
$\Rightarrow c = -20$

(c) **METHOD 1**

Given $-\alpha-\beta-\gamma=-6$
And $\alpha\beta+\beta\gamma+\gamma\alpha=18$

Let the three roots be α,β,γ.

So $\frac{\beta}{\alpha}=\frac{\gamma}{\beta}$

or $\beta^2=\alpha\gamma$

Attempt to solve simultaneous equations:

$\alpha\beta+\gamma\beta+\beta^2=18$
$\beta(\alpha+\beta+\gamma)=18$
$6\beta=18$
$\beta=3$

$\alpha+\gamma=3$, $\alpha=\frac{9}{\gamma}$
$\Rightarrow \gamma^2-3\gamma+9=0$
$\Rightarrow \gamma=\frac{3\pm i\sqrt{27}}{2}$

Therefore $c=-\alpha\beta\gamma=-\left(\frac{3+i\sqrt{27}}{2}\right)\left(\frac{3-i\sqrt{27}}{2}\right)3=-27$

METHOD 2

let the three roots be $a,\ ar,\ ar^2$
attempt at substitution of $a,\ ar,\ ar^2$ and *p* and *q* into equations from (a)
$6=a+ar+ar^2\left(=a\left(1+r+r^2\right)\right)$
$18=a^2r+a^2r^3+a^2r^2\left(=a^2r\left(1+r+r^2\right)\right)$

therefore $3=ar$
therefore $c=-a^3r^3=-3^3=-27$

20 – (MAT/10_HL_Winter_2016_Q3) – *Functions - Roots*

(a) $a = 1$ ***A1***

$c = 3$ ***A1***

[2 marks]

(b) use the coordinates of $(1, 0)$ on the graph ***M1***

$f(1) = 0 \Rightarrow 1 + \dfrac{b}{1-3} = 0 \Rightarrow b = 2$ ***A1***

[2 marks]

21 – (MAT/10_HL_Winter_2016_Q5) – *Functions - Roots*

$\alpha + \beta = 2k$ ***A1***

$\alpha\beta = k - 1$ ***A1***

$(\alpha + \beta)^2 = 4k^2 \Rightarrow \alpha^2 + \beta^2 + 2\underbrace{\alpha\beta}_{k-1} = 4k^2$ ***(M1)***

$\alpha^2 + \beta^2 = 4k^2 - 2k + 2$

$\alpha^2 + \beta^2 = 4 \Rightarrow 4k^2 - 2k - 2 = 0$ ***A1***

attempt to solve quadratic ***(M1)***

$k = 1, -\dfrac{1}{2}$ ***A1***

[6 marks]

22 – (MAT/12_HL_Summer_2017_Q2) – *Functions - Roots*

(a) $-11 \le f(x) \le 21$ ***A1A1***

Note: ***A1*** for correct end points, ***A1*** for correct inequalities.

[2 marks]

(b) $f^{-1}(x) = \sqrt[3]{\dfrac{x-5}{2}}$ ***(M1)A1***

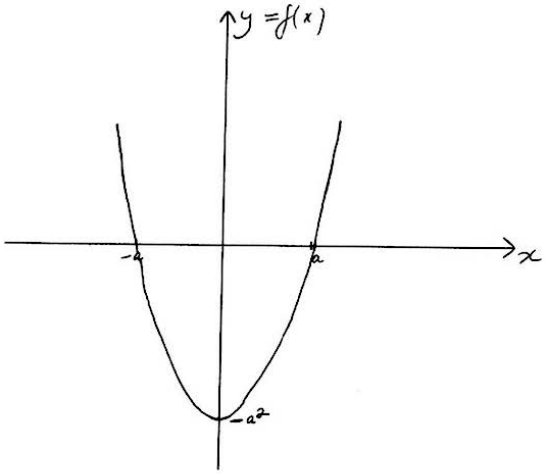
[2 marks]

(c) $-11 \le x \le 21,\ -2 \le f^{-1}(x) \le 2$ ***A1A1***

[2 marks]

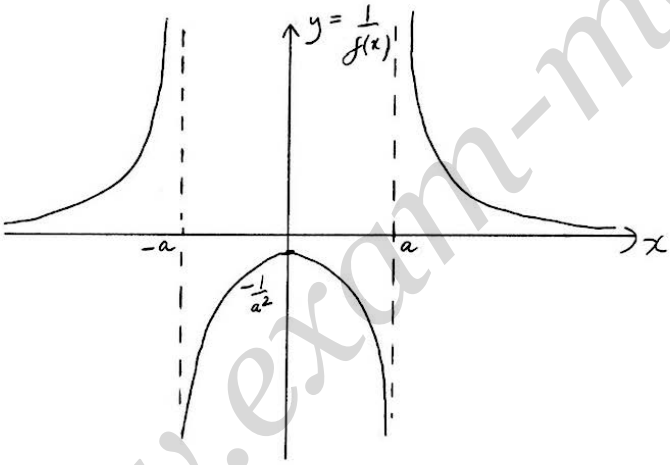
23 – (MAT/12_HL_Summer_2017_Q9) – *Functions - Roots, Differentiation, Integration*

(a) (i)



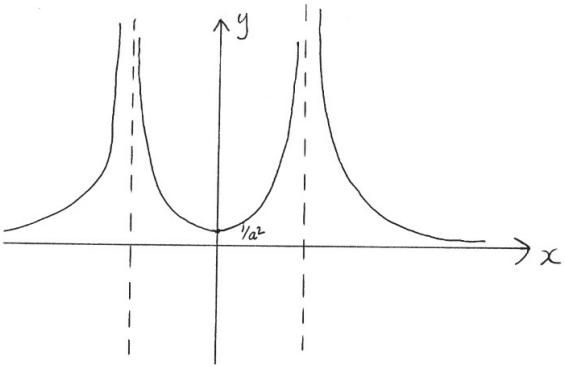
A1 for correct shape
A1 for correct x and y intercepts and minimum point

(ii)



A1 for correct shape
A1 for correct vertical asymptotes
A1 for correct implied horizontal asymptote
A1 for correct maximum point

(iii)



A1 for reflecting negative branch from (ii) in the x-axis
A1 for correctly labelled minimum point

[8 marks]

(b) **EITHER**

attempt at integration by parts ***(M1)***

$\int (x^2 - a^2)\cos x \mathrm{d}x = (x^2 - a^2)\sin x - \int 2x \sin x \mathrm{d}x$ ***A1A1***

$= (x^2 - a^2)\sin x - 2\left[-x\cos x + \int \cos x \mathrm{d}x\right]$ ***A1***

$= (x^2 - a^2)\sin x + 2x\cos x - 2\sin x + c$ ***A1***

OR

$\int (x^2 - a^2)\cos x \mathrm{d}x = \int x^2 \cos x \mathrm{d}x - \int a^2 \cos x \mathrm{d}x$

attempt at integration by parts ***(M1)***

$\int x^2 \cos x \mathrm{d}x = x^2 \sin x - \int 2x \sin x \mathrm{d}x$ ***A1A1***

$= x^2 \sin x - 2\left[-x\cos x + \int \cos x \mathrm{d}x\right]$ ***A1***

$= x^2 \sin x + 2x\cos x - 2\sin x$

$-\int a^2 \cos x \mathrm{d}x = -a^2 \sin x$

$\int (x^2 - a^2)\cos x \mathrm{d}x = (x^2 - a^2)\sin x + 2x\cos x - 2\sin x + c$ ***A1***

[5 marks]

(c) $g(x) = x\left(x^2 - a^2\right)^{\frac{1}{2}}$

$g'(x) = \left(x^2 - a^2\right)^{\frac{1}{2}} + \frac{1}{2}x\left(x^2 - a^2\right)^{-\frac{1}{2}}(2x)$ ***M1A1A1***

Note: Method mark is for differentiating the product. Award ***A1*** for each correct term.

$g'(x) = \left(x^2 - a^2\right)^{\frac{1}{2}} + x^2\left(x^2 - a^2\right)^{-\frac{1}{2}}$

both parts of the expression are positive hence $g'(x)$ is positive ***R1***

and therefore g is an increasing function (for $|x| > a$) ***AG***

[4 marks]

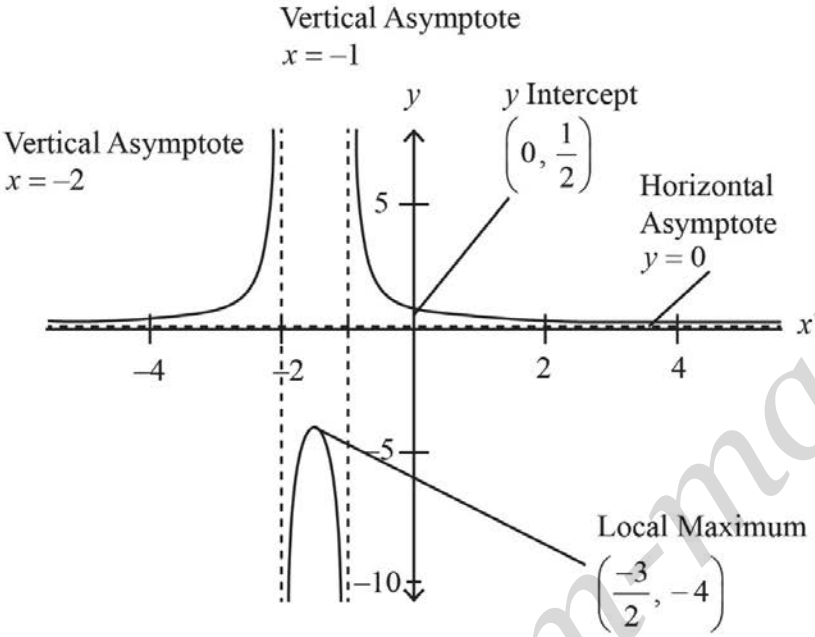
24 – (MAT/11_HL_Summer_2017_Q11) – *Functions - Roots, Graphs, Integration*

(a) (i) $x^2 + 3x + 2 = \left(x + \frac{3}{2}\right)^2 - \frac{1}{4}$ ***A1***

(ii) $x^2 + 3x + 2 = (x + 2)(x + 1)$ ***A1***

[2 marks]

(b)



A1 for the shape
A1 for the equation $y = 0$
A1 for asymptotes $x = -2$ and $x = -1$
A1 for coordinates $\left(-\frac{3}{2}, -4\right)$
A1 y-intercept $\left(0, \frac{1}{2}\right)$

[5 marks]

(c) $\frac{1}{x+1} - \frac{1}{x+2} = \frac{(x+2)-(x+1)}{(x+1)(x+2)}$ ***M1***

$= \frac{1}{x^2 + 3x + 2}$ ***AG***

[1 mark]

(d) $\int_0^1 \frac{1}{x+1} - \frac{1}{x+2}\,dx$

$= \left[\ln(x+1) - \ln(x+2)\right]_0^1$ ***A1***

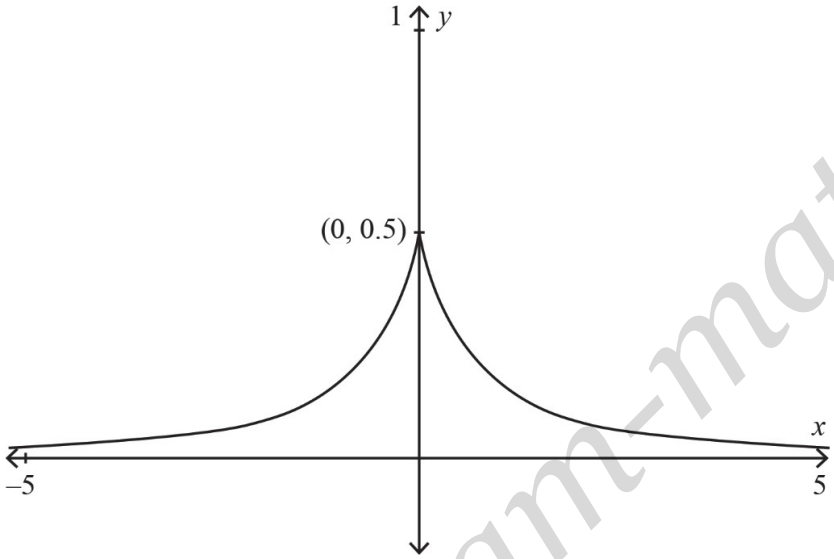
$= \ln 2 - \ln 3 - \ln 1 + \ln 2$ ***M1***

$= \ln\left(\frac{4}{3}\right)$ ***M1A1***

$\therefore\ p = \frac{4}{3}$

[4 marks]

(e)



symmetry about the y-axis ***M1***
correct shape ***A1***

Note: Allow ***FT*** from part (b).

[2 marks]

(f) $2\int_0^1 f(x)\,dx$ ***(M1)(A1)***

$= 2\ln\left(\frac{4}{3}\right)$ ***A1***

Note: Do not award ***FT*** from part (e).

[3 marks]

25 – (MAT/11_HL_Summer_2017_Q12) – *Functions - Roots, Graphs*

(a) sum $= 0$ ***A1***
product $= 6$ ***A1***
[2 marks]

(b) $P(1) = 1 - 10 + 15 - 6 = 0$ ***M1A1***
$\Rightarrow (z-1)$ is a factor of $P(z)$ ***AG***

Note: Accept use of division to show remainder is zero.

[2 marks]

(c) **METHOD 1**

$(z-1)^3\left(z^2 + bz + c\right) = z^5 - 10z^2 + 15z - 6$ ***(M1)***
by inspection $c = 6$ ***A1***
$\left(z^3 - 3z^2 + 3z - 1\right)\left(z^2 + bz + 6\right) = z^5 - 10z^2 + 15z - 6$ ***(M1)(A1)***
$b = 3$ ***A1***

METHOD 2

α, β are two roots of the quadratic
$b = -(\alpha + \beta),\ c = \alpha\beta$ ***(A1)***
from part (a) $1 + 1 + 1 + \alpha + \beta = 0$ ***(M1)***
$\Rightarrow b = 3$ ***A1***
$1 \times 1 \times 1 \times \alpha\beta = 6$ ***(M1)***
$\Rightarrow c = 6$ ***A1***

Note: Award ***FT*** if $b = -7$ following through from their sum $= 10$.

METHOD 3

$\left(z^5 - 10z^2 + 15z - 6\right) \div (z-1) = z^4 + z^3 + z^2 - 9z + 6$ ***(M1)A1***

Note: This may have been seen in part (b).

$z^4 + z^3 + z^2 - 9z + 6 \div (z-1) = z^3 + 2z^2 + 3z - 6$ ***(M1)***
$z^3 + 2z^2 + 3z - 6 \div (z-1) = z^2 + 3z + 6$ ***A1A1***

[5 marks]

(d) $z^2 + 3z + 6 = 0$ ***M1***

$z = \dfrac{-3 \pm \sqrt{9 - 4 \cdot 6}}{2}$ ***M1***

$= \dfrac{-3 \pm \sqrt{-15}}{2}$

$z = -\dfrac{3}{2} \pm \dfrac{\mathrm{i}\sqrt{15}}{2}$ ***A1***

(or $z = 1$)

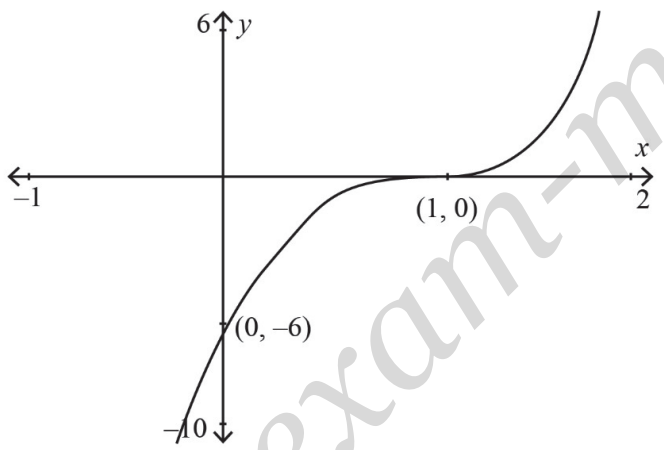
Notes: Award the second ***M1*** for an attempt to use the quadratic formula or to complete the square. Do not award ***FT*** from (c).

[3 marks]

(e) (i) $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 20x^3 - 20$ ***M1A1***

for $x > 1$, $20x^3 - 20 > 0 \Rightarrow$ concave up ***R1AG***

(ii)



x-intercept at $(1, 0)$ ***A1***

y-intercept at $(0, -6)$ ***A1***

stationary point of inflexion at $(1, 0)$ with correct curvature either side ***A1***

[6 marks]

26 – (MAT/12_HL_Summer_2018_Q10) – *Functions - Roots*

(a) attempt to make x the subject of $y = \frac{ax+b}{cx+d}$ ***M1***

$y(cx+d) = ax + b$ ***A1***

$x = \frac{dy-b}{a-cy}$ ***A1***

$f^{-1}(x) = \frac{dx-b}{a-cx}$, ***A1***

Note: Do not allow $y =$ in place of $f^{-1}(x)$.

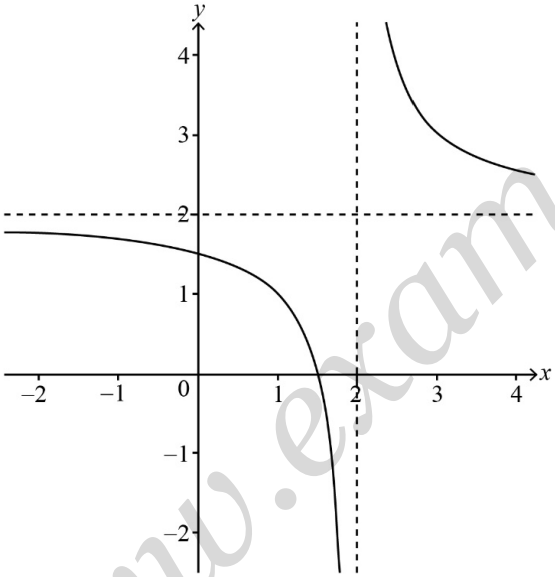
$x \neq \frac{a}{c}$, $(x \in \mathbb{R})$ ***A1***

Note: The final ***A*** mark is independent.

[5 marks]

(b) (i) $g(x) = 2 + \frac{1}{x-2}$ ***A1A1***

(ii)



hyperbola shape, with single curves in second and fourth quadrants and third quadrant blank, including vertical asymptote $x = 2$ ***A1***

horizontal asymptote $y = 2$ ***A1***

intercepts $\left(\frac{3}{2}, 0\right), \left(0, \frac{3}{2}\right)$ ***A1***

[5 marks]

(c) the domain of $h \circ g$ is $x \leq \frac{3}{2}$, $x > 2$ ***A1A1***

the range of $h \circ g$ is $y \geq 0$, $y \neq \sqrt{2}$ ***A1A1***

[4 marks]

27 – (MAT/12_HL_Summer_2019_Q3) – *Functions - Roots*

$g(x) = f(x+2)\left(=(x+2)^4 - 6(x+2)^2 - 2(x+2) + 4\right)$ ***M1***

attempt to expand $(x+2)^4$ ***M1***

$(x+2)^4 = x^4 + 4(2x^3) + 6(2^2x^2) + 4(2^3x) + 2^4$ ***(A1)***

$= x^4 + 8x^3 + 24x^2 + 32x + 16$ ***A1***

$g(x) = x^4 + 8x^3 + 24x^2 + 32x + 16 - 6(x^2 + 4x + 4) - 2x - 4 + 4$

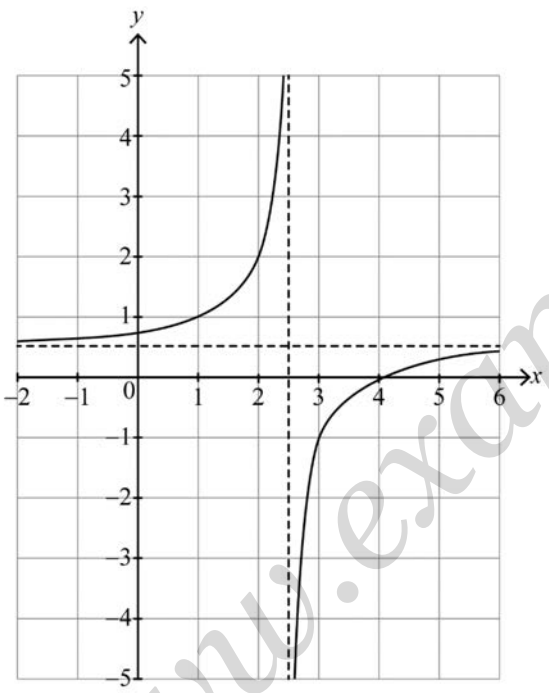
$= x^4 + 8x^3 + 18x^2 + 6x - 8$ ***A1***

Note: For correct expansion of $f(x-2) = x^4 - 8x^3 + 18x^2 - 10x$ award max ***M0M1(A1)A0A1***.

[5 marks]

28 – (MAT/12_HL_Summer_2019_Q5) – *Graphs, Functions - Roots*

(a)



correct shape: two branches in correct quadrants with asymptotic behaviour ***A1***

crosses at $(4, 0)$ and $\left(0, \frac{4}{5}\right)$ ***A1A1***

asymptotes at $x = \frac{5}{2}$ and $y = \frac{1}{2}$ ***A1A1***

[5 marks]

(b) (i) $x < \frac{5}{2}, x \geq 4$ ***A1A1***

(ii) $f(x) \geq 0, f(x) \neq \frac{1}{\sqrt{2}} \left(f(x) \in \mathbb{R}\right)$ ***A1***

Note: Follow through from their graph, as long as it is a rectangular hyperbola.

Note: Allow range expressed in terms of y.

[3 marks]

29 – (MathAA/12_HL_Summer_2021_Q1) – *Algebra-functions - Roots*

attempt to subtract squares of integers ***(M1)***

$(n+1)^2 - n^2$

EITHER

correct order of subtraction and correct expansion of $(n+1)^2$, seen anywhere ***A1A1***

$= n^2 + 2n + 1 - n^2 \left(= 2n + 1\right)$

OR

correct order of subtraction and correct factorization of difference of squares ***A1A1***

$= (n+1-n)(n+1+n)\left(= 2n+1\right)$

THEN

$= n + n + 1$ =RHS ***A1***

Note: Do not award final ***A1*** unless all previous working is correct.

which is the sum of n and $n+1$ ***AG***

Note: If expansion and order of subtraction are correct, award full marks for candidates who find the sum of the integers as $2n+1$ and then show that the difference of the squares (subtracted in the correct order) is $2n+1$.

[4 marks]

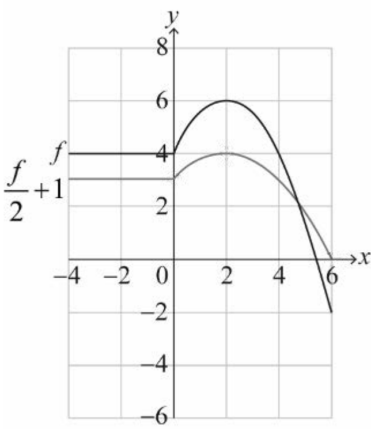
30 – (MathAA/11_HL_Summer_2021_Q1) – *Algebra-functions - Roots*

(a) (i) $f(2)=6$ ***A1***

(ii) $(f\circ f)(2)=-2$ ***A1***

[2 marks]

(b)



M1A1A1

Note: Award ***M1*** for an attempt to apply any vertical stretch or vertical translation,
A1 for a correct horizontal line segment between −4 and 0 (located roughly at $y=3$),
A1 for a correct concave down parabola including max point at (2,4) and for correct end points at (0,3) and (6,0) (within circles). Points do not need to be labelled.

[3 marks]

31 – (MathAA/12_HL_Summer_2021_Q7) – *Algebra-functions - Roots*

$\alpha+\beta+\alpha+\beta=k$ ***(A1)***

$\alpha+\beta=\frac{k}{2}$

$\alpha\beta(\alpha+\beta)=-3k$ ***(A1)***

$\left(-\frac{k^2}{4}\right)\left(\frac{k}{2}\right)=-3k\ \left(-\frac{k^3}{8}=-3k\right)$ ***M1***

attempting to solve $-\frac{k^3}{8}+3k=0$ (or equivalent) for k ***(M1)***

$k=2\sqrt{6}\ \left(=\sqrt{24}\right)(k>0)$ ***A1***

Note: Award ***A0*** for $k=\pm2\sqrt{6}\ \left(\pm\sqrt{24}\right)$.

[5 marks]

32 – (MathAA/10_HL_Winter_2021_Q2) – *Algebra-functions - Roots, Graphs*

(a) (i) $x = 3$ ***A1***

(ii) $y = -2$ ***A1***

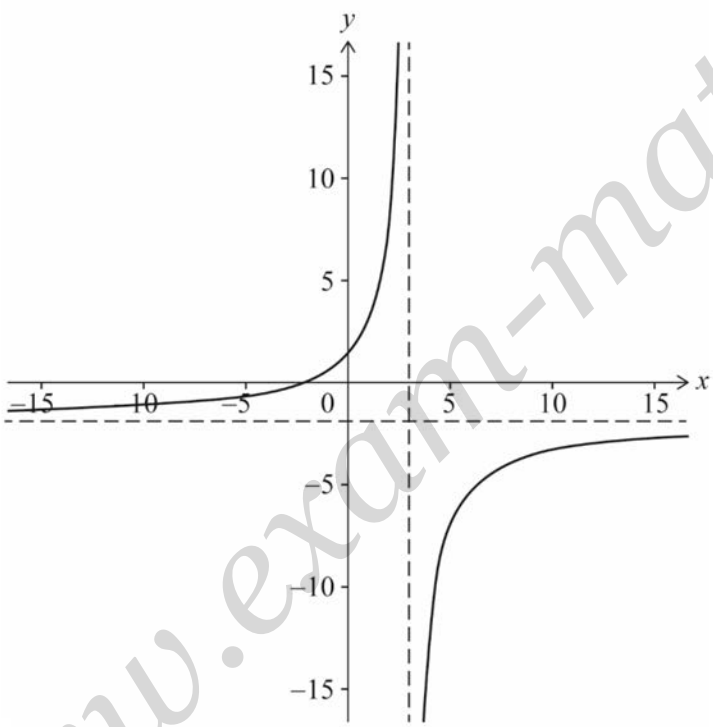
[2 marks]

(b) (i) $(-2, 0)$ (accept $x = -2$) ***A1***

(ii) $\left(0, \frac{4}{3}\right)$ (accept $y = \frac{4}{3}$ and $f(0) = \frac{4}{3}$) ***A1***

[2 marks]

(c)



A1

Note: Award ***A1*** for completely correct shape: two branches in correct quadrants with asymptotic behaviour.

[1 mark]

(d) **METHOD 1**

$\left(g(x)=\right)y=\dfrac{ax+4}{3-x}$

attempt to find x in terms of y ***(M1)***

OR exchange x and y and attempt to find y in terms of x

$3y-xy=ax+4$ ***A1***

$ax+xy=3y-4$

$x(a+y)=3y-4$

$x=\dfrac{3y-4}{y+a}$

$g^{-1}(x)=\dfrac{3x-4}{x+a}$ ***A1***

Note: Condone use of $y=$

$g(x)\equiv g^{-1}(x)$

$\dfrac{ax+4}{3-x}\equiv\dfrac{3x-4}{x+a}$

$\Rightarrow a=-3$ ***A1***

[4 marks]

METHOD 2

$g(x)=\dfrac{ax+4}{3-x}$

attempt to find an expression for $g(g(x))$ and equate to x ***(M1)***

$gg(x)=\dfrac{a\left(\dfrac{ax+4}{3-x}\right)+4}{3-\left(\dfrac{ax+4}{3-x}\right)}=x$ ***A1***

$\dfrac{a(ax+4)+4(3-x)}{(9-3x)-(ax+4)}=x$

$\dfrac{a(ax+4)+4(3-x)}{5-(3+a)x}=x$

$a(ax+4)+4(3-x)=x(5-(3+a)x)$ ***A1***

equating coefficients of x^2 (or similar)

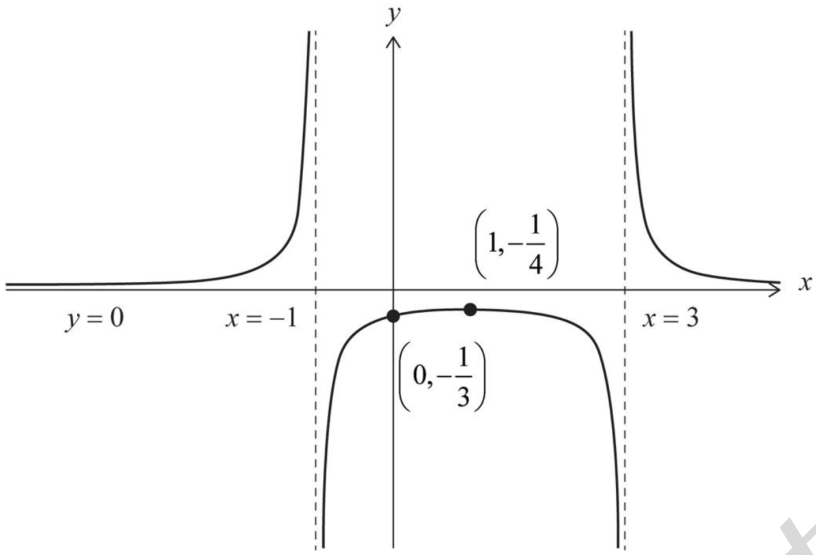
$a=-3$ ***A1***

[4 marks]

Total [9 marks]

33 – (MathAA/12_HL_Summer_2022_Q11) – *Algebra-functions - Roots*

(a)



y- intercept $\left(0,-\frac{1}{3}\right)$ ***A1***

Note: Accept an indication of $-\frac{1}{3}$ on the $y-$axis.

vertical asymptotes $x=-1$ and $x=3$ ***A1***

horizontal asymptote $y=0$ ***A1***

uses a valid method to find the x- coordinate of the local maximum point ***(M1)***

Note: For example, uses the axis of symmetry or attempts to solve $f'(x)=0$.

local maximum point $\left(1,-\frac{1}{4}\right)$ ***A1***

Note: Award ***(M1)A0*** for a local maximum point at $x=1$ and coordinates not given.

three correct branches with correct asymptotic behaviour and the key features in approximately correct relative positions to each other ***A1***

[6 marks]

(b) (i) $x = \dfrac{1}{y^2 - 2y - 3}$ ***M1***

Note: Award ***M1*** for interchanging x and y (this can be done at a later stage).

EITHER

attempts to complete the square ***M1***

$y^2 - 2y - 3 = (y-1)^2 - 4$ ***A1***

$x = \dfrac{1}{(y-1)^2 - 4}$

$(y-1)^2 - 4 = \dfrac{1}{x} \quad \left((y-1)^2 = 4 + \dfrac{1}{x}\right)$ ***A1***

$y - 1 = \pm\sqrt{4 + \dfrac{1}{x}} \quad \left(= \pm\sqrt{\dfrac{4x+1}{x}}\right)$

OR

attempts to solve $xy^2 - 2xy - 3x - 1 = 0$ for y ***M1***

$y = \dfrac{-(-2x) \pm \sqrt{(-2x)^2 + 4x(3x+1)}}{2x}$ ***A1***

Note: Award ***A1*** even if – (in $\pm$) is missing

$= \dfrac{2x \pm \sqrt{16x^2 + 4x}}{2x}$ ***A1***

THEN

$= 1 \pm \dfrac{\sqrt{4x^2 + x}}{x}$ ***A1***

$y > 3$ and hence $y = 1 - \dfrac{\sqrt{4x^2 + x}}{x}$ is rejected ***R1***

Note: Award ***R1*** for concluding that the expression for y must have the '+' sign.
The ***R1*** may be awarded earlier for using the condition $x > 3$.

$y = 1 + \dfrac{\sqrt{4x^2 + x}}{x}$

$g^{-1}(x) = 1 + \dfrac{\sqrt{4x^2 + x}}{x}$ ***AG***

(ii) domain of g^{-1} is $x > 0$ ***A1***

[7 marks]

(c) attempts to find $(h \circ g)(a)$ ***(M1)***

$(h \circ g)(a) = \arctan\left(\frac{g(a)}{2}\right)$ $\left((h \circ g)(a) = \arctan\left(\frac{1}{2(a^2 - 2a - 3)}\right)\right)$ ***(A1)***

$\arctan\left(\frac{g(a)}{2}\right) = \frac{\pi}{4}$ $\left(\arctan\left(\frac{1}{2(a^2 - 2a - 3)}\right) = \frac{\pi}{4}\right)$

attempts to solve for $g(a)$ ***M1***

$\Rightarrow g(a) = 2$ $\left(\frac{1}{(a^2 - 2a - 3)} = 2\right)$

EITHER

$\Rightarrow a = g^{-1}(2)$ ***A1***

attempts to find their $g^{-1}(2)$ ***M1***

$a = 1 + \frac{\sqrt{4(2)^2 + 2}}{2}$ ***A1***

Note: Award all available marks to this stage if x is used instead of a.

OR

$\Rightarrow 2a^2 - 4a - 7 = 0$ ***A1***

attempts to solve their quadratic equation ***M1***

$a = \frac{-(-4) \pm \sqrt{(-4)^2 + 4(2)(7)}}{4}$ $\left(= \frac{4 \pm \sqrt{72}}{4}\right)$ ***A1***

Note: Award all available marks to this stage if x is used instead of a.

THEN

$a = 1 + \frac{3}{2}\sqrt{2}$ (as $a > 3$) ***A1***

$(p = 1,\ q = 3,\ r = 2)$

Note: Award ***A1*** for $a = 1 + \frac{1}{2}\sqrt{18}$ $(p = 1,\ q = 1,\ r = 18)$.

[7 marks]

34 – (MathAA/12_HL_Summer_2022_Q9) – *Algebra-functions - Roots*

METHOD 1 (rearranging the equation)

assume there exists some $\alpha \in \mathbb{Z}$ such that $2\alpha^3 + 6\alpha + 1 = 0$ ***M1***

Note: Award ***M1*** for equivalent statements such as 'assume that α is an integer root of $2\alpha^3 + 6\alpha + 1 = 0$'. Condone the use of x throughout the proof.

Award ***M1*** for an assumption involving $\alpha^3 + 3\alpha + \frac{1}{2} = 0$.

Note: Award ***M0*** for statements such as "let's consider the equation has integer roots…" ,"let $\alpha \in \mathbb{Z}$ be a root of $2\alpha^3 + 6\alpha + 1 = 0$…"

Note: Subsequent marks after this ***M1*** are independent of this ***M1*** and can be awarded.

attempts to rearrange their equation into a suitable form ***M1***

EITHER

$2\alpha^3 + 6\alpha = -1$ ***A1***

$\alpha \in \mathbb{Z} \Rightarrow 2\alpha^3 + 6\alpha$ is even ***R1***

$2\alpha^3 + 6\alpha = -1$ which is not even and so α cannot be an integer ***R1***

Note: Accept '$2\alpha^3 + 6\alpha = -1$ which gives a contradiction'.

OR

$1 = 2\left(-\alpha^3 - 3\alpha\right)$ ***A1***

$\alpha \in \mathbb{Z} \Rightarrow \left(-\alpha^3 - 3\alpha\right) \in \mathbb{Z}$ ***R1***

$\Rightarrow 1$ is even which is not true and so α cannot be an integer ***R1***

Note: Accept '$\Rightarrow 1$ is even which gives a contradiction'.

OR

$\frac{1}{2} = -\alpha^3 - 3\alpha$ ***A1***

$\alpha \in \mathbb{Z} \Rightarrow \left(-\alpha^3 - 3\alpha\right) \in \mathbb{Z}$ ***R1***

$-\alpha^3 - 3\alpha$ is not an integer $\left(=\frac{1}{2}\right)$ and so α cannot be an integer ***R1***

Note: Accept '$-\alpha^3 - 3\alpha$ is not an integer $\left(=\frac{1}{2}\right)$ which gives a contradiction'.

OR

$\alpha = -\frac{1}{2\left(\alpha^2+3\right)}$ ***A1***

$\alpha \in \mathbb{Z} \Rightarrow -\frac{1}{2\left(\alpha^2+3\right)} \in \mathbb{Z}$ ***R1***

$-\frac{1}{2\left(\alpha^2+3\right)}$ is not an integer and so α cannot be an integer ***R1***

Note: Accept $-\frac{1}{2\left(\alpha^2+3\right)}$ is not an integer which gives a contradiction'.

THEN

so the equation $2x^3 + 6x + 1 = 0$ has no integer roots ***AG***

[5 marks]

METHOD 2

assume there exists some $\alpha \in \mathbb{Z}$ such that $2\alpha^3 + 6\alpha + 1 = 0$ ***M1***

Note: Award ***M1*** for statements such as 'assume that α is an integer root of $2\alpha^3 + 6\alpha + 1 = 0$'. Condone the use of x throughout the proof. Award ***M1*** for an assumption involving $\alpha^3 + 3\alpha + \frac{1}{2} = 0$ and award subsequent marks based on this.

Note: Award ***M0*** for statements such as "let's consider the equation has integer roots…", "let $\alpha \in \mathbb{Z}$ be a root of $2\alpha^3 + 6\alpha + 1 = 0$ …"

Note: Subsequent marks after this ***M1*** are independent of this ***M1*** and can be awarded.

let $f(x) = 2x^3 + 6x + 1$ (and $f(\alpha) = 0$)

$f'(x) = 6x^2 + 6 > 0$ for all $x \in \mathbb{R}$ $\Rightarrow f$ is a (strictly) increasing function ***M1A1***

$f(0) = 1$ and $f(-1) = -7$ ***R1***

thus $f(x) = 0$ has only one real root between -1 and 0, which gives a contradiction

(or therefore, contradicting the assumption that $f(\alpha) = 0$ for some $\alpha \in \mathbb{Z}$), ***R1***

so the equation $2x^3 + 6x + 1 = 0$ has no integer roots ***AG***

[5 marks]

35 – (MathAA/12_HL_Summer_2022_Q6) – *Algebra-functions - Roots*

(a) attempts to replace x with $-x$ ***M1***

$f(-x) = -x\sqrt{1-(-x)^2}$

$= -x\sqrt{1-(-x)^2}\ (= -f(x))$ ***A1***

Note: Award ***M1A1*** for an attempt to calculate both $f(-x)$ and $-f(-x)$ independently, showing that they are equal.

Note: Award ***M1A0*** for a graphical approach including evidence that **either** the graph is invariant after rotation by $180°$ about the origin **or** the graph is invariant after a reflection in the y- axis and then in the x- axis (or vice versa).

so f is an odd function ***AG***

[2 marks]

(b) attempts both product rule and chain rule differentiation to find $f'(x)$ ***M1***

$f'(x) = x \times \frac{1}{2} \times (-2x) \times (1-x^2)^{\frac{1}{2}} + (1-x^2)^{\frac{1}{2}} \times 1 \quad \left(= \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}}\right)$ ***A1***

$= \frac{1-2x^2}{\sqrt{1-x^2}}$

sets their $f'(x) = 0$ ***M1***

$\Rightarrow x = \pm\frac{1}{\sqrt{2}}$ ***A1***

attempts to find at least one of $f\left(\pm\frac{1}{\sqrt{2}}\right)$ ***(M1)***

Note: Award ***M1*** for an attempt to evaluate $f(x)$ at least at one of their $f'(x) = 0$ roots.

$a = -\frac{1}{2}$ and $b = \frac{1}{2}$ ***A1***

Note: Award ***A1*** for $-\frac{1}{2} \le y \le \frac{1}{2}$.

[6 marks]

36 – (MathAA/12_HL_Summer_2022_Q2) – *Algebra-functions - Roots*

(a) $(n-1)+n+(n+1)$ ***(A1)***

$=3n$ ***A1***

which is always divisible by 3 ***AG***

[2 marks]

(b) $(n-1)^2+n^2+(n+1)^2$ $(=n^2-2n+1+n^2+n^2+2n+1)$ ***A1***

attempts to expand either $(n-1)^2$ or $(n+1)^2$ (do not accept n^2-1 or n^2+1) ***(M1)***

$=3n^2+2$ ***A1***

demonstrating recognition that 2 is not divisible by 3 or $\frac{2}{3}$ seen after correct expression divided by 3 ***R1***

$3n^2$ is divisible by 3 and so $3n^2+2$ is never divisible by 3

OR the first term is divisible by 3, the second is not

OR $3\left(n^2+\frac{2}{3}\right)$ OR $\frac{3n^2+2}{3}=n^2+\frac{2}{3}$

hence the sum of the squares is never divisible by 3 ***AG***

[4 marks]

37 – (MathAA/11_HL_Summer_2022_Q8) – *Algebra-functions - Roots*

Assume that a and b are both odd. ***M1***

Note: Award ***M0*** for statements such as "let a and b be both odd".
Note: Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

Then $a = 2m+1$ and $b = 2n+1$ ***A1***

$a^2 + b^2 \equiv (2m+1)^2 + (2n+1)^2$

$= 4m^2 + 4m + 1 + 4n^2 + 4n + 1$ ***A1***

$= 4\left(m^2 + m + n^2 + n\right) + 2$ ***(A1)***

($4\left(m^2 + m + n^2 + n\right)$ is always divisible by 4) but 2 is not divisible by 4. (or equivalent) ***R1***

$\Rightarrow a^2 + b^2$ is not divisible by 4, a contradiction. (or equivalent) ***R1***

hence a and b cannot both be odd. ***AG***

Note: Award a maximum of ***M1A0A0(A0)R1R1*** for considering identical or two consecutive odd numbers for a and b.

[6 marks]

38 – (MathAA/11_HL_Summer_2022_Q3) – *Algebra-functions - Roots*

(a) $(f \circ g)(x) = f(2x)$ ***(A1)***

$f(2x) = \sqrt{3}\sin 2x + \cos 2x$ ***A1***

[2 marks]

(b) $\sqrt{3}\sin 2x + \cos 2x = 2\cos 2x$

$\sqrt{3}\sin 2x = \cos 2x$

recognizing to use $\tan$ or $\cot$ ***M1***

$\tan 2x = \frac{1}{\sqrt{3}}$ OR $\cot 2x = \sqrt{3}$ (values may be seen in right triangle) ***(A1)***

$\left(\arctan\left(\frac{1}{\sqrt{3}}\right) =\right) \frac{\pi}{6}$ (seen anywhere) (accept degrees) ***(A1)***

$2x = \frac{\pi}{6}, \frac{7\pi}{6}$

$x = \frac{\pi}{12}, \frac{7\pi}{12}$ ***A1A1***

Note: Do not award the final ***A1*** if any additional solutions are seen. Award ***A1A0*** for correct answers in degrees. Award ***A0A0*** for correct answers in degrees with additional values.

[5 marks]

39 – (MathAA/10_HL_Winter_2022_Q8) – *Trigonometry, Algebra-functions - Roots*

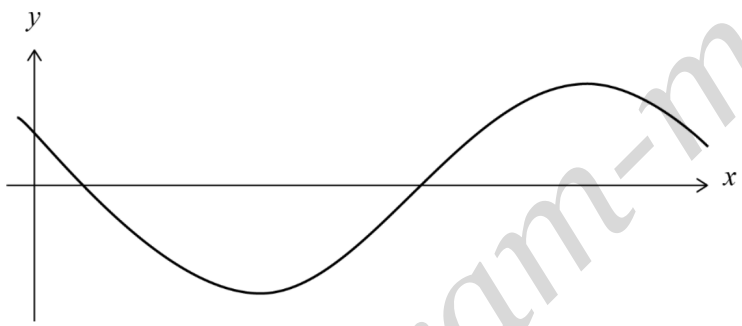
(a)

$a = \frac{\pi}{2}$ ***A2***

Note: For sinusoidal graph through the origin seen with incorrect a, or use of horizontal line test with incorrect a, award ***A1A0***

(b) $a = \pi$ ***A1***

(c)



sketch showing sinusoidal shape decreasing as it crosses the y-axis (below or above the origin) ***(A1)***

$a = k - \pi$ ***A1***

[5 marks]

40 – (MathAA/10_HL_Winter_2022_Q3) – *Algebra-functions - Roots*

(a)

Note: Award a maximum of ***M1A0A0*** if the candidate manipulates both sides of the equation (such as moving terms from one side to the other).

METHOD 1 (working with LHS)

attempting to expand $\left(a^2-1\right)^2$ (do not accept a^4+1 or a^4-1) ***(M1)***

$\text{LHS} = a^2 + \dfrac{a^4-2a^2+1}{4}$ or $\dfrac{4a^2+a^4-2a^2+1}{4}$ ***A1***

$= \dfrac{a^4+2a^2+1}{4}$ ***A1***

$= \left(\dfrac{a^2+1}{2}\right)^2 \ (= \text{RHS})$ ***AG***

Note: Do not award the final A1 if further working contradicts the AG.

METHOD 2 (working with RHS)

attempting to expand $\left(a^2+1\right)^2$ ***(M1)***

$\text{RHS} = \dfrac{a^4+2a^2+1}{4}$

$= \dfrac{4a^2+a^4-2a^2+1}{4}$ ***A1***

$= a^2 + \dfrac{a^4-2a^2+1}{4}$ ***A1***

$= a^2 + \left(\dfrac{a^2-1}{2}\right)^2$ (=LHS) ***AG***

Note: Do not award the final A1 if further working contradicts the AG.

[3 marks]

(b) recognise base and height as a and $\left(\frac{a^2-1}{2}\right)$ (may be seen in diagram) ***(M1)***

correct substitution into triangle area formula ***A1***

$$\text{Area} = \frac{a}{2}\left(\frac{a^2-1}{2}\right) \text{ (or equivalent) } \left(= \frac{a\left(a^2-1\right)}{4} = \frac{a^3-a}{4}\right)$$

[2 marks]

1 – (MAT/11_HL_Summer_2013_Q12) – *Quadratics, Graphs, Integration*

(a) $4(x-0.5)^2+4$

Note: ***A1*** for two correct parameters, ***A2*** for all three correct.

(b) translation $\begin{pmatrix}0.5\\0\end{pmatrix}$ (allow "0.5 to the right")

stretch parallel to y-axis, scale factor 4 (allow vertical stretch or similar)

translation $\begin{pmatrix}0\\4\end{pmatrix}$ (allow "4 up")

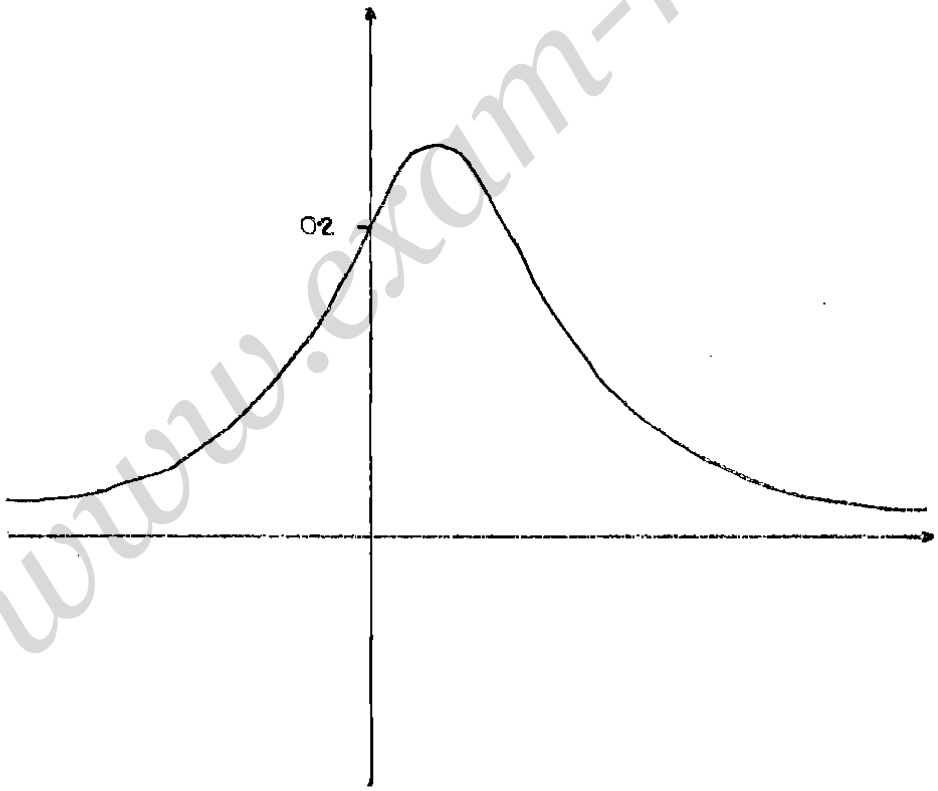
Note: All transformations must state magnitude and direction.

Note: First two transformations can be in either order.

It could be a stretch followed by a single translation

of $\begin{pmatrix}0.5\\4\end{pmatrix}$. If the vertical translation is before the stretch it is $\begin{pmatrix}0\\1\end{pmatrix}$.

(c)



general shape (including asymptote and single maximum in first quadrant),

intercept $\left(0,\frac{1}{5}\right)$ **or** maximum $\left(\frac{1}{2},\frac{1}{4}\right)$ shown

(d) $0 < f(x) \le \frac{1}{4}$

Note: *A1* for $\le \frac{1}{4}$, *A1* for $0 <$.

(e) let $u = x - \frac{1}{2}$

$\frac{du}{dx} = 1$ (or $du = dx$)

$$\int \frac{1}{4x^2 - 4x + 5} dx = \int \frac{1}{4\left(x - \frac{1}{2}\right)^2 + 4} dx$$

$$\int \frac{1}{4u^2 + 4} du = \frac{1}{4} \int \frac{1}{u^2 + 1} du$$

Note: If following through an incorrect answer to part (a), do not award final *A1* mark.

(f) $\int_1^{3.5} \frac{1}{4x^2 - 4x + 5} dx = \frac{1}{4} \int_{0.5}^{3} \frac{1}{u^2 + 1} du$

Note: *A1* for correct change of limits. Award also if they do not change limits but go back to x values when substituting the limit (even if there is an error in the integral).

$$\frac{1}{4}\left[\arctan(u)\right]_{0.5}^{3}$$

$$\frac{1}{4}\left(\arctan(3) - \arctan\left(\frac{1}{2}\right)\right)$$

let the integral $= I$

$$\tan 4I = \tan\left(\arctan(3) - \arctan\left(\frac{1}{2}\right)\right)$$

$$\frac{3 - 0.5}{1 + 3 \times 0.5} = \frac{2.5}{2.5} = 1$$

$$4I = \frac{\pi}{4} \Rightarrow I = \frac{\pi}{16}$$

2 – (MAT/12_HL_Summer_2014_Q4) – *Quadratics*

(a) using the formulae for the sum and product of roots:

$\alpha+\beta=-2$

$\alpha\beta=-\frac{1}{2}$

$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$=(-2)^2-2\left(-\frac{1}{2}\right)$

$=5$

Note: Award ***M0*** for attempt to solve quadratic equation.

(b) $(x-\alpha^2)(x-\beta^2)=x^2-(\alpha^2+\beta^2)x+\alpha^2\beta^2$

$x^2-5x+\left(-\frac{1}{2}\right)^2=0$

$x^2-5x+\frac{1}{4}=0$

3 – (MAT/10_HL_Winter_2014_Q2) – *Quadratics*

(a) using the formulae for the sum and product of roots:

(i) $\alpha + \beta = 4$

(ii) $\alpha\beta = \frac{1}{2}$

Note: Award ***A0A0*** if the above results are obtained by solving the original equation (except for the purpose of checking).

(b) **METHOD 1**

required quadratic is of the form $x^2 - \left(\frac{2}{\alpha} + \frac{2}{\beta}\right)x + \left(\frac{2}{\alpha}\right)\left(\frac{2}{\beta}\right)$

$$q = \frac{4}{\alpha\beta}$$
$$q = 8$$
$$p = -\left(\frac{2}{\alpha} + \frac{2}{\beta}\right)$$
$$= -\frac{2(\alpha+\beta)}{\alpha\beta}$$
$$= -\frac{2 \times 4}{\frac{1}{2}}$$
$$p = -16$$

METHOD 2

replacing x with $\frac{2}{x}$

$$2\left(\frac{2}{x}\right)^2 - 8\left(\frac{2}{x}\right) + 1 = 0$$
$$\frac{8}{x^2} - \frac{16}{x} + 1 = 0$$
$$x^2 - 16x + 8 = 0$$
$p = -16$ and $q = 8$

4 – (MAT/10_HL_Winter_2020_Q6) – *Quadratics, Trigonometry*

METHOD 1

$\sin\theta\cos\theta = \frac{c}{a}$ and $\sin\theta + \cos\theta = -\frac{b}{a}$ ***A1***

attempt to square $\sin\theta + \cos\theta$ ***M1***

$\left(\frac{b^2}{a^2} =\right)(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$ ***A1***

$\frac{b^2}{a^2}(= 1 + 2\sin\theta\cos\theta) = 1 + \frac{2c}{a}$ ***A1***

$b^2 = a^2 + 2ac$ ***AG***

[4 marks]

METHOD 2

$a\sin^2\theta + b\sin\theta + c = 0$ and $a\cos^2\theta + b\cos\theta + c = 0$ ***A1***

adding the two equations ***M1***

$a + b(\sin\theta + \cos\theta) + 2c = 0$ ***A1***

$a + b \times -\frac{b}{a} + 2c = 0$ ***A1***

$a^2 - b^2 + 2ac = 0$

$b^2 = a^2 + 2ac$ ***AG***

[4 marks]

5 - (MathAA/11_SL_Summer_2021_Q7) - *Quadratics*

(a) **METHOD 1 (discriminant)**

$mx^2 - 2mx = mx - 9$ ***(M1)***

$mx^2 - 3mx + 9 = 0$

recognizing $\Delta = 0$ (seen anywhere) ***M1***

$\Delta = (-3m)^2 - 4(m)(9)$ (do not accept only in quadratic formula for x) ***A1***

valid approach to solve quadratic for m ***(M1)***

$9m(m-4) = 0$ OR $m = \dfrac{36 \pm \sqrt{36^2 - 4 \times 9 \times 0}}{2 \times 9}$

both solutions $m = 0, 4$ ***A1***

$m \neq 0$ with a valid reason ***R1***

the two graphs would not intersect OR $0 \neq -9$

$m = 4$ ***AG***

METHOD 2 (equating slopes)

$mx^2 - 2mx = mx - 9$ (seen anywhere) ***(M1)***

$f'(x) = 2mx - 2m$ ***A1***

equating slopes, $f'(x) = m$ (seen anywhere) ***M1***

$2mx - 2m = m$

$x = \dfrac{3}{2}$ ***A1***

substituting their x value ***(M1)***

$\left(\dfrac{3}{2}\right)^2 m - 2m \times \dfrac{3}{2} = m \times \dfrac{3}{2} - 9$

$\dfrac{9}{4}m - \dfrac{12}{4}m = \dfrac{6}{4}m - 9$ ***A1***

$\dfrac{-9m}{4} = -9$

$m = 4$ ***AG***

METHOD 3 (using $\frac{-b}{2a}$)

$mx^2 - 2mx = mx - 9$ ***(M1)***

$mx^2 - 3mx + 9 = 0$

attempt to find x-coord of vertex using $\frac{-b}{2a}$ ***(M1)***

$\frac{-(-3m)}{2m}$ ***A1***

$x = \frac{3}{2}$ ***A1***

substituting their x value ***(M1)***

$\left(\frac{3}{2}\right)^2 m - 3m \times \frac{3}{2} + 9 = 0$

$\frac{9}{4}m - \frac{9}{2}m + 9 = 0$ ***A1***

$-9m = -36$

$m = 4$ ***AG***

[6 marks]

(b) $4x(x-2)$ ***(A1)***

$p = 0$ and $q = 2$ OR $p = 2$ and $q = 0$ ***A1***

[2 marks]

(c) attempt to use valid approach ***(M1)***

$\frac{0+2}{2}$, $\frac{-(-8)}{2\times 4}$, $f(1)$, $8x - 8 = 0$ OR $4(x^2 - 2x + 1 - 1)\left(= 4(x-1)^2 - 4\right)$

$h = 1,\ k = -4$ ***A1A1***

[3 marks]

(d) **EITHER**

recognition $x = h$ to 2 (may be seen on sketch) ***(M1)***

OR

recognition that $f(x) < 0$ and $f'(x) > 0$ ***(M1)***

THEN

$1 < x < 2$ ***A1A1***

Note: Award ***A1*** for two correct values, ***A1*** for correct inequality signs.

[3 marks]

6 – (MathAA/10_SL_Winter_2021_Q1) – *Quadratics*

(a) (i) setting $f(x)=0$ ***(M1)***

$x=1$, $x=-3$ (accept $(1,0),(-3,0)$) ***A1***

(ii) **METHOD 1**

$x=-1$ ***A1***

substituting their x-coordinate into f ***(M1)***

$y=8$ ***A1***

$(-1,8)$

METHOD 2

attempt to complete the square ***(M1)***

$-2\left((x+1)^2-4\right)$

$x=-1$, $y=8$ ***A1A1***

$(-1,8)$

[5 marks]

(b) $h=-1$ ***A1***

$k=8$ ***A1***

[2 marks]

Total [7 marks]

7 – (MathAA/10_HL_Winter_2021_Q7) – *Quadratics*

(a) attempt to use discriminant $b^2 - 4ac(>0)$ ***M1***

$(2p)^2 - 4(3p)(1-p)(>0)$

$16p^2 - 12p(>0)$ ***(A1)***

$p(4p-3)(>0)$

attempt to find critical values $\left(p=0,\ p=\frac{3}{4}\right)$ ***M1***

recognition that discriminant >0 ***(M1)***

$p<0$ or $p>\frac{3}{4}$ ***A1***

Note: Condone 'or' replaced with 'and', a comma, or no separator

[5 marks]

(b) $p=4 \Rightarrow 12x^2+8x-3=0$

valid attempt to use $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ (or equivalent) ***M1***

$x=\frac{-8\pm\sqrt{208}}{24}$

$x=\frac{-2\pm\sqrt{13}}{6}$

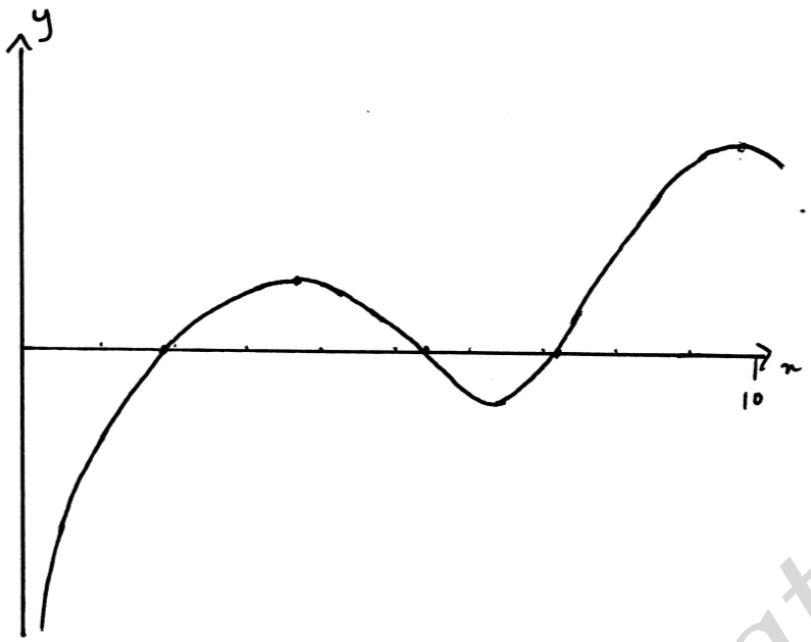
$a=-2$ ***A1***

[2 marks]

Total [7 marks]

1 – (MAT/20_HL_Winter_2013_Q3) – *Graphs, Exponentials - Logarithms*

(a)



A correct graph shape for $0 < x \leq 10$.
maxima $(3.78,\ 0.882)$ and $(9.70,\ 1.89)$
minimum $(6.22,\ -0.885)$
x-axis intercepts $(1.97, 0)$, $(5.24, 0)$ and $(7.11, 0)$

Note: Award ***A1*** if two x-axis intercepts are correct.

(b) $0 < x \leq 1.97$
$5.24 \leq x \leq 7.11$

2 – (MAT/20_HL_Winter_2013_Q13) – *Functions - Roots, Exponentials - Logarithms, Integration*

(a) (i) either counterexample or sketch or
recognising that $y = k$ $(k > 1)$ intersects the graph of $y = f(x)$ twice .
function is not 1–1 (does not obey horizontal line test)
so f^{-1} does not exist

(ii) $f'(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right)$

$f'(\ln 3) = \frac{4}{3}$ $(= 1.33)$

$m = -\frac{3}{4}$

$f(\ln 3) = \frac{5}{3}$ $(= 1.67)$

EITHER

$$\frac{y - \frac{5}{3}}{x - \ln 3} = -\frac{3}{4}$$

$$4y - \frac{20}{3} = -3x + 3\ln 3$$

OR

$$\frac{5}{3} = -\frac{3}{4}\ln 3 + c$$

$$c = \frac{5}{3} + \frac{3}{4}\ln 3$$

$$y = -\frac{3}{4}x + \frac{5}{3} + \frac{3}{4}\ln 3$$

$$12y = -9x + 20 + 9\ln 3$$

THEN

$$9x + 12y - 9\ln 3 - 20 = 0$$

(iii) The tangent at $(a, f(a))$ has equation $y - f(a) = f'(a)(x - a)$.

$f'(a) = \frac{f(a)}{a}$ (or equivalent)

$e^{a} - e^{-a} = \frac{e^{a} + e^{-a}}{a}$ (or equivalent)

attempting to solve for a

$a = \pm 1.20$

(b) (i) $2y = e^x + e^{-x}$

$e^{2x} - 2ye^x + 1 = 0$ ***M1A1***

Note: Award ***M1*** for either attempting to rearrange or interchanging x and y.

$e^x = \dfrac{2y \pm \sqrt{4y^2 - 4}}{2}$ ***A1***

$e^x = y \pm \sqrt{y^2 - 1}$

$x = \ln\left(y \pm \sqrt{y^2 - 1}\right)$ ***A1***

$f^{-1}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$ ***A1***

Note: Award ***A1*** for correct notation and for stating the positive "branch".

(ii) $V = \pi\int_1^5 \left(\ln\left(y + \sqrt{y^2 - 1}\right)\right)^2 \mathrm{d}y$ ***(M1)(A1)***

Note: Award ***M1*** for attempting to use $V = \pi\int_c^d x^2 \mathrm{d}y$.

$= 37.1\ \left(\text{units}^3\right)$ ***A1***

3 – (MAT/22_HL_Summer_2016_Q3) – *Exponentials - Logarithms*

METHOD 1

$\ln\frac{y}{x} = 2 \Rightarrow -\ln x + \ln y = 2$

$\ln x^2 + \ln y^3 = 7 \Rightarrow 2\ln x + 3\ln y = 7$

attempting to solve for x and y (to obtain $\ln x = \frac{1}{5}$ and $\ln y = \frac{11}{5}$)

$x = e^{\frac{1}{5}}\ (= 1.22)$

$y = e^{\frac{11}{5}}\ (= 9.03)$

METHOD 2

$\ln\frac{y}{x} = 2 \Rightarrow y = e^2 x$

$\ln x^2 + \ln e^6 x^3 = 7$

attempting to solve for x

$x = e^{\frac{1}{5}}\ (= 1.22)$

$y = e^{\frac{11}{5}}\ (= 9.03)$

METHOD 3

$\ln\frac{y}{x} = 2 \Rightarrow y = e^2 x$

$\ln x^2 + \ln y^3 = 7 \Rightarrow \ln\left(x^2 y^3\right) = 7$

$x^2 y^3 = e^7$

substituting $y = e^2 x$ into $x^2 y^3 = e^7$ (to obtain $e^6 x^5 = e^7$)

$x = e^{\frac{1}{5}}\ (= 1.22)$

$y = e^{\frac{11}{5}}\ (= 9.03)$

4 – (MAT/21_HL_Summer_2016_Q7) – *Exponentials - Logarithms*

(a) $a = 420.65$

$390.94 = a \times 2^b$

$2^b = \dfrac{390.94}{420.65} = 0.929...$

$b = -0.10567$

(b) $N = 8 \;\; T = 337.67$

Note: Accept 5sf answers between 337.44 and 337.67.

(c) $N = 8$ Percentage error 1.29%

Note: Accept negative values of the above.

(d) likely not to be a good fit for larger values of N
likely to be quite a good fit for values close to 8

5 – (MAT/22_HL_Summer_2017_Q6) – *Exponentials - Logarithms*

$\log_{10} \dfrac{1}{2\sqrt{2}}(p+2q) = \dfrac{1}{2}\left(\log_{10} p + \log_{10} q\right)$

$\log_{10} \dfrac{1}{2\sqrt{2}}(p+2q) = \dfrac{1}{2}\log_{10} pq$ ***(M1)***

$\log_{10} \dfrac{1}{2\sqrt{2}}(p+2q) = \log_{10}(pq)^{\frac{1}{2}}$ ***(M1)***

$\dfrac{1}{2\sqrt{2}}(p+2q) = (pq)^{\frac{1}{2}}$ ***(A1)***

$(p+2q)^2 = 8pq$

$p^2 + 4pq + 4q^2 = 8pq$

$p^2 - 4pq + 4q^2 = 0$

$(p-2q)^2 = 0$ ***M1***

hence $p = 2q$ ***A1***

[5 marks]

6 – (MAT/21_HL_Summer_2019_Q11) – *Exponentials - Logarithms*

(a) recognition of the other root $= -d\mathrm{i}$ ***(A1)***

$\log_2 a + \log_2 b + \log_2 c + d\mathrm{i} - d\mathrm{i} = 3$ ***M1A1***

Note: Award ***M1*** for sum of the roots, ***A1*** for 3. Award ***A0M1A0*** for just $\log_2 a + \log_2 b + \log_2 c = 3$.

$\log_2 abc = 3$ ***(M1)***

$\Rightarrow abc = 2^3$ ***A1***

$abc = 8$ ***AG***

[5 marks]

(b) **METHOD 1**

let the geometric series be u_1, $u_1 r$, $u_1 r^2$

$(u_1 r)^3 = 8$ ***M1***

$u_1 r = 2$ ***A1***

hence one of the roots is $\log_2 2 = 1$ ***R1***

METHOD 2

$\frac{b}{a} = \frac{c}{b}$

$b^2 = ac \Rightarrow b^3 = abc = 8$ ***M1***

$b = 2$ ***A1***

hence one of the roots is $\log_2 2 = 1$ ***R1***

[3 marks]

(c) **METHOD 1**

product of the roots is $r_1 \times r_2 \times 1 \times d\mathrm{i} \times -d\mathrm{i} = -8d^2$ ***(M1)(A1)***

$r_1 \times r_2 = -8$ ***A1***

sum of the roots is $r_1 + r_2 + 1 + d\mathrm{i} + -d\mathrm{i} = 3$ ***(M1)(A1)***

$r_1 + r_2 = 2$ ***A1***

solving simultaneously ***(M1)***

$r_1 = -2$, $r_2 = 4$ ***A1A1***

7 – (MAT/10_HL_Winter_2020_Q5) – *Exponentials - Logarithms, Sequences - Series*

$u_5 = 4 + 4d = \log_2 625$ ***(A1)***

$4d = \log_2 625 - 4$

attempt to write an integer (*eg* 4 or 1) in terms of $\log_2$ ***M1***

$4d = \log_2 625 - \log_2 16$

attempt to combine two logs into one ***M1***

$4d = \log_2\left(\frac{625}{16}\right)$

$d = \frac{1}{4}\log_2\left(\frac{625}{16}\right)$

attempt to use power rule for logs ***M1***

$d = \log_2\left(\frac{625}{16}\right)^{\frac{1}{4}}$

$d = \log_2\left(\frac{5}{2}\right)$ ***A1***

[5 marks]

Note: Award method marks in any order.

8 – (MathAA/10_HL_Winter_2021_Q3) – *Exponentials - Logarithms*

attempt to use change the base ***(M1)***

$$\log_3 \sqrt{x} = \frac{\log_3 2}{2} + \log_3\left(4x^3\right)$$

attempt to use the power rule ***(M1)***

$$\log_3 \sqrt{x} = \log_3 \sqrt{2} + \log_3\left(4x^3\right)$$

attempt to use product or quotient rule for logs, $\ln a + \ln b = \ln ab$ ***(M1)***

$$\log_3 \sqrt{x} = \log_3\left(4\sqrt{2}x^3\right)$$

Note: The ***M*** marks are for attempting to use the relevant log rule and may be applied in any order and at any time during the attempt seen.

$$\sqrt{x} = 4\sqrt{2}x^3$$

$$x = 32x^6$$

$$x^5 = \frac{1}{32}$$ ***(A1)***

$$x = \frac{1}{2}$$ ***A1***

[5 marks]

1 – (MAT/21_HL_Summer_2016_Q5) – *Graphs*

(a) $f(-x) = \dfrac{3(-x)^2 + 10}{(-x)^2 - 4}$ ***A1***

$= \dfrac{3x^2 + 10}{x^2 - 4} = f(x)$

$f(x) = f(-x)$ ***R1***

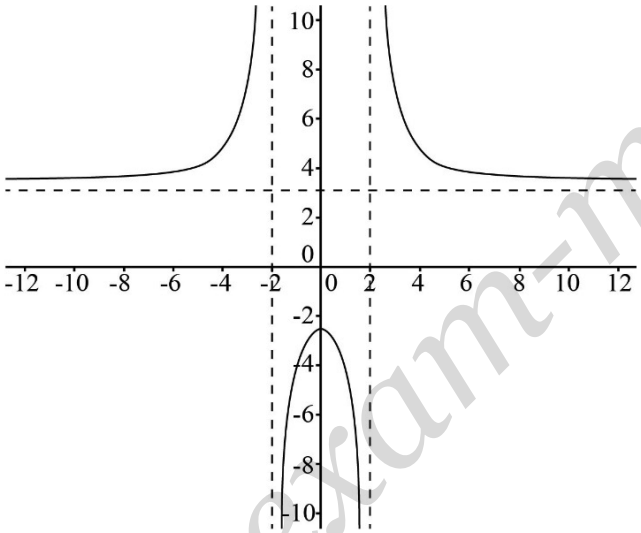
hence this is an even function ***AG***

Note: Award ***A1R1*** for the statement, all the powers are even hence $f(x) = f(-x)$.

Note: Just stating all the powers are even is ***A0R0***.

Note: Do not accept arguments based on the symmetry of the graph.

(b) (i)



correct shape in 3 parts which are asymptotic and symmetrical ***A1***

correct vertical asymptotes clear at 2 and -2 ***A1***

correct horizontal asymptote clear at 3 ***A1***

(ii) $f(x) > 3$

$f(x) \le -2.5$

2 – (MAT/21_HL_Summer_2017_Q6) – *Graphs*

(a) $x = \frac{\pi}{4}$ ***A1***

$x = \frac{5\pi}{4},\ x = -\frac{3\pi}{4}$ ***A1***

[2 marks]

(b) reflection in the y-axis results in $y = \tan\left(-x + \frac{\pi}{4}\right)\left(= \cot\left(x + \frac{\pi}{4}\right)\right)$ ***(A1)***

vertical stretch gives $y = \frac{1}{2}\tan\left(-x + \frac{\pi}{4}\right)\left(= \frac{1}{2}\cot\left(x + \frac{\pi}{4}\right)\right)$ ***(A1)***

translation

$y = \frac{1}{2}\tan\left[-\left(x - \frac{\pi}{4} - \frac{\pi}{4}\right)\right] - 3$

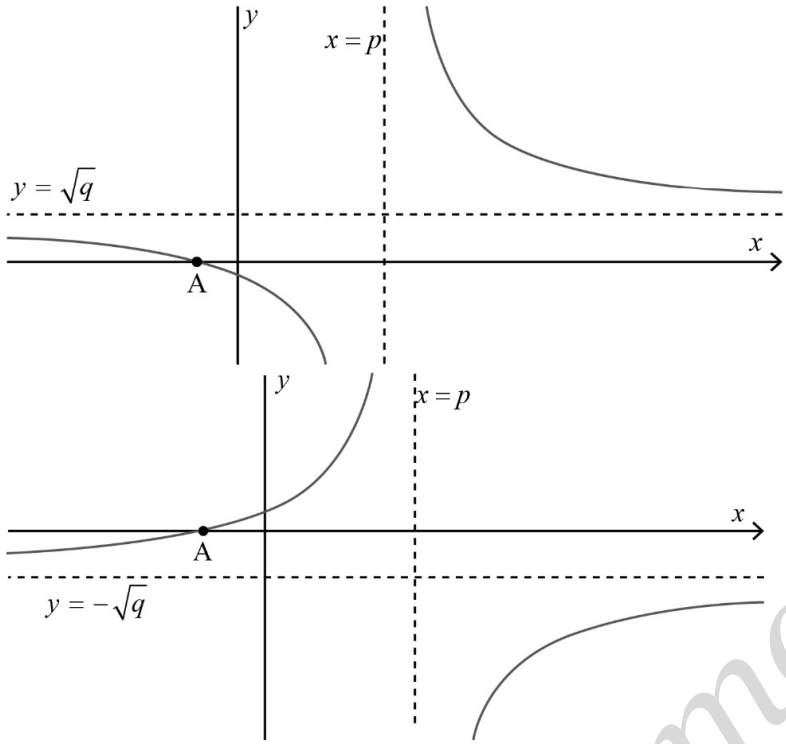
$= \frac{1}{2}\tan\left(-x + \frac{\pi}{2}\right) - 3\left(= \frac{1}{2}\cot(x) - 3\right)$ ***A1A1***

Notes: Award the ***A1***s independently of each other.
Do not penalize the absence of $y =$.

[4 marks]

3 – (MAT/20_HL_Winter_2018_Q8) – *Graphs*

(a)



either graph passing through (or touching) A	***A1***
correct shape and vertical asymptote with correct equation for either graph	***A1***
correct horizontal asymptote with correct equation for either graph	***A1***
two completely correct sketches	***A1***
	[4 marks]

(b) $a\left(-\frac{1}{2}\right)+1=0 \Rightarrow a=2$ ***A1***

from horizontal asymptote, $\left(\frac{a}{b}\right)^2=\frac{4}{9}$ ***(M1)***

$\frac{a}{b}=\pm\frac{2}{3} \Rightarrow b=\pm 3$ ***A1***

from vertical asymptote, $b\left(\frac{4}{3}\right)+c=0$

$b=3,\ c=-4$ or $b=-3,\ c=4$ ***A1***

[4 marks]

4 – (MAT/22_HL_Summer_2019_Q9) – *Graphs*

(a) **METHOD 1**

attempt to find roots or factors ***(M1)***

roots are $-3,\ 1,\ (4+\mathrm{i}),(4-\mathrm{i})$ ***A1A1***

Note: Award ***A1*** for each pair of roots or factors, real and complex.

attempt to form quadratic ***M1***

$(z-1)(z+3)=z^2+2z-3$ ***A1***

$(z-(4+\mathrm{i}))(z-(4-\mathrm{i}))$

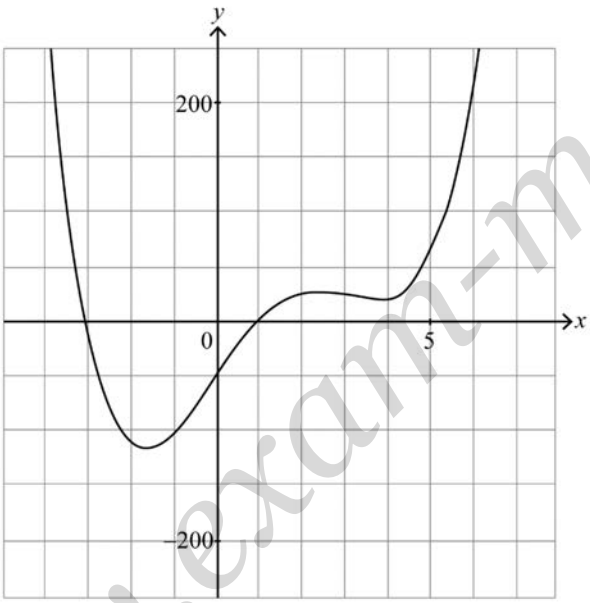
$=z^2-(4-\mathrm{i})z-(4+\mathrm{i})z+17$ ***(A1)***

$=z^2-8z+17$ ***A1***

$z^4-6z^3-2z^2+58z-51=(z^2-8z+17)(z^2+2z-3)$

[7 marks]

(b)



shape ***A1***

x-axis intercepts at $(-3,\ 0)$, $(1,\ 0)$ and y-axis intercept at $(0,\ -51)$ ***A1A1***

minimum points at $(-1.62,\ -118)$ and $(3.72,\ 19.7)$ ***A1A1***

maximum point at $(2.40,\ 26.9)$ ***A1***

Note: Coordinates may be seen on the graph or elsewhere.

Note: Accept -3, 1 and -51 marked on the axes.

[6 marks]

(c) from graph, $19.7 \le k \le 26.9$ ***A1A1***

Note: Award ***A1*** for correct endpoints and ***A1*** for correct inequalities.

[2 marks]

5 – (MAT/20_HL_Winter_2019_Q3) – *Graphs*

(a) $f(1)=0$ ***(A1)***

$f(0)=-1$ ***A1***

[2 marks]

(b) $a=f(3)$ ***(M1)***

$\Rightarrow a=4$ ***A1***

[2 marks]

(c) domain is $-2\le x\le 6$ ***A1***

range is $-6\le y\le 10$ ***A1***

[2 marks]

6 – (MAT/10_HL_Winter_2020_Q10) – *Differentiation, Graphs*

(a) (i) $f'(x)=3ax^2+2bx+c$ ***A1***

(ii) since $f^{\prime}$ does not exist, there must be two turning points ***R1***

($\Rightarrow f'(x)=0$ has more than one solution)

using the discriminant $\Delta>0$ ***M1***

$4b^2-12ac>0$ ***A1***

$b^2-3ac>0$ ***AG***

[4 marks]

(b) (i) **METHOD 1**

$b^2-3ac=(-3)^2-3\times\frac{1}{2}\times 6$ ***M1***

$=9-9$

$=0$ ***A1***

hence $g^{\prime}$ exists ***AG***

METHOD 2

$g'(x)=\frac{3}{2}x^2-6x+6$ ***M1***

$\Delta=(-6)^2-4\times\frac{3}{2}\times 6$

$\Delta=36-36=0\Rightarrow$ there is (only) one point with gradient of 0 and this must be a point of inflexion (since $g(x)$ is a cubic.) ***R1***

hence g^{-1} exists ***AG***

(ii) $p=\frac{1}{2}$ ***A1***

$(x-2)^3=x^3-6x^2+12x-8$ ***(M1)***

$\frac{1}{2}(x^3-6x^2+12x-8)=\frac{1}{2}x^3-3x^2+6x-4$

$g(x)=\frac{1}{2}(x-2)^3-4\Rightarrow q=-4$ ***A1***

(iii) $x=\frac{1}{2}(y-2)^3-4$ ***(M1)***

Note: Interchanging x and y can be done at any stage.

$2(x+4)=(y-2)^3$ ***(M1)***

$\sqrt[3]{2(x+4)}=y-2$

$y=\sqrt[3]{2(x+4)}+2$

$g^{-1}(x)=\sqrt[3]{2(x+4)}+2$ ***A1***

Note: $g^{-1}(x)=\ldots$ must be seen for the final ***A*** mark.

[8 marks]

(c) translation through $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, ***A1***

Note: This can be seen anywhere.

EITHER

a stretch scale factor $\frac{1}{2}$ parallel to the y-axis then a translation through $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$ ***A2***

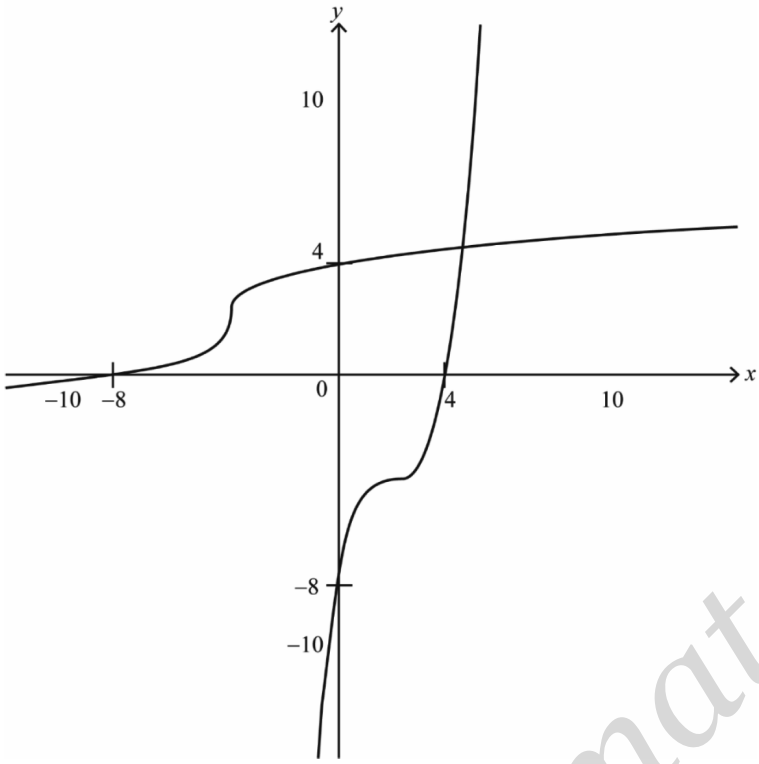
OR

a translation through $\begin{pmatrix} 0 \\ -8 \end{pmatrix}$ then a stretch scale factor $\frac{1}{2}$ parallel to the y-axis ***A2***

Note: Accept 'shift' for translation, but do not accept 'move'. Accept 'scaling' for 'stretch'.

[3 marks]

(d)



A1A1A1
M1A1

Note: Award ***A1*** for correct 'shape' of g (allow non-stationary point of inflexion)
Award ***A1*** for each correct intercept of g
Award ***M1*** for attempt to reflect their graph in $y = x$, ***A1*** for completely correct g^{-1} including intercepts

[5 marks]

7 – (MAT/10_HL_Winter_2020_Q12) – *Graphs, Integration*

(a) $x = k$ ***A1***

[1 mark]

(b) $y = k$ ***A1***

[1 mark]

(c) **METHOD 1**

$$(f \circ f)(x) = \frac{k\left(\frac{kx-5}{x-k}\right)-5}{\left(\frac{kx-5}{x-k}\right)-k}$$ ***M1***

$$= \frac{k(kx-5)-5(x-k)}{kx-5-k(x-k)}$$ ***A1***

$$= \frac{k^2x-5k-5x+5k}{kx-5-kx+k^2}$$

$$= \frac{k^2x-5x}{k^2-5}$$ ***A1***

$$= \frac{x(k^2-5)}{k^2-5}$$

$$= x$$

$(f \circ f)(x) = x$, (hence f is self-inverse) ***R1***

Note: The statement $f(f(x)) = x$ could be seen anywhere in the candidate's working to award ***R1***.

[4 marks]

METHOD 2

$f(x)=\dfrac{kx-5}{x-k}$

$x=\dfrac{ky-5}{y-k}$ ***M1***

Note: Interchanging x and y can be done at any stage.

$x(y-k)=ky-5$ ***A1***

$xy-xk=ky-5$

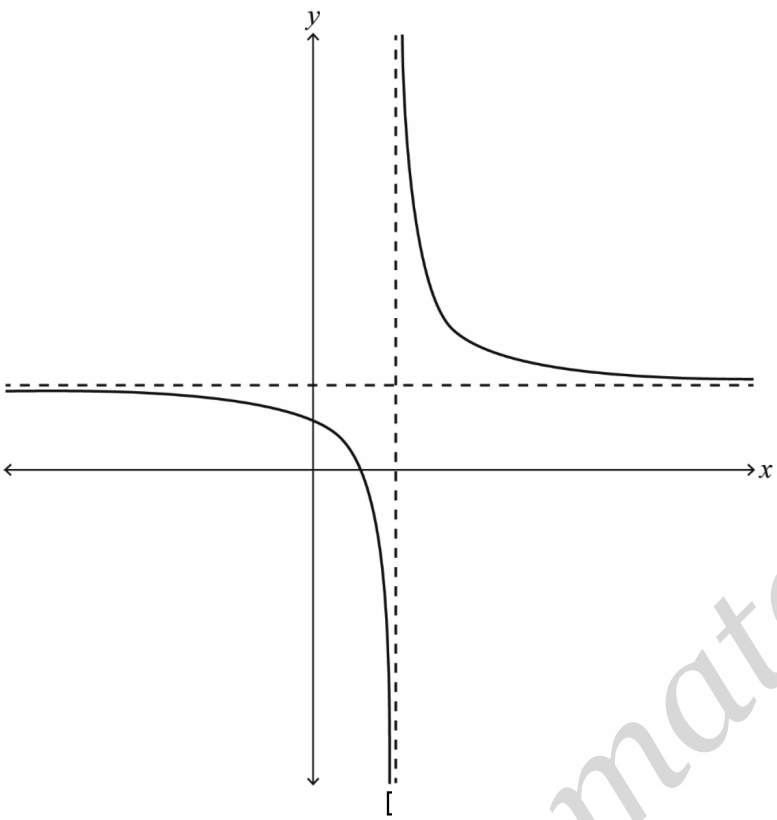
$xy-ky=xk-5$

$y(x-k)=kx-5$ ***A1***

$y=f^{-1}(x)=\dfrac{kx-5}{x-k}$ (hence f is self-inverse.) ***R1***

[4 marks]

(d)



attempt to draw both branches of a rectangular hyperbola ***M1***

$x = 3$ and $y = 3$ ***A1***

$\left(0, \frac{5}{3}\right)$ and $\left(\frac{5}{3}, 0\right)$ ***A1***

[3 marks]

(e) **METHOD 1**

$$\text{volume} = \pi\int_5^7 \left(\frac{3x-5}{x-3}\right)^2 \mathrm{d}x$$ ***(M1)***

EITHER

attempt to express $\frac{3x-5}{x-3}$ in the form $p+\frac{q}{x-3}$ ***M1***

$$\frac{3x-5}{x-3} = 3+\frac{4}{x-3}$$ ***A1***

OR

attempt to expand $\left(\frac{3x-5}{x-3}\right)^2$ or $(3x-5)^2$ and divide out ***M1***

$$\left(\frac{3x-5}{x-3}\right)^2 = 9+\frac{24x-56}{(x-3)^2}$$ ***A1***

THEN

$$\left(\frac{3x-5}{x-3}\right)^2 = 9+\frac{24}{x-3}+\frac{16}{(x-3)^2}$$ ***A1***

$$\text{volume} = \pi\int_5^7 \left(9+\frac{24}{x-3}+\frac{16}{(x-3)^2}\right)\mathrm{d}x$$

$$= \pi\left[9x+24\ln(x-3)-\frac{16}{x-3}\right]_5^7$$ ***A1***

$$= \pi\left[(63+24\ln 4-4)-(45+24\ln 2-8)\right]$$

$$= \pi(22+24\ln 2)$$ ***A1***

[6 marks]

METHOD 2

$\text{volume} = \pi\int_5^7 \left(\frac{3x-5}{x-3}\right)^2 \mathrm{d}x$ ***(M1)***

substituting $u = x-3 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$ ***A1***

$3x-5 = 3(u+3)-5 = 3u+4$

$\text{volume} = \pi\int_2^4 \left(\frac{3u+4}{u}\right)^2 \mathrm{d}u$ ***M1***

$= \pi\int_2^4 9 + \frac{16}{u^2} + \frac{24}{u}\,\mathrm{d}u$ ***A1***

$= \pi\left[9u - \frac{16}{u} + 24\ln u\right]_2^4$ ***A1***

Note: Ignore absence of or incorrect limits seen up to this point.

$= \pi(22 + 24\ln 2)$ ***A1***

[6 marks]

8 – (MathAA/10_HL_Winter_2021_Q2) – *Algebra-functions - Roots, Graphs*

(a) (i) $x = 3$ ***A1***

(ii) $y = -2$ ***A1***

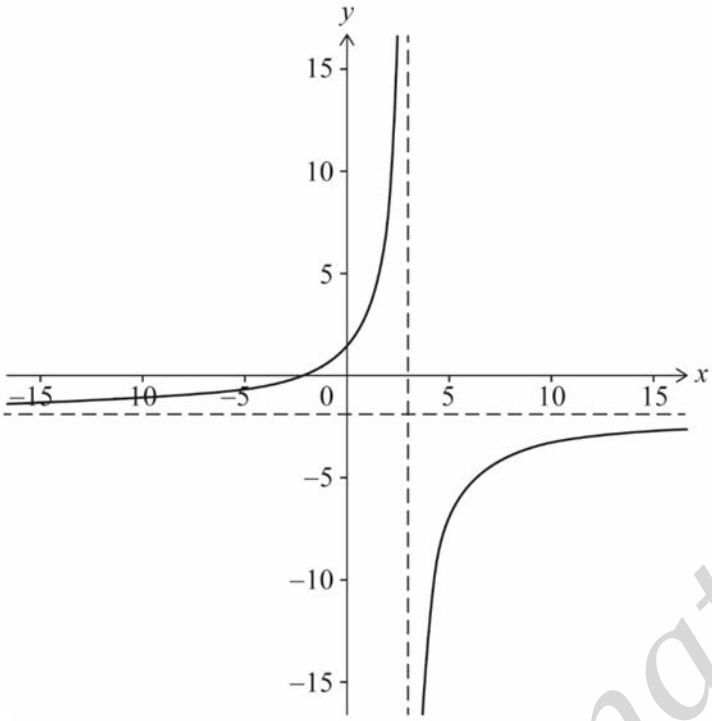
[2 marks]

(b) (i) $(-2, 0)$ (accept $x = -2$) ***A1***

(ii) $\left(0, \frac{4}{3}\right)$ (accept $y = \frac{4}{3}$ and $f(0) = \frac{4}{3}$) ***A1***

[2 marks]

(c)



A1

Note: Award ***A1*** for completely correct shape: two branches in correct quadrants with asymptotic behaviour.

[1 mark]

(d) **METHOD 1**

$(g(x)=)y=\dfrac{ax+4}{3-x}$

attempt to find x in terms of y ***(M1)***

OR exchange x and y and attempt to find y in terms of x

$3y-xy=ax+4$ ***A1***

$ax+xy=3y-4$

$x(a+y)=3y-4$

$x=\dfrac{3y-4}{y+a}$

$g^{-1}(x)=\dfrac{3x-4}{x+a}$ ***A1***

Note: Condone use of $y=$

$g(x)\equiv g^{-1}(x)$

$\dfrac{ax+4}{3-x}\equiv\dfrac{3x-4}{x+a}$

$\Rightarrow a=-3$ ***A1***

[4 marks]

METHOD 2

$g(x)=\frac{ax+4}{3-x}$

attempt to find an expression for $g(g(x))$ and equate to x ***(M1)***

$gg(x)=\frac{a\left(\frac{ax+4}{3-x}\right)+4}{3-\left(\frac{ax+4}{3-x}\right)}=x$ ***A1***

$\frac{a(ax+4)+4(3-x)}{(9-3x)-(ax+4)}=x$

$\frac{a(ax+4)+4(3-x)}{5-(3+a)x}=x$

$a(ax+4)+4(3-x)=x(5-(3+a)x)$ ***A1***

equating coefficients of x^2 (or similar)

$a=-3$ ***A1***

[4 marks]

Total [9 marks]

9 – (MathAA/20_HL_Winter_2021_Q10) – *Algebra-functions - Roots, Graphs, Integration*

(a) (i)

Note: In part (a), penalise once only, if correct values are given instead of correct coordinates.

attempts to solve $x^2 - x - 12 = 0$ ***(M1)***

$(-3, 0)$ and $(4, 0)$ ***A1***

(ii) $\left(0, \frac{4}{5}\right)$ ***A1***

[3 marks]

(b) $x = \frac{15}{2}$ ***A1***

Note: Award ***A0*** for $x \neq \frac{15}{2}$.

Award ***A1*** in part (b), if $x = \frac{15}{2}$ is seen on their graph in part (d).

[1 mark]

(c) **METHOD 1**

$(ax+b)(2x-15) \equiv x^2 - x - 12$

attempts to expand $(ax+b)(2x-15)$ ***(M1)***

$2ax^2 - 15ax + 2bx - 15b \equiv x^2 - x - 12$

$a = \frac{1}{2}$ ***A1***

equates coefficients of x ***(M1)***

$-1 = -\frac{15}{2} + 2b$

$b = \frac{13}{4}$ ***A1***

$\left(y = \frac{x}{2} + \frac{13}{4}\right)$

METHOD 2

attempts division on $\frac{x^2-x-12}{2x-15}$ ***M1***

$\frac{x}{2}+\frac{13}{4}+...$ ***M1***

$a=\frac{1}{2}$ ***A1***

$b=\frac{13}{4}$ ***A1***

$\left(y=\frac{x}{2}+\frac{13}{4}\right)$

METHOD 3

$a=\frac{1}{2}$ ***A1***

$\frac{x^2-x-12}{2x-15}\equiv\frac{x}{2}+b+\frac{c}{2x-15}$ ***M1***

$x^2-x-12\equiv\frac{(2x-15)x}{2}+(2x-15)b+c$

equates coefficients of x : ***(M1)***

$-1=-\frac{15}{2}+2b$

$b=\frac{13}{4}$ ***A1***

$\left(y=\frac{x}{2}+\frac{13}{4}\right)$

METHOD 4

attempts division on $\frac{x^2-x-12}{2x-15}$ ***M1***

$\frac{x^2-x-12}{2x-15}=\frac{x}{2}+\frac{\frac{13x}{2}-12}{2x-15}$

$a=\frac{1}{2}$ ***A1***

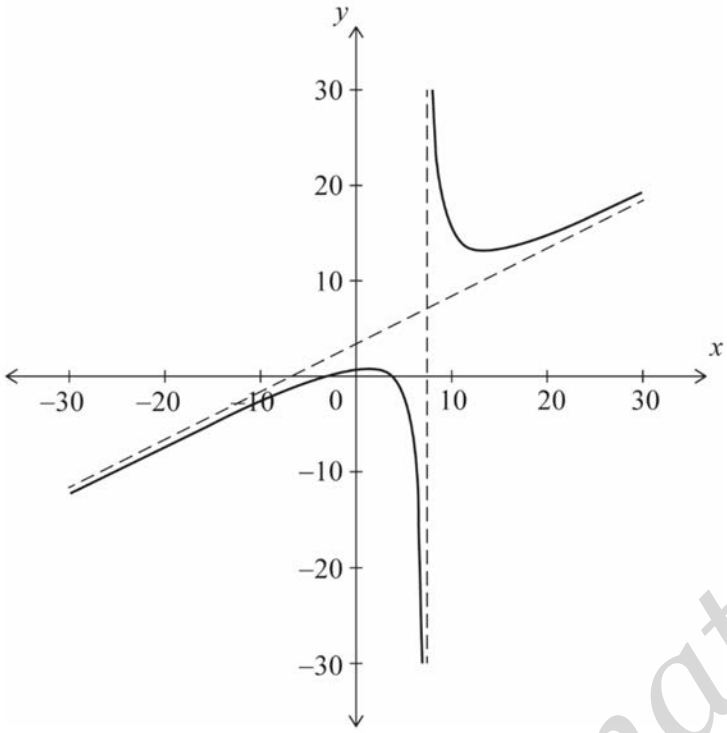
$\frac{\frac{13x}{2}-12}{2x-15}=\frac{13}{4}+...$ ***M1***

$b=\frac{13}{4}$ ***A1***

$\left(y=\frac{x}{2}+\frac{13}{4}\right)$

[4 marks]

(d)



two branches with approximately correct shape (for $-30 \le x \le 30$) ***A1***

their vertical and oblique asymptotes in approximately correct positions with both branches showing correct asymptotic behaviour to these asymptotes ***A1***

their axes intercepts in approximately the correct positions ***A1***

Note: Points of intersection with the axes and the equations of asymptotes are not required to be labelled.

[3 marks]

(e) (i) attempts to split into partial fractions: ***(M1)***

$$\frac{2x-15}{(x+3)(x-4)} \equiv \frac{A}{x+3} + \frac{B}{x-4}$$

$$2x-15 \equiv A(x-4)+B(x+3)$$

$A=3$ ***A1***

$B=-1$ ***A1***

$$\left(\frac{3}{x+3} - \frac{1}{x-4}\right)$$

(ii) $\int_0^3 \left(\frac{3}{x+3} - \frac{1}{x-4}\right) \mathrm{d}x$

attempts to integrate and obtains two terms involving 'ln' ***(M1)***

$= \left[3\ln|x+3| - \ln|x-4|\right]_0^3$ ***A1***

$= 3\ln 6 - \ln 1 - 3\ln 3 + \ln 4$ ***A1***

$= 3\ln 2 + \ln 4 \ (= \ln 8 + \ln 4)$

$= \ln 32 \ (= 5\ln 2)$ ***A1***

Note: The final ***A1*** is dependent on the previous two ***A*** marks.

[7 marks]

Total [18 marks]

10 – (MathAA/21_HL_Summer_2021_Q11) – *Algebra-functions - Roots, Graphs*

(a) attempt to solve $4x^2 - 1 = 0$ e.g. by factorising $4x^2 - 1$ ***(M1)***

$p = \frac{1}{2}, q = -\frac{1}{2}$ or vice versa ***A1***

[2 marks]

(b) attempt to use quotient rule or product rule ***(M1)***

EITHER

$$f'(x) = \frac{3(4x^2-1) - 8x(3x+2)}{(4x^2-1)^2} \left(= \frac{-12x^2 - 16x - 3}{(4x^2-1)^2} \right)$$ ***A1A1***

Note: Award ***A1*** for each term in the numerator with correct signs, provided correct denominator is seen.

OR

$$f'(x) = -8x(3x+2)(4x^2-1)^{-2} + 3(4x^2-1)^{-1}$$ ***A1A1***

Note: Award ***A1*** for each term.

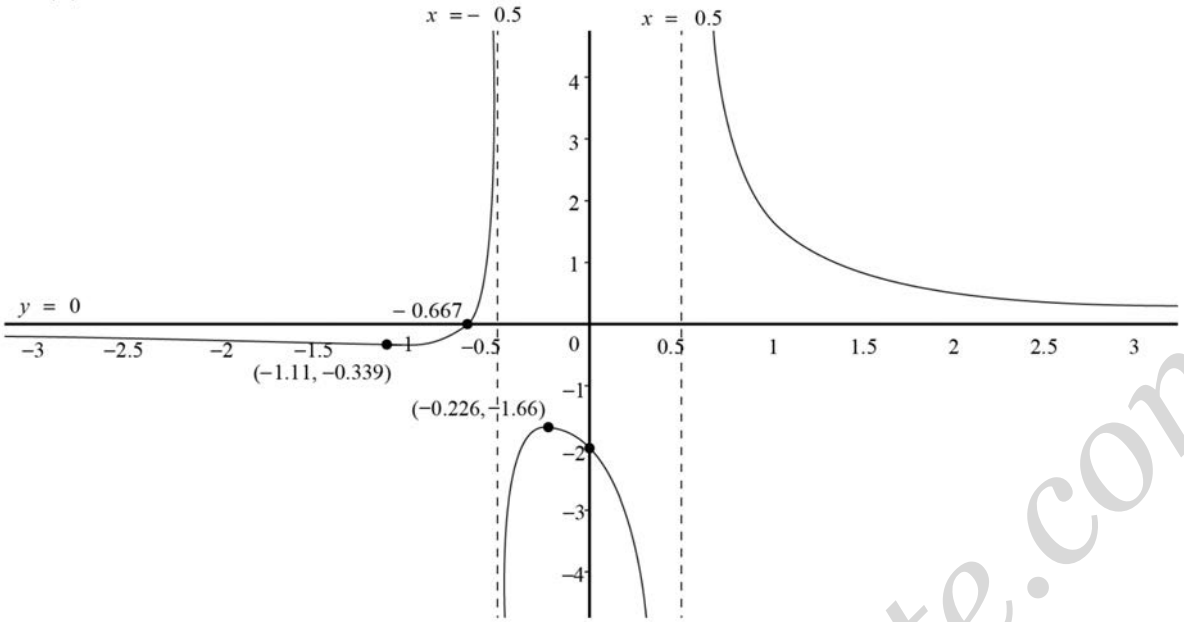
[3 marks]

(c) attempt to find the local min point on $y = f'(x)$ OR solve $f''(x) = 0$ ***(M1)***

$x = -1.60$ ***A1***

[2 marks]

(d)



A1A1A1A1A1

Note: Award ***A1*** for both vertical asymptotes with their equations, award ***A1*** for horizontal asymptote with equation, award ***A1*** for each correct branch including asymptotic behaviour, coordinates of minimum and maximum points (may be seen next to the graph) and values of axes intercepts.
If vertical asymptotes are absent (or not vertical) and the branches overlap as a consequence, award maximum ***A0A1A0A1A1.***

[5 marks]

e) $x = -\frac{2}{3}\left(= -0.667\right)$ ***A1***

(oblique asymptote has) gradient $\frac{4}{3}\left(= 1.33\right)$ ***(A1)***

appropriate method to find complete equation of oblique asymptote ***M1***

$$\begin{array}{r} \frac{4}{3}x - \frac{8}{9} \\ 3x+2\overline{\big)\,4x^2 + 0x - 1} \\ \underline{4x^2 + \frac{8}{3}x} \\ -\frac{8}{3}x - 1 \\ \underline{-\frac{8}{3}x - \frac{16}{9}} \\ \frac{7}{9} \end{array}$$

$y = \frac{4}{3}x - \frac{8}{9}\left(= 1.33x - 0.889\right)$ ***A1***

Note: Do not award the final ***A1*** if the answer is not given as an equation.

[4 marks]

f) attempting to find at least one critical value $\left(x = -0.568729..., x = 1.31872...\right)$ ***(M1)***

$-\frac{2}{3} < x < -0.569$ OR $-0.5 < x < 0.5$ OR $x > 1.32$ ***A1A1A1***

Note: Only penalize once for use of $\leq$ rather than $<$.

[4 marks]

11 – (MathAA/11_HL_Summer_2022_Q5) – *Graphs*

(a) translation (shift) by $\frac{3\pi}{2}$ to the right OR positive horizontal direction by $\frac{3\pi}{2}$ ***A1***

translation (shift) by q upwards OR positive vertical direction by q ***A1***

Note: accept translation by $\begin{pmatrix} \frac{3\pi}{2} \\ q \end{pmatrix}$

Do not accept 'move' for translation/shift.

[2 marks]

(b) **METHOD 1**

minimum of $4\sin\left(x-\frac{3\pi}{2}\right)$ is -4 (may be seen in sketch) ***(M1)***

$-4+2.5+q \geq 7$

$q \geq 8.5$ (accept $q=8.5$) ***A1***

substituting $x=0$ and their $q(=8.5)$ to find r ***(M1)***

$(r=)\quad 4\sin\left(\frac{-3\pi}{2}\right)+2.5+8.5$

$4+2.5+8.5$ ***(A1)***

smallest value of r is 15 ***A1***

METHOD 2

substituting $x=0$ to find an expression (for r) in terms of q ***(M1)***

$\left(g(0)=r=\right)\quad 4\sin\left(\frac{-3\pi}{2}\right)+2.5+q$

$\left(r=\right)\ 6.5+q$ ***A1***

minimum of $4\sin\left(x-\frac{3\pi}{2}\right)$ is -4 ***(M1)***

$-4+2.5+q\geq 7$

$-4+2.5+(r-6.5)\geq 7$ (accept =) ***(A1)***

smallest value of r is 15 ***A1***

METHOD 3

$4\sin\left(x-\frac{3\pi}{2}\right)+2.5+q=4\cos x+2.5+q$ ***A1***

y-intercept of $4\cos x+2.5+q$ is a maximum ***(M1)***

amplitude of $g(x)$ is 4 ***(A1)***

attempt to find least maximum ***(M1)***

$r=2\times 4+7$

smallest value of r is 15 ***A1***

[5 marks]

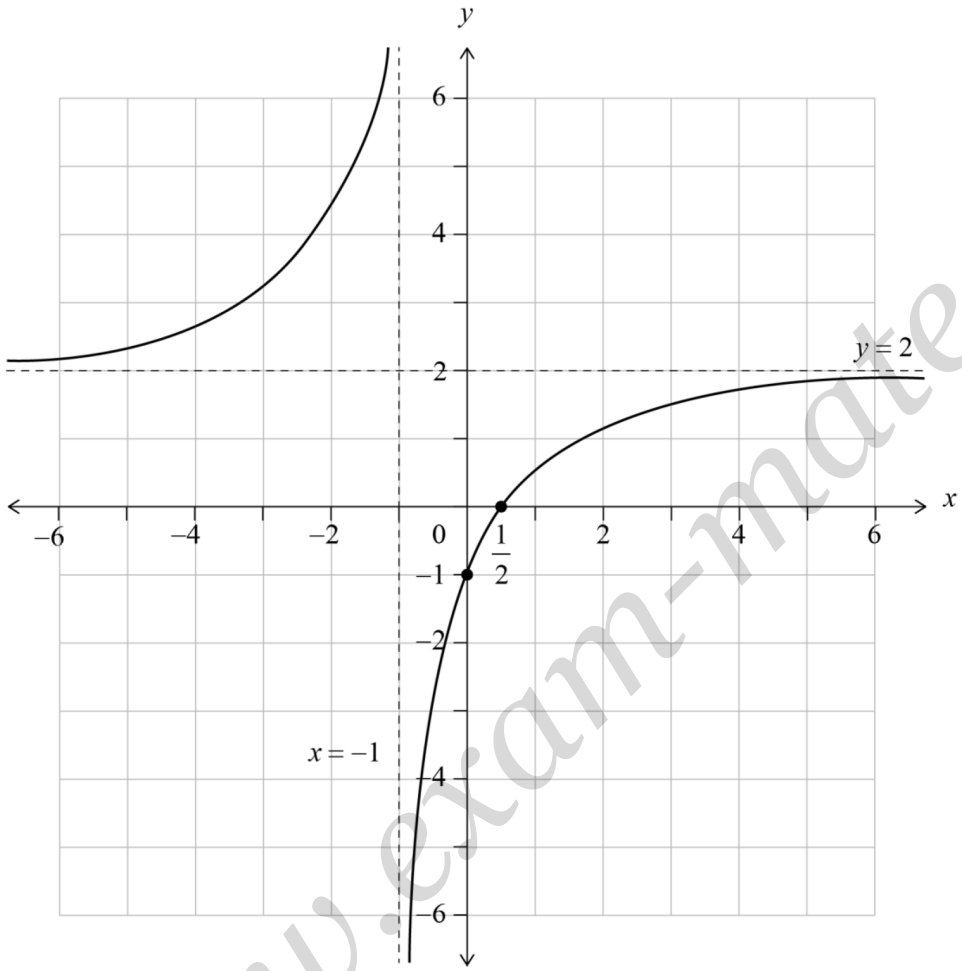
12 – (MathAA/12_HL_Summer_2022_Q3) – *Graphs*

(a) (i) $x = -1$ ***A1***

(ii) $y = 2$ ***A1***

[2 marks]

(b)



rational function shape with two branches in opposite quadrants, with two correctly positioned asymptotes and asymptotic behaviour shown ***A1***

axes intercepts clearly shown at $x = \frac{1}{2}$ and $y = -1$ ***A1A1***

[3 marks]

(c) $x > \frac{1}{2}$ ***A1***

Note: Accept correct alternative correct notation, such as $\left(\frac{1}{2}, \infty\right)$ and $\left]\frac{1}{2}, \infty\right[$.

[1 mark]

(d) **EITHER**

attempts to sketch $y = \frac{2|x|-1}{|x|+1}$ ***(M1)***

OR

attempts to solve $2|x|-1=0$ ***(M1)***

Note: Award the ***(M1)*** if $x = \frac{1}{2}$ and $x = -\frac{1}{2}$ are identified.

THEN

$x < -\frac{1}{2}$ or $x > \frac{1}{2}$ ***A1***

Note: Accept the use of a comma. Condone the use of 'and'. Accept correct alternative notation.

[2 marks]

13 – (MathAA/10_HL_Winter_2022_Q10) – *Differentiation, Graphs*

(a) $\cos^2 x - 3\sin^2 x = 0$

valid attempt to reduce equation to one involving one trigonometric function ***(M1)***

$\frac{\sin^2 x}{\cos^2 x} = \frac{1}{3}$ OR $1 - \sin^2 x - 3\sin^2 x = 0$ OR $\cos^2 x - 3\left(1 - \cos^2 x\right) = 0$

OR $\cos 2x - 1 + \cos 2x = 0$

correct equation ***(A1)***

$\tan^2 x = \frac{1}{3}$ OR $\cos^2 x = \frac{3}{4}$ OR $\sin^2 x = \frac{1}{4}$ OR $\cos 2x = \frac{1}{2}$

$\tan x = \pm\frac{1}{\sqrt{3}}$ OR $\cos x = \pm\frac{\sqrt{3}}{2}$ OR $\sin x = (\pm)\frac{1}{2}$ OR $2x = \frac{\pi}{3}\left(, \frac{5\pi}{3}\right)$ ***(A1)***

$x = \frac{\pi}{6},\ x = \frac{5\pi}{6}$ ***A1A1***

Note: Award ***M1A1A0A1A0*** for candidates who omit the $\pm$ (for tan or cos) and give only $x = \frac{\pi}{6}$.

Award ***M1A1A0A0A0*** for candidates who omit the $\pm$ (for tan or cos) and give only $x = 30°$.

Award ***M1A1A1A1A0*** for candidates who give both answers in degrees.

Award ***M1A1A1A1A0*** for candidates who give both correct answers in radians, but who include additional solutions outside the domain.

Award a maximum of ***M1A0A0A1A1*** for correct answers with no working.

[5 marks]

(b) (i) attempt to use the chain rule (may be evidenced by at least one $\cos x \sin x$ term) ***(M1)***

$f'(x) = -2\cos x \sin x - 6\sin x \cos x \left(= -8\sin x \cos x = -4\sin 2x\right)$ ***A1***

(ii) valid attempt to solve their $f'(x) = 0$ ***(M1)***

at least 2 correct x-coordinates (may be seen in coordinates) ***(A1)***

$x = 0$, $x = \frac{\pi}{2}$, $x = \pi$

Note: Accept additional correct solutions outside the domain. Award ***A0*** if any additional incorrect solutions are given.

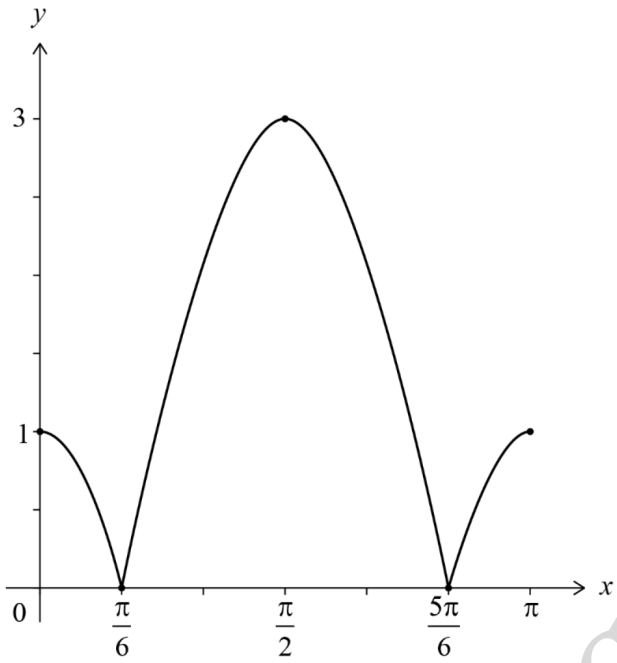
correct coordinates (may be seen in graph for part (c)) ***A1A1A1***

$(0,1)$, $(\pi,1)$, $\left(\frac{\pi}{2},-3\right)$

Note: Award a maximum of ***M1A1A1A1A0*** if any additional solutions are given. **Note**: If candidates do not find at least two correct x-coordinates, it is possible to award the appropriate final marks for their correct coordinates, such as ***M1A0A0A1A0.***

[7 marks]

(c)



attempt to reflect the negative part of the graph of f in the x-axis ***M1***

endpoints have coordinates $(0,1)$, $(\pi,1)$ ***A1***

smooth maximum at $\left(\frac{\pi}{2},3\right)$ ***A1***

sharp points (cusps) at x-intercepts $\frac{\pi}{6}$, $\frac{5\pi}{6}$ ***A1***

[4 marks]

(d) considers points of intersection of $y=\left|f(x)\right|$ and $y=1$ on graph or algebraically ***(M1)***

$-\left(\cos^2 x-3\sin^2 x\right)=1$ or $-\left(1-4\sin^2 x\right)=1$ or $-\left(4\cos^2 x-3\right)=1$ or $-\left(2\cos 2x-1\right)=1$

$\tan^2 x=1$ or $\sin^2 x=\frac{1}{2}$ or $\cos^2 x=\frac{1}{2}$ or $\cos 2x=0$ ***(A1)***

$x=\frac{\pi}{4},\frac{3\pi}{4}$ ***(A1)***

For $\left|f(x)\right|>1$

$\frac{\pi}{4}<x<\frac{3\pi}{4}$ ***A1***

[4 marks]

1 - (MAT/11_HL_Summer_2012_Q3) - *Complex Numbers*

$$z_1 = 2a\,\text{cis}\left(\frac{\pi}{3}\right),\ z_2 = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$$

EITHER

$$\left(\frac{z_1}{z_2}\right)^6 = \frac{2^6 a^6 \text{cis}(0)}{\sqrt{2}^6 \text{cis}\left(\frac{\pi}{2}\right)} \quad \left(= 8a^6 \text{cis}\left(-\frac{\pi}{2}\right)\right)$$

OR

$$\left(\frac{z_1}{z_2}\right)^6 = \left(\frac{2a}{\sqrt{2}}\text{cis}\left(\frac{7\pi}{12}\right)\right)^6$$

$$= 8a^6 \text{cis}\left(-\frac{\pi}{2}\right)$$

THEN

$$= -8a^6 \text{i}$$

2 - (MAT/12_HL_Summer_2012_Q6) - *Complex Numbers*

(a) attempt to equate real and imaginary parts
equate real parts: $4m + 4n = 16$; equate imaginary parts: $-5m = 15$
$\Rightarrow m = -3,\ n = 7$

(b) let $m = x + \text{i}y,\ n = x - \text{i}y$
$\Rightarrow (4 - 5\text{i})(x + \text{i}y) + 4(x - \text{i}y) = 16 + 15\text{i}$
$\Rightarrow 4x - 5\text{i}x + 4\text{i}y + 5y + 4x - 4\text{i}y = 16 + 15\text{i}$
attempt to equate real and imaginary parts
$8x + 5y = 16$, $-5x = 15$
$\Rightarrow x = -3,\ y = 8$
$(\Rightarrow m = -3 + 8\text{i},\ n = -3 - 8\text{i})$

3 - (MAT/11_HL_Summer_2012_Q7) - *Complex Numbers*

$\sqrt{x^2+y^2}+x+y\mathrm{i}=6-2\mathrm{i}$

equating real and imaginary parts

$y=-2$

$\sqrt{x^2+4}+x=6$

$x^2+4=(6-x)^2$

$-32=-12x \Rightarrow x=\frac{8}{3}$

4 – (MAT/12_HL_Summer_2012_Q12) – *Complex Numbers*

Part A

(a) (i) $(x+\mathrm{i}y)^2 = -5+12\mathrm{i}$

$x^2 + 2\mathrm{i}xy + \mathrm{i}^2y^2 = -5+12\mathrm{i}$

(ii) equating real and imaginary parts

$x^2 - y^2 = -5$

$xy = 6$

(b) substituting

EITHER

$x^2 - \dfrac{36}{x^2} = -5$

$x^4 + 5x^2 - 36 = 0$

$x^2 = 4, -9$

$x = \pm 2$ and $y = \pm 3$

OR

$\dfrac{36}{y^2} - y^2 = -5$

$y^4 - 5y^2 - 36 = 0$

$y^2 = 9, -4$

$y = \pm 3$ **and** $x = \pm 2$

Note: Accept solution by inspection if completely correct.

THEN

the square roots are $(2+3\mathrm{i})$ and $(-2-3\mathrm{i})$

(c) **EITHER**

consider $z = x + \mathrm{i}y$

$z^* = x - \mathrm{i}y$

$(z^*)^2 = x^2 - y^2 - 2\mathrm{i}xy$

$(z^2) = x^2 - y^2 + 2\mathrm{i}xy$

$(z^2)^* = x^2 - y^2 - 2\mathrm{i}xy$

$(z^*)^2 = (z^2)^*$

OR

$z^* = r\mathrm{e}^{-\mathrm{i}\theta}$

$(z^*)^2 = r^2\mathrm{e}^{-2\mathrm{i}\theta}$

$z^2 = r^2\mathrm{e}^{2\mathrm{i}\theta}$

$(z^2)^* = r^2 e^{-2i\theta}$

$(z^*)^2 = (z^2)^*$

(d) $(2-3i)$ and $(-2+3i)$

Part B

(a) the graph crosses the x-axis twice, indicating two real roots
since the quartic equation has four roots and only two are real, the other two roots must be complex

(b) $f(x) = (x+4)(x-2)(x^2+cx+d)$

$f(0) = -32 \Rightarrow d = 4$

Since the curve passes through (– 1, – 18),

$-18 = 3 \times (-3)(5-c)$

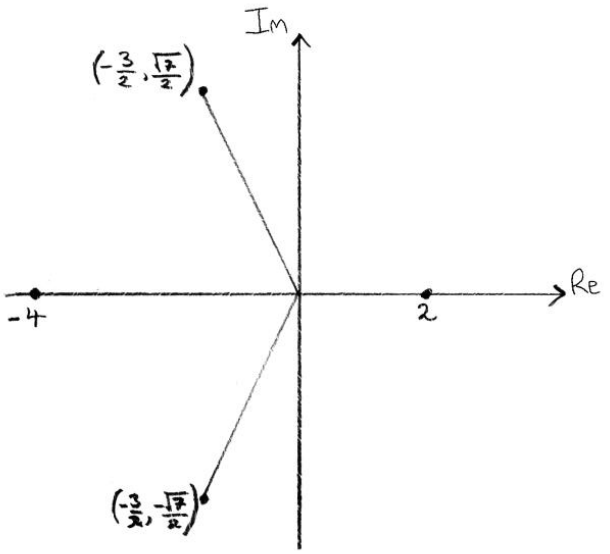
$c = 3$

Hence $f(x) = (x+4)(x-2)(x^2+3x+4)$

(c) $x = \dfrac{-3 \pm \sqrt{9-16}}{2}$

$\Rightarrow x = -\dfrac{3}{2} \pm i\dfrac{\sqrt{7}}{2}$

(d)



Note:	Accept points or vectors on complex plane. Award ***A1*** for two real roots and ***A1*** for two complex roots.

(e) real roots are $4e^{i\pi}$ and $2e^{i0}$

considering $-\frac{3}{2} \pm i\frac{\sqrt{7}}{2}$

$r = \sqrt{\frac{9}{4} + \frac{7}{4}} = 2$

finding θ using $\arctan\left(\frac{\sqrt{7}}{3}\right)$

$\theta = \arctan\left(\frac{\sqrt{7}}{3}\right) + \pi$ or $\theta = \arctan\left(-\frac{\sqrt{7}}{3}\right) + \pi$

$\Rightarrow z = 2e^{i\left(\arctan\left(\frac{\sqrt{7}}{3}\right)+\pi\right)}$ or $\Rightarrow z = 2e^{i\left(\arctan\left(\frac{-\sqrt{7}}{3}\right)+\pi\right)}$

5 – (MAT/10_HL_Winter_2012_Q10) – *Complex Numbers*

(a) (i) $z_1 = 2\sqrt{3}\,\text{cis}\,\frac{3\pi}{2} \Rightarrow z_1 = -2\sqrt{3}\text{i}$

(ii) $z_1 + z_2 = -2\sqrt{3}\text{i} - 1 + \sqrt{3}\text{i} = -1 - \sqrt{3}\text{i}$

$(z_1 + z_2)^* = -1 + \sqrt{3}\text{i}$

(b) (i) $|z_2| = 2$

$\tan\theta = -\sqrt{3}$

z_2 lies on the second quadrant

$\theta = \arg z_2 = \frac{2\pi}{3}$

$z_2 = 2\text{cis}\,\frac{2\pi}{3}$

(ii) attempt to use De Moivre's theorem

$z = \sqrt[3]{2}\,\text{cis}\,\frac{\frac{2\pi}{3} + 2k\pi}{3}$, $k = 0, 1$ and 2

$z = \sqrt[3]{2}\,\text{cis}\,\frac{2\pi}{9}, \sqrt[3]{2}\,\text{cis}\,\frac{8\pi}{9}, \sqrt[3]{2}\,\text{cis}\,\frac{14\pi}{9}\left(= \sqrt[3]{2}\,\text{cis}\left(\frac{-4\pi}{9}\right)\right)$

Note: Award ***A1*** for modulus, ***A1*** for arguments.

Note: Allow equivalent forms for z.

(c) (i) **METHOD 1**

$z^2 = \left(1 - 1 + \sqrt{3}\text{i}\right)^2 = -3\left(\Rightarrow z = \pm\sqrt{3}\text{i}\right)$

$z = \sqrt{3}\,\text{cis}\,\frac{\pi}{2}$ or $z_1 = \sqrt{3}\,\text{cis}\,\frac{3\pi}{2}\left(= \sqrt{3}\,\text{cis}\left(\frac{-\pi}{2}\right)\right)$

so $r = \sqrt{3}$ and $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}\left(= \frac{-\pi}{2}\right)$

Note: Accept $r\,\text{cis}(\theta)$ form.

METHOD 2

$z^2 = \left(1 - 1 + \sqrt{3}\text{i}\right)^2 = -3 \Rightarrow z^2 = 3\text{cis}\left((2n+1)\pi\right)$

$r^2 = 3 \Rightarrow r = \sqrt{3}$

$2\theta = (2n+1)\pi \Rightarrow \theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$ (as $0 \le \theta < 2\pi$)

(ii) **METHOD 1**

$$z = -\frac{1}{2cis\frac{2\pi}{3}} \Rightarrow z = \frac{\text{cis}\pi}{2\text{cis}\frac{2\pi}{3}}$$

$$\Rightarrow z = \frac{1}{2}\text{cis}\frac{\pi}{3}$$

so $r = \frac{1}{2}$ and $\theta = \frac{\pi}{3}$

METHOD 2

$$z_1 = -\frac{1}{-1+\sqrt{3}\text{i}} \Rightarrow z_1 = -\frac{-1-\sqrt{3}\text{i}}{\left(-1+\sqrt{3}\text{i}\right)\left(-1-\sqrt{3}\text{i}\right)}$$

$$z = \frac{1+\sqrt{3}\text{i}}{4} \Rightarrow z = \frac{1}{2}\text{cis}\frac{\pi}{3}$$

so $r = \frac{1}{2}$ and $\theta = \frac{\pi}{3}$

(d) $\frac{z_1}{z_2} = \sqrt{3}\,\text{cis}\frac{5\pi}{6}$

$$\left(\frac{z_1}{z_2}\right)^n = \sqrt{3}^n\,\text{cis}\frac{5n\pi}{6}$$

equating imaginary part to zero and attempting to solve
obtain $n = 12$

6 - (MAT/11_HL_Summer_2013_Q1) - *Complex Numbers*

(a) modulus $=\sqrt{8}$

argument $=\frac{\pi}{4}$ (accept 45°)

Note: ***A0*** **if extra values given.**

(b) **METHOD 1**

$w^4 z^6 = 64e^{\pi i} \times e^{5\pi i}$

Note: Allow alternative notation.

$= 64e^{6\pi i}$
$= 64$

METHOD 2

$w^4 = -64$
$z^6 = -1$
$w^4 z^6 = 64$

7 - (MAT/12_HL_Summer_2013_Q7) - *Complex Numbers*

(a) $|z_1| = \sqrt{10}$; $\arg(z_2) = -\frac{3\pi}{4}$ $\left(\text{accept } \frac{5\pi}{4}\right)$

(b) $|z_1 + \alpha z_2| = \sqrt{(1-\alpha)^2 + (3-\alpha)^2}$ or the squared modulus
attempt to minimise $2\alpha^2 - 8\alpha + 10$ or their quadratic or its half or its square root
obtain $\alpha = 2$ at minimum
state $\sqrt{2}$ as final answer

8 – (MAT/12_HL_Summer_2013_Q13) – *Complex Numbers*

(a) (i) $z_1 = 2\text{cis}\left(\frac{\pi}{6}\right)$, $z_2 = 2\text{cis}\left(\frac{5\pi}{6}\right)$, $z_3 = 2\text{cis}\left(-\frac{\pi}{2}\right)$ or $2\text{cis}\left(\frac{3\pi}{2}\right)$

Note:	Accept modulus and argument given separately, or the use of exponential (Euler) form.

Note:	Accept arguments given in rational degrees, except where exponential form is used.

(ii) the points lie on a circle of radius 2 centre the origin

differences are all $\frac{2\pi}{3}$ (mod 2π)

$\Rightarrow$ points equally spaced $\Rightarrow$ triangle is equilateral

Note:	Accept an approach based on a clearly marked diagram.

(iii) $z_1^{3n} + z_2^{3n} = 2^{3n}\text{cis}\left(\frac{n\pi}{2}\right) + 2^{3n}\text{cis}\left(\frac{5n\pi}{2}\right)$

$= 2 \times 2^{3n}\text{cis}\left(\frac{n\pi}{2}\right)$

$2z_3^{3n} = 2 \times 2^{3n}\text{cis}\left(\frac{9n\pi}{2}\right) = 2 \times 2^{3n}\text{cis}\left(\frac{n\pi}{2}\right)$

(b) (i) attempt to obtain **seven** solutions in modulus argument form

$z = \text{cis}\left(\frac{2k\pi}{7}\right), k = 0, 1 \ldots 6$

(ii) w has argument $\frac{2\pi}{7}$ and $1 + w$ has argument ϕ,

then $\tan(\phi) = \dfrac{\sin\left(\frac{2\pi}{7}\right)}{1 + \cos\left(\frac{2\pi}{7}\right)}$

$= \dfrac{2\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{\pi}{7}\right)}{2\cos^2\left(\frac{\pi}{7}\right)}$

$= \tan\left(\frac{\pi}{7}\right) \Rightarrow \phi = \frac{\pi}{7}$

(iii) since roots occur in conjugate pairs,

$z^7 - 1$ has a quadratic factor $\left(z - \text{cis}\left(\frac{2\pi}{7}\right)\right) \times \left(z - \text{cis}\left(-\frac{2\pi}{7}\right)\right)$

$= z^2 - 2z\cos\left(\frac{2\pi}{7}\right) + 1$

other quadratic factors are $z^2 - 2z\cos\left(\frac{4\pi}{7}\right) + 1$

and $z^2 - 2z\cos\left(\frac{6\pi}{7}\right) + 1$

9 – (MAT/10_HL_Winter_2013_Q12) – *Complex Numbers, Trigonometry, Integration*

(a) $z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$
$= \cos n\theta + \cos n\theta + i\sin n\theta - i\sin n\theta$
$= 2\cos n\theta$

(b) $\left(z + z^{-1}\right)^4 = z^4 + 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z^2}\right) + 4z\left(\frac{1}{z^3}\right) + \frac{1}{z^4}$

Note: Accept $\left(z + z^{-1}\right)^4 = 16\cos^4\theta$.

(c) **METHOD 1**

$\left(z + z^{-1}\right)^4 = \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$

$(2\cos\theta)^4 = 2\cos 4\theta + 8\cos 2\theta + 6$

Note: Award ***A1*** for RHS, ***A1*** for LHS independent of the ***M1***.

$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

$\left(\text{or } p = \frac{1}{8},\ q = \frac{1}{2},\ r = \frac{3}{8}\right)$

METHOD 2

$\cos^4\theta = \left(\frac{\cos 2\theta + 1}{2}\right)^2$

$= \frac{1}{4}(\cos^2 2\theta + 2\cos 2\theta + 1)$

$= \frac{1}{4}\left(\frac{\cos 4\theta + 1}{2} + 2\cos 2\theta + 1\right)$

$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

$\left(\text{or } p = \frac{1}{8},\ q = \frac{1}{2},\ r = \frac{3}{8}\right)$

(d) $\left(z+z^{-1}\right)^6 = z^6 + 6z^5\left(\frac{1}{z}\right) + 15z^4\left(\frac{1}{z^2}\right) + 20z^3\left(\frac{1}{z^3}\right) + 15z^2\left(\frac{1}{z^4}\right) + 6z\left(\frac{1}{z^5}\right) + \frac{1}{z^6}$

$\left(z+z^{-1}\right)^6 = \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$

$(2\cos\theta)^6 = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$

Note: Award ***A1*** for RHS, ***A1*** for LHS, independent of the ***M1***.

$\cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$

Note: Accept a purely trigonometric solution as for (c).

(e) $\int_0^{\frac{\pi}{2}} \cos^6\theta \, d\theta = \int_0^{\frac{\pi}{2}} \left(\frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}\right) d\theta$

$= \left[\frac{1}{192}\sin 6\theta + \frac{3}{64}\sin 4\theta + \frac{15}{64}\sin 2\theta + \frac{5}{16}\theta\right]_0^{\frac{\pi}{2}}$

$= \frac{5\pi}{32}$

(f) $V = \pi\int_0^{\frac{\pi}{2}} \sin^2 x \cos^4 x \, dx$

$= \pi\int_0^{\frac{\pi}{2}} \cos^4 x \, dx - \pi\int_0^{\frac{\pi}{2}} \cos^6 x \, dx$

$\int_0^{\frac{\pi}{2}} \cos^4 x \, dx = \frac{3\pi}{16}$

$V = \frac{3\pi^2}{16} - \frac{5\pi^2}{32} = \frac{\pi^2}{32}$

(g) (i) constant term $= \binom{2k}{k} = \frac{(2k)!}{k!k!} = \frac{(2k)!}{(k!)^2}$ (accept C_k^{2k})

(ii) $2^{2k}\int_0^{\frac{\pi}{2}} \cos^{2k}\theta \mathrm{d}\theta = \frac{(2k)!}{(k!)^2}\frac{\pi}{2}$

$\int_0^{\frac{\pi}{2}} \cos^{2k}\theta \mathrm{d}\theta = \frac{(2k)!\pi}{2^{2k+1}(k!)^2}$ $\left(\text{or } \frac{\binom{2k}{k}\pi}{2^{2k+1}}\right)$

10 – (MAT/12_HL_Summer_2014_Q7) – *Complex Numbers*

(a) **METHOD 1**

$$\frac{1}{2+3\mathrm{i}}+\frac{1}{3+2\mathrm{i}}=\frac{2-3\mathrm{i}}{4+9}+\frac{3-2\mathrm{i}}{9+4}$$
$$\frac{10}{w}=\frac{5-5\mathrm{i}}{13}$$
$$w=\frac{130}{5-5\mathrm{i}}$$
$$=\frac{130\times5\times(1+\mathrm{i})}{50}$$
$$w=13+13\mathrm{i}$$

METHOD 2

$$\frac{1}{2+3\mathrm{i}}+\frac{1}{3+2\mathrm{i}}=\frac{3+2\mathrm{i}+2+3\mathrm{i}}{(2+3\mathrm{i})(3+2\mathrm{i})}$$
$$\frac{10}{w}=\frac{5+5\mathrm{i}}{13\mathrm{i}}$$
$$\frac{w}{10}=\frac{13\mathrm{i}}{5+5\mathrm{i}}$$
$$w=\frac{130\mathrm{i}}{(5+5\mathrm{i})}\times\frac{(5-5\mathrm{i})}{(5-5\mathrm{i})}$$
$$=\frac{650+650\mathrm{i}}{50}$$
$$=13+13\mathrm{i}$$

(b) $w^*=13-13\mathrm{i}$

$$z=\sqrt{338}e^{-\frac{\pi}{4}\mathrm{i}}\left(=13\sqrt{2}\,e^{-\frac{\pi}{4}\mathrm{i}}\right)$$

11 - (MAT/11_HL_Summer_2014_Q13) - *Complex Numbers, Sequences - Series*

(a) $r = 1 + \mathrm{i}$

$u_4 = 3(1+\mathrm{i})^3$

$= -6 + 6\mathrm{i}$

(b) $S_{20} = \dfrac{3\left((1+\mathrm{i})^{20} - 1\right)}{\mathrm{i}}$

$= \dfrac{3\left(\left(2i\right)^{10} - 1\right)}{\mathrm{i}}$

Note: Only one of the two ***M1***s can be implied. Other algebraic methods may be seen.

$= \dfrac{3\left(-2^{10} - 1\right)}{\mathrm{i}}$

$= 3\mathrm{i}\left(2^{10} + 1\right)$

(c) (i) **METHOD 1**

$v_n = \left(3(1+\mathrm{i})^{n-1}\right)\left(3(1+\mathrm{i})^{n-1+k}\right)$

$9(1+\mathrm{i})^k (1+i)^{2n-2}$

$= 9(1+\mathrm{i})^k ((1+i)^2)^{n-1} \left(= 9(1+\mathrm{i})^k (2\mathrm{i})^{n-1}\right)$

this is the general term of a geometrical sequence

Notes: Do not accept the statement that the product of terms in a geometric sequence is also geometric unless justified further. If the final expression for v_n is $9(1+\mathrm{i})^k (1+i)^{2n-2}$ award ***M1A1R0***.

METHOD 2

$\dfrac{v_{n+1}}{v_n} = \dfrac{u_{n+1}u_{n+k+1}}{u_n u_{n+k}}$

$= (1+\mathrm{i})(1+\mathrm{i})$

this is a constant, hence sequence is geometric

Note: Do not allow methods that do not consider the general term.

(ii) $9(1+\mathrm{i})^k$

(iii) common ratio is $(1+i)^2 (= 2i)$ (which is independent of k)

(d) (i) **METHOD 1**

$$w_n = \left|3(1+i)^{n-1} - 3(1+\mathrm{i})^n\right|$$
$$= 3\left|1+i\right|^{n-1}\left|1-(1+\mathrm{i})\right|$$
$$= 3\left|1+i\right|^{n-1}$$
$$\left(= 3\left(\sqrt{2}\right)^{n-1}\right)$$

this is the general term for a geometric sequence

METHOD 2

$$w_n = \left|u_n - (1+\mathrm{i})u_n\right|$$
$$= \left|u_n\right|\left|-\mathrm{i}\right|$$
$$= \left|u_n\right|$$
$$= \left|3(1+\mathrm{i})^{n-1}\right|$$
$$= 3\left|(1+\mathrm{i})\right|^{n-1}$$
$$\left(= 3\left(\sqrt{2}\right)^{n-1}\right)$$

this is the general term for a geometric sequence

Note: Do not allow methods that do not consider the general term.

(ii) distance between successive points representing u_n in the complex plane forms a geometric sequence

12 – (MAT/10_HL_Winter_2014_Q13) – *Complex Numbers, Trigonometry, Integration*

(a) (i) METHOD 1

$$(1+\mathrm{i}\tan\theta)^n+(1-\mathrm{i}\tan\theta)^n=(1+\mathrm{i}\frac{\sin\theta}{\cos\theta})^n+(1-\mathrm{i}\frac{\sin\theta}{\cos\theta})^n$$

$$=\left(\frac{\cos\theta+i\sin\theta}{\cos\theta}\right)^n+\left(\frac{\cos\theta-i\sin\theta}{\cos\theta}\right)^n$$

by de Moivre's theorem

$$\left(\frac{\cos\theta+i\sin\theta}{\cos\theta}\right)^n=\frac{\cos n\theta+i\sin n\theta}{\cos^n\theta}$$

recognition that $\cos\theta-i\sin\theta$ is the complex conjugate of $\cos\theta+i\sin\theta$

use of the fact that the operation of complex conjugation commutes with the operation of raising to an integer power:

$$\left(\frac{\cos\theta-i\sin\theta}{\cos\theta}\right)^n=\frac{\cos n\theta-i\sin n\theta}{\cos^n\theta}$$

$$(1+\mathrm{i}\tan\theta)^n+(1-\mathrm{i}\tan\theta)^n=\frac{2\cos n\theta}{\cos^n\theta}$$

METHOD 2

$$(1+\mathrm{i}\tan\theta)^n+(1-\mathrm{i}\tan\theta)^n=(1+\mathrm{i}\tan\theta)^n+(1+i\tan(-\theta))^n$$

$$=\frac{(\cos\theta+i\sin\theta)^n}{\cos^n\theta}+\frac{\left(\cos(-\theta)+i\sin(-\theta)\right)^n}{\cos^n\theta}$$

Note: Award ***M1*** for converting to cosine and sine terms.

use of de Moivre's theorem

$$=\frac{1}{\cos^n\theta}\left(\cos n\theta+\mathrm{i}\sin n\theta+\cos(-n\theta)+\mathrm{i}\sin(-n\theta)\right)$$

$$=\frac{2\cos n\theta}{\cos^n\theta}\text{ as }\cos(-n\theta)=\cos n\theta\text{ and }\sin(-n\theta)=-\sin n\theta$$

(ii) $\left(1+\mathrm{i}\tan\frac{3\pi}{8}\right)^4+\left(1-\mathrm{i}\tan\frac{3\pi}{8}\right)^4=\dfrac{2\cos\left(4\times\frac{3\pi}{8}\right)}{\cos^4\frac{3\pi}{8}}$

$=\dfrac{2\cos\frac{3\pi}{2}}{\cos^4\frac{3\pi}{8}}$

$=0$ as $\cos\frac{3\pi}{2}=0$

Note: The above working could involve theta and the solution of $\cos(4\theta)=0$.

so $\mathrm{i}\tan\frac{3\pi}{8}$ is a root of the equation

(iii) either $-\mathrm{i}\tan\frac{3\pi}{8}$ or $-\mathrm{i}\tan\frac{\pi}{8}$ or $\mathrm{i}\tan\frac{\pi}{8}$

Note: Accept $\mathrm{i}\tan\frac{5\pi}{8}$ or $\mathrm{i}\tan\frac{7\pi}{8}$.
Accept $-\left(1+\sqrt{2}\right)\mathrm{i}$ or $\left(1-\sqrt{2}\right)\mathrm{i}$ or $\left(-1+\sqrt{2}\right)\mathrm{i}$.

(b) (i) $\tan\frac{\pi}{4}=\dfrac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$

$\tan^2\frac{\pi}{8}+2\tan\frac{\pi}{8}-1=0$

let $t=\tan\frac{\pi}{8}$

attempting to solve $t^2+2t-1=0$ for t

$t=-1\pm\sqrt{2}$

$\frac{\pi}{8}$ is a first quadrant angle and tan is positive in this quadrant, so

$\tan\frac{\pi}{8}>0$

$\tan\frac{\pi}{8}=\sqrt{2}-1$

(ii) $\cos 4x = 2\cos^2 2x - 1$

$$= 2\left(2\cos^2 x - 1\right)^2 - 1$$

$$= 2\left(4\cos^4 x - 4\cos^2 x + 1\right) - 1$$

$$= 8\cos^4 x - 8\cos^2 x + 1$$

Note: Accept equivalent complex number derivation.

(iii) $\int_0^{\frac{\pi}{8}} \frac{2\cos 4x}{\cos^2 x}\,\mathrm{d}x = 2\int_0^{\frac{\pi}{8}} \frac{8\cos^4 x - 8\cos^2 x + 1}{\cos^2 x}\,\mathrm{d}x$

$$= 2\int_0^{\frac{\pi}{8}} 8\cos^2 x - 8 + \sec^2 x\,\mathrm{d}x$$

Note: The ***M1*** is for an integrand involving no fractions.

use of $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$

$$= 2\int_0^{\frac{\pi}{8}} 4\cos 2x - 4 + \sec^2 x\,\mathrm{d}x$$

$$= \left[4\sin 2x - 8x + 2\tan x\right]_0^{\frac{\pi}{8}}$$

$$= 4\sqrt{2} - \pi - 2 \text{ (or equivalent)}$$

13 – (MAT/12_HL_Summer_2015_Q7) – *Complex Numbers*

(a) **METHOD 1**

$$z^3 = -\frac{27}{8} = \frac{27}{8}(\cos\pi + \mathrm{i}\sin\pi)$$

$$= \frac{27}{8}\left(\cos(\pi + 2n\pi) + \mathrm{i}\sin(\pi + 2n\pi)\right)$$

$$z = \frac{3}{2}\left(\cos\left(\frac{\pi + 2n\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi + 2n\pi}{3}\right)\right)$$

$$z_1 = \frac{3}{2}\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right),$$

$$z_2 = \frac{3}{2}\left(\cos\pi + \mathrm{i}\sin\pi\right),$$

$$z_3 = \frac{3}{2}\left(\cos\frac{5\pi}{3} + \mathrm{i}\sin\frac{5\pi}{3}\right).$$

METHOD 2

$8z^3 + 27 = 0$

$\Rightarrow z = -\frac{3}{2}$ so $(2z+3)$ is a factor

Attempt to use long division or factor theorem:

$\Rightarrow 8z^3 + 27 \equiv (2z+3)(4z^2 - 6z + 9)$

$\Rightarrow 4z^2 - 6z + 9 = 0$

Attempt to solve quadratic:

$z = \frac{3 \pm 3\sqrt{3}\,\mathrm{i}}{4}$

$z_1 = \frac{3}{2}\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$,

$z_2 = \frac{3}{2}(\cos\pi + \mathrm{i}\sin\pi)$,

$z_3 = \frac{3}{2}\left(\cos\frac{5\pi}{3} + \mathrm{i}\sin\frac{5\pi}{3}\right)$.

METHOD 3

$8z^3 + 27 = 0$

Substitute $z = x + \mathrm{i}\,y$

$8\left(x^3 + 3\mathrm{i}\,x^2 y - 3xy^2 - \mathrm{i}\,y^3\right) + 27 = 0$

$\Rightarrow 8x^3 - 24xy^2 + 27 = 0$ and $24x^2 y - 8y^3 = 0$

Attempt to solve simultaneously:

$8y\left(3x^2 - y^2\right) = 0$

$y = 0, y = x\sqrt{3}, y = -x\sqrt{3}$

$\Rightarrow \left(x = -\frac{3}{2}, y = 0\right), x = \frac{3}{4}, y = \pm\frac{3\sqrt{3}}{4}$

$z_1 = \frac{3}{2}\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right),$

$z_2 = \frac{3}{2}\left(\cos\pi + \mathrm{i}\sin\pi\right),$

$z_3 = \frac{3}{2}\left(\cos\frac{5\pi}{3} + \mathrm{i}\sin\frac{5\pi}{3}\right).$

(b) **EITHER**

Valid attempt to use area $= 3\left(\frac{1}{2}ab\sin C\right)$

$= 3 \times \frac{1}{2} \times \frac{3}{2} \times \frac{3}{2} \times \frac{\sqrt{3}}{2}$

Note: Award ***A1*** for correct sides, ***A1*** for correct $\sin C$.

OR

Valid attempt to use area $= \frac{1}{2}$ base $\times$ height

area $= \frac{1}{2} \times \left(\frac{3}{4} + \frac{3}{2}\right) \times \frac{6\sqrt{3}}{4}$

Note: ***A1*** for correct height, ***A1*** for correct base.

THEN

$= \frac{27\sqrt{3}}{16}$

14 – (MAT/10_HL_Winter_2015_Q11) – *Complex Numbers*

(a) $z^3 = 8\left(\cos\left(\frac{\pi}{2} + 2\pi k\right) + \mathrm{i}\sin\left(\frac{\pi}{2} + 2\pi k\right)\right)$

attempt the use of De Moivre's Theorem in reverse

$z = 2\left(\cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right)\right);\ 2\left(\cos\left(\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(\frac{5\pi}{6}\right)\right);$

$2\left(\cos\left(\frac{9\pi}{6}\right) + \mathrm{i}\sin\left(\frac{9\pi}{6}\right)\right)$

Note: Accept cis form.

$z = \pm\sqrt{3} + \mathrm{i},\ -2\mathrm{i}$

Note: Award ***A1*** for two correct solutions in each of the two lines above.

(b) (i) $z_1 = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right)\right)$

(ii) $\left(z_2 = \left(\sqrt{3} + \mathrm{i}\right)\right)$

$z_1 z_2 = (1 + \mathrm{i})\left(\sqrt{3} + \mathrm{i}\right)$

$= \left(\sqrt{3} - 1\right) + \mathrm{i}\left(1 + \sqrt{3}\right)$

(iii) $z_1 z_2 = 2\sqrt{2}\left(\cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{6} + \frac{\pi}{4}\right)\right)$

$\tan\frac{5\pi}{12} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$

$= 2 + \sqrt{3}$

Note: Award final ***M1*** for an attempt to rationalise the fraction.

(iv) $z_2^{\,p} = 2^p\left(\mathrm{cis}\left(\frac{p\pi}{6}\right)\right)$

$z_2^{\,p}$ is a positive real number when $p = 12$

15 - (MAT/11_HL_Summer_2016_Q12) - *Complex Numbers*

(a) $\left(\cos\left(\frac{\pi}{3}\right)+\mathrm{i}\sin\left(\frac{\pi}{3}\right)\right)^3=\cos\pi+\mathrm{i}\sin\pi$ ***M1***

$=-1$ ***A1***

(b) show the expression is true for $n=1$ ***R1***

assume true for $n=k$, $(\cos\theta-\mathrm{i}\sin\theta)^k=\cos k\theta-\mathrm{i}\sin k\theta$ ***M1***

Note: Do not accept "let $n=k$" or "assume $n=k$", assumption of truth must be present.

$(\cos\theta-\mathrm{i}\sin\theta)^{k+1}=(\cos\theta-\mathrm{i}\sin\theta)^k(\cos\theta-\mathrm{i}\sin\theta)$

$=(\cos k\theta-\mathrm{i}\sin k\theta)(\cos\theta-\mathrm{i}\sin\theta)$ ***M1***

$=\cos k\theta\cos\theta-\sin k\theta\sin\theta-\mathrm{i}(\cos k\theta\sin\theta+\sin k\theta\cos\theta)$ ***A1***

Note: Award ***A1*** for any correct expansion.

$=\cos((k+1)\theta)-\mathrm{i}\sin((k+1)\theta)$ ***A1***

therefore if true for $n=k$ true for $n=k+1$, true for $n=1$, so true for all $n(\in\mathbb{Z}^+)$ ***R1***

Note: To award the final ***R*** mark the first 4 marks must be awarded.

(c) $(z)^n+(z^*)^n=(\cos\theta+\mathrm{i}\sin\theta)^n+(\cos\theta-\mathrm{i}\sin\theta)^n$

$=\cos n\theta+\mathrm{i}\sin n\theta+\cos n\theta-\mathrm{i}\sin n\theta=2\cos(n\theta)$ ***(M1)A1***

(d) (i) $zz^*=(\cos\theta+\mathrm{i}\sin\theta)(\cos\theta-\mathrm{i}\sin\theta)$

$=\cos^2\theta+\sin^2\theta$ ***A1***

$=1$ ***AG***

Note: Allow justification starting with $|z|=1$.

(ii) $(z+z^*)^3=z^3+3z^2z^*+3z(z^*)^2+(z^*)^3\left(=z^3+3z+3z^*+(z^*)^3\right)$ ***A1***

(iii) $(z+z^*)^3=(2\cos\theta)^3$ ***A1***

$z^3+3z+3z^*+(z^*)^3=2\cos3\theta+6\cos\theta$ ***M1A1***

$\cos3\theta=4\cos^3\theta-3\cos\theta$ ***AG***

(e) $4\cos^3\theta - 2\cos^2\theta - 3\cos\theta + 1 = 0$

$4\cos^3\theta - 3\cos\theta = 2\cos^2\theta - 1$

$\cos(3\theta) = \cos(2\theta)$

Note: ***A1*** for $\cos(3\theta)$ and ***A1*** for $\cos(2\theta)$.

$\theta = 0$

or $3\theta = 2\pi - 2\theta$ (or $3\theta = 4\pi - 2\theta$)

$\theta = \frac{2\pi}{5}, \frac{4\pi}{5}$

16 - (MAT/12_HL_Summer_2016_Q12) - *Complex Numbers*

(a) **EITHER**

$w^7 = \left(\cos\frac{2\pi}{7} + \mathrm{i}\sin\frac{2\pi}{7}\right)^7$

$= \cos 2\pi + \mathrm{i}\sin 2\pi$

$= 1$

so w is a root

OR

$z^7 = 1 = \cos(2\pi k) + i\sin(2\pi k)$

$z = \cos\left(\frac{2\pi k}{7}\right) + i\sin\left(\frac{2\pi k}{7}\right)$

$k = 1 \Rightarrow z = \cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)$

so w is a root

(b) (i) $(w-1)\left(1 + w + w^2 + w^3 + w^4 + w^5 + w^6\right)$

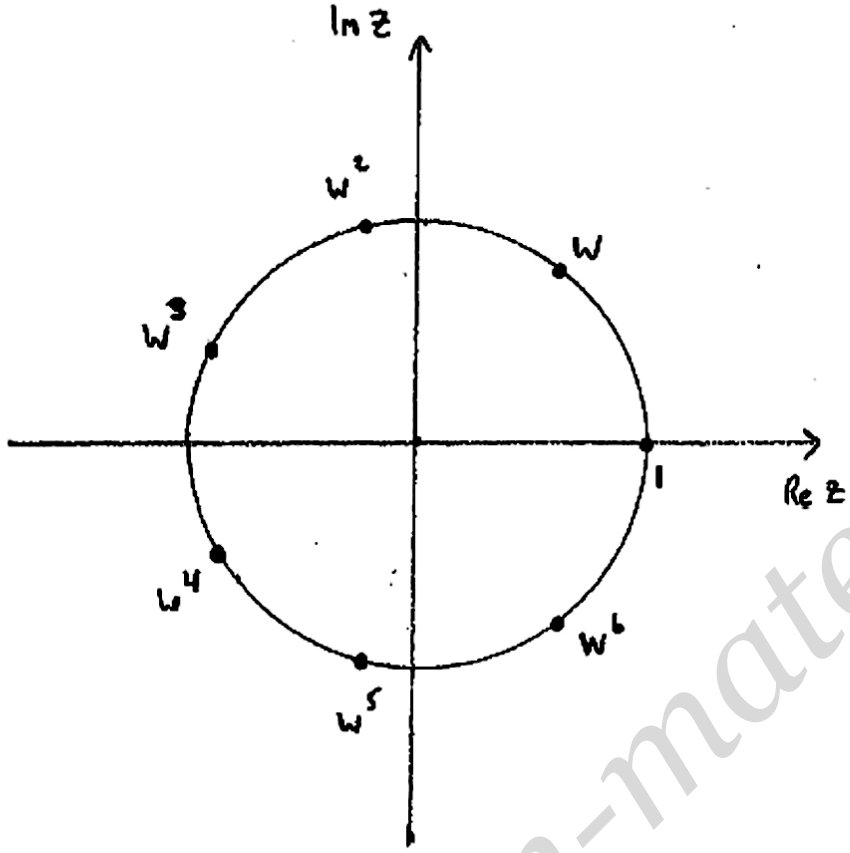
$= w + w^2 + w^3 + w^4 + w^5 + w^6 + w^7 - 1 - w - w^2 - w^3 - w^4 - w^5 - w^6$

$= w^7 - 1\ (= 0)$

(ii) $w^7 - 1 = 0$ and $w - 1 \neq 0$

so $1 + w + w^2 + w^3 + w^4 + w^5 + w^6 = 0$

(c) the roots are 1, w, w^2, w^3, w^4, w^5 and w^6 ***A1***



7 points equidistant from the origin ***A1***

approximately correct angular positions for 1, w, w^2, w^3, w^4, w^5 and w^6 ***A1***

Note: Condone use of *cis* notation for the final two ***A*** marks.

Note: For the final ***A*** mark there should be one root in the first quadrant, two in the second, two in the third, one in the fourth, and one on the real axis.

(d) (i) $\alpha^* = \left(w + w^2 + w^4\right)^*$

$= w^* + \left(w^2\right)^* + \left(w^4\right)^*$ ***A1***

since $w^* = w^6$, $\left(w^2\right)^* = w^5$ and $\left(w^4\right)^* = w^3$ ***R1***

$\Rightarrow \alpha^* = w^6 + w^5 + w^3$ ***AG***

(ii) $b = -\left(\alpha + \alpha^*\right)$ (using sum of roots (or otherwise))

$b = -\left(w + w^2 + w^3 + w^4 + w^5 + w^6\right)$

$= -(-1)$

$= 1$

$c = \alpha\alpha^*$ (using product of roots (or otherwise))

$c = \left(w + w^2 + w^4\right)\left(w^6 + w^5 + w^3\right)$

EITHER

$= w^{10} + w^9 + w^8 + 3w^7 + w^6 + w^5 + w^4$

$= \left(w^6 + w^5 + w^4 + w^3 + w^2 + w\right) + 3$

$= 3 - 1$

OR

$= w^{10} + w^9 + w^8 + 3w^7 + w^6 + w^5 + w^4 \left(= w^4\left(1 + w + w^3\right)\left(w^3 + w^2 + 1\right)\right)$

$= w^4\left(w^6 + w^5 + w^4 + w^2 + w + 1 + 3w^3\right)$

$= w^4\left(w^6 + w^5 + w^4 + w^3 + w^2 + w + 1 + 2w^3\right)$

$= w^4\left(2w^3\right)$

THEN

$= 2$

(e) $z^2 + z + 2 = 0 \Rightarrow z = \dfrac{-1 \pm \mathrm{i}\sqrt{7}}{2}$

$\operatorname{Im}\left(w + w^2 + w^4\right) > 0$

$\operatorname{Im}\alpha = \dfrac{\sqrt{7}}{2}$

17 – (MAT/10_HL_Winter_2016_Q12) – *Complex Numbers*

(a) (i) **METHOD 1**

$1+\omega+\omega^2=\dfrac{1-\omega^3}{1-\omega}=0$ ***A1***

as $\omega \neq 1$ ***R1***

METHOD 2

solutions of $1-\omega^3=0$ are $\omega=1,\ \omega=\dfrac{-1\pm\sqrt{3}\,\mathrm{i}}{2}$ ***A1***

verification that the sum of these roots is 0 ***R1***

(ii) $1+\omega^*+\left(\omega^*\right)^2=0$ ***A2***

[4 marks]

(b) $\left(\omega-3\omega^2\right)\left(\omega^2-3\omega\right)=-3\omega^4+10\omega^3-3\omega^2$ ***M1A1***

EITHER

$=-3\omega^2\left(\omega^2+\omega+1\right)+13\omega^3$ ***M1***

$=-3\omega^2\times 0+13\times 1$ ***A1***

OR

$=-3\omega+10-3\omega^2=-3(\omega^2+\omega+1)+13$ ***M1***

$=-3\times 0+13$ ***A1***

OR

substitution by $\omega=\dfrac{-1\pm\sqrt{3}\,\mathrm{i}}{2}$ in any form ***M1***

numerical values of each term seen ***A1***

THEN

$=13$ ***AG***

[4 marks]

(c) $|p|=|q|\Rightarrow\sqrt{1^2+3^2}=\sqrt{x^2+(2x+1)^2}$ ***(M1)(A1)***

$5x^2+4x-9=0$ ***A1***

$(5x+9)(x-1)=0$ ***(M1)***

$x=1,\ x=-\dfrac{9}{5}$ ***A1***

[5 marks]

(d) $pq = (1-3\text{i})\big(x + (2x+1)\text{i}\big) = (7x+3) + (1-x)\text{i}$ ***M1A1***

$\text{Re}(pq) + 8 < \big(\text{Im}(pq)\big)^2 \Rightarrow (7x+3) + 8 < (1-x)^2$ ***M1***

$\Rightarrow x^2 - 9x - 10 > 0$ ***A1***

$\Rightarrow (x+1)(x-10) > 0$ ***M1***

$x < -1,\ x > 10$ ***A1***

[6 marks]

18 – (MAT/11_HL_Summer_2017_Q2) – *Complex Numbers*

(a) $z_1 = 2\text{cis}\left(\frac{\pi}{3}\right)$ and $z_2 = \sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$ ***A1A1***

Note: Award ***A1A0*** for correct moduli and arguments found, but not written in mod-arg form.

(i) $|w| = \sqrt{2}$ ***A1***

(ii) $\arg w = \frac{\pi}{12}$ ***A1***

Notes: Allow ***FT*** from incorrect answers for z_1 and z_2 in modulus-argument form.

[4 marks]

(b) **EITHER**

$\sin\left(\frac{\pi n}{12}\right) = 0$ ***(M1)***

OR

$\arg\left(w^n\right) = \pi$ ***(M1)***

$\frac{n\pi}{12} = \pi$

THEN

$\therefore n = 12$ ***A1***

[2 marks]

19 – (MAT/12_HL_Summer_2017_Q5) – *Complex Numbers*

C represents the complex number $1 - 2\text{i}$ ***A2***

D represents the complex number $3 + 2\text{i}$ ***A2***

[4 marks]

20 – (MAT/12_HL_Summer_2017_Q11) – *Trigonometry, Complex Numbers*

(a) $2\sin(x+60^\circ)=\cos(x+30^\circ)$

$2(\sin x\cos 60^\circ+\cos x\sin 60^\circ)=\cos x\cos 30^\circ-\sin x\sin 30^\circ$ ***(M1)(A1)***

$2\sin x\times\frac{1}{2}+2\cos x\times\frac{\sqrt{3}}{2}=\cos x\times\frac{\sqrt{3}}{2}-\sin x\times\frac{1}{2}$ ***A1***

$\Rightarrow\frac{3}{2}\sin x=-\frac{\sqrt{3}}{2}\cos x$

$\Rightarrow\tan x=-\frac{1}{\sqrt{3}}$ ***M1***

$\Rightarrow x=150^\circ$ ***A1***

[5 marks]

(b) **EITHER**

choosing two appropriate angles, for example 60° and 45° ***M1***

$\sin 105^\circ=\sin 60^\circ\cos 45^\circ+\cos 60^\circ\sin 45^\circ$ and

$\cos 105^\circ=\cos 60^\circ\cos 45^\circ-\sin 60^\circ\sin 45^\circ$ ***(A1)***

$\sin 105^\circ+\cos 105^\circ=\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}+\frac{1}{2}\times\frac{1}{\sqrt{2}}+\frac{1}{2}\times\frac{1}{\sqrt{2}}-\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}$ ***A1***

$=\frac{1}{\sqrt{2}}$ ***AG***

OR

attempt to square the expression ***M1***

$\left(\sin 105^\circ+\cos 105^\circ\right)^2=\sin^2 105^\circ+2\sin 105^\circ\cos 105^\circ+\cos^2 105^\circ$

$\left(\sin 105^\circ+\cos 105^\circ\right)^2=1+\sin 210^\circ$ ***A1***

$=\frac{1}{2}$ ***A1***

$\sin 105^\circ+\cos 105^\circ=\frac{1}{\sqrt{2}}$ ***AG***

[3 marks]

(c) (i) **EITHER**

$z = (1 - \cos 2\theta) - \mathrm{i}\sin 2\theta$

$|z| = \sqrt{(1-\cos 2\theta)^2 + (\sin 2\theta)^2}$ ***M1***

$|z| = \sqrt{1 - 2\cos 2\theta + \cos^2 2\theta + \sin^2 2\theta}$ ***A1***

$= \sqrt{2}\sqrt{(1-\cos 2\theta)}$ ***A1***

$= \sqrt{2\left(2\sin^2\theta\right)}$

$= 2\sin\theta$ ***A1***

let $\arg(z) = \alpha$

$\tan\alpha = -\dfrac{\sin 2\theta}{1-\cos 2\theta}$ ***M1***

$= \dfrac{-2\sin\theta\cos\theta}{2\sin^2\theta}$ ***(A1)***

$= -\cot\theta$ ***A1***

$\arg(z) = \alpha = -\arctan\left(\tan(\frac{\pi}{2} - \theta)\right)$ ***A1***

$= \theta - \dfrac{\pi}{2}$ ***A1***

OR

$z = (1 - \cos 2\theta) - \mathrm{i}\sin 2\theta$

$= 2\sin^2\theta - 2i\sin\theta\cos\theta$ ***M1A1***

$= 2\sin\theta\left(\sin\theta - i\cos\theta\right)$ ***(A1)***

$= -2i\sin\theta\left(\cos\theta + i\sin\theta\right)$ ***M1A1***

$= 2\sin\theta\left(\cos\left(\theta - \dfrac{\pi}{2}\right) + i\sin\left(\theta - \dfrac{\pi}{2}\right)\right)$ ***M1A1***

$|z| = 2\sin\theta$ ***A1***

$\arg(z) = \theta - \dfrac{\pi}{2}$ ***A1***

(ii) attempt to apply De Moivre's theorem ***M1***

$$(1-\cos 2\theta - \mathrm{i}\sin 2\theta)^{\frac{1}{3}} = 2^{\frac{1}{3}}(\sin\theta)^{\frac{1}{3}}\left[\cos\left(\frac{\theta-\frac{\pi}{2}+2n\pi}{3}\right)+\mathrm{i}\sin\left(\frac{\theta-\frac{\pi}{2}+2n\pi}{3}\right)\right]$$

A1A1A1

Note: ***A1*** for modulus, ***A1*** for dividing argument of z by 3 and ***A1*** for $2n\pi$.

Hence cube roots are the above expression when $n=-1,\ 0,\ 1$.
Equivalent forms are acceptable. ***A1***

[14 marks]

21 – (MAT/10_HL_Winter_2017_Q8) – *Complex Numbers*

METHOD 1

$$216\mathrm{i} = 216\left(\cos\frac{\pi}{2}+\mathrm{i}\sin\frac{\pi}{2}\right)$$ ***A1***

$$z+2\mathrm{i} = \sqrt[3]{216}\left(\cos\left(\frac{\pi}{2}+2\pi k\right)+\mathrm{i}\sin\left(\frac{\pi}{2}+2\pi k\right)\right)^{\frac{1}{3}}$$ ***(M1)***

$$z+2\mathrm{i} = 6\left(\cos\left(\frac{\pi}{6}+\frac{2\pi k}{3}\right)+\mathrm{i}\sin\left(\frac{\pi}{6}+\frac{2\pi k}{3}\right)\right)$$ ***A1***

$$z_1+2\mathrm{i} = 6\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right) = 6\left(\frac{\sqrt{3}}{2}+\frac{\mathrm{i}}{2}\right) = 3\sqrt{3}+3\mathrm{i}$$

$$z_2+2\mathrm{i} = 6\left(\cos\frac{5\pi}{6}+\mathrm{i}\sin\frac{5\pi}{6}\right) = 6\left(\frac{-\sqrt{3}}{2}+\frac{\mathrm{i}}{2}\right) = -3\sqrt{3}+3\mathrm{i}$$

$$z_3+2\mathrm{i} = 6\left(\cos\frac{3\pi}{2}+\mathrm{i}\sin\frac{3\pi}{2}\right) = -6\mathrm{i}$$ ***A2***

Note: Award ***A1A0*** for one correct root.

so roots are $z_1 = 3\sqrt{3}+\mathrm{i}$, $z_2 = -3\sqrt{3}+\mathrm{i}$ and $z_3 = -8\mathrm{i}$ ***M1A1***

Note: Award ***M1*** for subtracting 2i from their three roots.

[7 marks]

METHOD 2

$\left(a\sqrt{3}+(b+2)\mathrm{i}\right)^3 = 216\mathrm{i}$

$\left(a\sqrt{3}\right)^3 + 3\left(a\sqrt{3}\right)^2(b+2)\mathrm{i} - 3\left(a\sqrt{3}\right)(b+2)^2 - \mathrm{i}(b+2)^3 = 216\mathrm{i}$ ***M1A1***

$\left(a\sqrt{3}\right)^3 - 3\left(a\sqrt{3}\right)(b+2)^2 + \mathrm{i}\left(3\left(a\sqrt{3}\right)^2(b+2) - (b+2)^3\right) = 216\mathrm{i}$

$\left(a\sqrt{3}\right)^3 - 3\left(a\sqrt{3}\right)(b+2)^2 = 0$ and $3\left(a\sqrt{3}\right)^2(b+2) - (b+2)^3 = 216$ ***M1A1***

$a\left(a^2 - (b+2)^2\right) = 0$ and $9a^2(b+2) - (b+2)^3 = 216$

$a = 0$ or $a^2 = (b+2)^2$

if $a = 0$, $-(b+2)^3 = 216 \Rightarrow b + 2 = -6$

$\therefore b = -8$ ***A1***

$(a, b) = (0, -8)$

if $a^2 = (b+2)^2$, $9(b+2)^2(b+2) - (b+2)^3 = 216$

$8(b+2)^3 = 216$

$(b+2)^3 = 27$

$b + 2 = 3$

$b = 1$

$\therefore a^2 = 9 \Rightarrow a = \pm 3$

$\therefore (a, b) = (\pm 3, 1)$ ***A1A1***

so roots are $z_1 = 3\sqrt{3} + \mathrm{i}$, $z_2 = -3\sqrt{3} + \mathrm{i}$ and $z_3 = -8\mathrm{i}$

METHOD 3

$(z+2i)^3 - (-6i)^3 = 0$

attempt to factorise: ***M1***

$\left((z+2i)-(-6i)\right)\left((z+2i)^2 + (z+2i)(-6i) + (-6i)^2\right) = 0$ ***A1***

$(z+8i)\left(z^2 - 2iz - 28\right) = 0$ ***A1***

$z + 8i = 0 \Rightarrow z = -8i$ ***A1***

$z^2 - 2iz - 28 = 0 \Rightarrow z = \dfrac{2i \pm \sqrt{-4-(4\times 1\times -28)}}{2}$ ***M1***

$z = \dfrac{2i \pm \sqrt{108}}{2}$

$z = \dfrac{2i \pm 6\sqrt{3}}{2}$

$z = i \pm 3\sqrt{3}$ ***A1A1***

Special Case:

Note: If a candidate recognises that $\sqrt[3]{216i} = -6i$ (anywhere seen), and makes no valid progress in finding three roots, award ***A1*** only.

[7 marks]

22 - (MAT/12_HL_Summer_2018_Q7) - *Complex Numbers*

(a) $\dfrac{z+w}{z-w} = \dfrac{(a+c)+\mathrm{i}(b+d)}{(a-c)+\mathrm{i}(b-d)}$

$= \dfrac{(a+c)+\mathrm{i}(b+d)}{(a-c)+\mathrm{i}(b-d)} \times \dfrac{(a-c)-\mathrm{i}(b-d)}{(a-c)-\mathrm{i}(b-d)}$ ***M1A1***

real part $= \dfrac{(a+c)(a-c)+(b+d)(b-d)}{(a-c)^2+(b-d)^2} \left(= \dfrac{a^2-c^2+b^2-d^2}{(a-c)^2+(b-d)^2} \right)$ ***A1A1***

Note: Award ***A1*** for numerator, ***A1*** for denominator.

[4 marks]

(b) $|z| = |w| \Rightarrow a^2+b^2 = c^2+d^2$ ***R1***

hence real part $= 0$ ***A1***

Note: Do not award ***R0A1***.

[2 marks]

23 – (MAT/11_HL_Summer_2018_Q11) – *Complex Numbers*

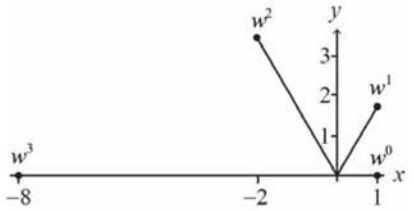
(a) (i) $w^2 = 4cis\left(\frac{2\pi}{3}\right)$; $w^3 = 8\text{cis}(\pi)$ ***(M1)A1A1***

Note: Accept Euler form.

Note: ***M1*** can be awarded for either both correct moduli or both correct arguments.

Note: Allow multiplication of correct Cartesian form for ***M1***, final answers must be in modulus-argument form.

(ii)



A1A1

[5 marks]

(b) use of area $= \frac{1}{2}ab\sin C$ ***M1***

$\frac{1}{2}\times 1\times 2\times \sin\frac{\pi}{3} + \frac{1}{2}\times 2\times 4\times \sin\frac{\pi}{3} + \frac{1}{2}\times 4\times 8\times \sin\frac{\pi}{3}$ ***A1A1***

Note: Award ***A1*** for $C = \frac{\pi}{3}$, ***A1*** for correct moduli.

$= \frac{21\sqrt{3}}{2}$ ***AG***

Note: Other methods of splitting the area may receive full marks.

[3 marks]

(c) $\frac{1}{2}\times 2^0\times 2^1\times \sin\frac{\pi}{n} + \frac{1}{2}\times 2^1\times 2^2\times \sin\frac{\pi}{n} + \frac{1}{2}\times 2^2\times 2^3\times \sin\frac{\pi}{n} + \ldots + \frac{1}{2}\times 2^{n-1}\times 2^n\times \sin\frac{\pi}{n}$

M1A1

Note: Award ***M1*** for powers of 2, ***A1*** for any correct expression including both the first and last term.

$= \sin\frac{\pi}{n}\times\left(2^0 + 2^2 + 2^4 + \ldots + 2^{2n-2}\right)$

identifying a geometric series with common ratio $2^2(=4)$ ***(M1)A1***

$= \frac{1-2^{2n}}{1-4}\times\sin\frac{\pi}{n}$ ***M1***

Note: Award ***M1*** for use of formula for sum of geometric series.

$= \frac{1}{3}\left(4^n - 1\right)\sin\frac{\pi}{n}$ ***A1***

[6 marks]

24 – (MAT/10_HL_Winter_2018_Q8) – *Complex Numbers*

$-\mathrm{i}\sqrt{3}$ is a root ***(A1)***

$3+\log_2 3-\log_2 6\left(=3+\log_2\frac{1}{2}=3-1=2\right)$ is a root ***(A1)***

sum of roots: $-a=3+\log_2 3 \Rightarrow a=-3-\log_2 3$ ***M1***

Note: Award ***M1*** for use of $-a$ is equal to the sum of the roots, do not award if minus is missing.

Note: If expanding the factored form of the equation, award ***M1*** for equating a to the coefficient of z^3.

product of roots: $(-1)^4 d \quad = 2(\log_2 6)\left(\mathrm{i}\sqrt{3}\right)\left(-\mathrm{i}\sqrt{3}\right)$ ***M1***

$= 6\log_2 6$ ***A1***

Note: Award ***M1A0*** for $d=-6\log_2 6$.

$6a+d+12=-18-6\log_2 3+6\log_2 6+12$

EITHER

$=-6+6\log_2 2=0$ ***M1A1AG***

Note: ***M1*** is for a correct use of one of the log laws.

OR

$=-6-6\log_2 3+6\log_2 3+6\log_2 2=0$ ***M1A1AG***

Note: ***M1*** is for a correct use of one of the log laws.

[7 marks]

25 – (MAT/10_HL_Winter_2018_Q11) – *Complex Numbers*

(a) $\left(r(\cos\theta + \mathrm{i}\sin\theta)\right)^{24} = 1(\cos 0 + \mathrm{i}\sin 0)$

use of De Moivre's theorem ***(M1)***

$r^{24} = 1 \Rightarrow r = 1$ ***(A1)***

$24\theta = 2\pi n \Rightarrow \theta = \frac{\pi n}{12}, (n \in \mathbb{Z})$ ***(A1)***

$0 < \arg z < \frac{\pi}{2} \Rightarrow n = 1, 2, 3, 4, 5$

$z = \mathrm{e}^{\frac{\pi\mathrm{i}}{12}}$ or $\mathrm{e}^{\frac{2\pi\mathrm{i}}{12}}$ or $\mathrm{e}^{\frac{3\pi\mathrm{i}}{12}}$ or $\mathrm{e}^{\frac{4\pi\mathrm{i}}{12}}$ or $\mathrm{e}^{\frac{5\pi\mathrm{i}}{12}}$ ***A2***

Note: Award ***A1*** if additional roots are given or if three correct roots are given with no incorrect (or additional) roots.

[5 marks]

(b) (i) $\operatorname{Re} S = \cos\frac{\pi}{12} + \cos\frac{2\pi}{12} + \cos\frac{3\pi}{12} + \cos\frac{4\pi}{12} + \cos\frac{5\pi}{12}$

$\operatorname{Im} S = \sin\frac{\pi}{12} + \sin\frac{2\pi}{12} + \sin\frac{3\pi}{12} + \sin\frac{4\pi}{12} + \sin\frac{5\pi}{12}$ ***A1***

Note: Award ***A1*** for both parts correct.

but $\sin\frac{5\pi}{12} = \cos\frac{\pi}{12}$, $\sin\frac{4\pi}{12} = \cos\frac{2\pi}{12}$, $\sin\frac{3\pi}{12} = \cos\frac{3\pi}{12}$,

$\sin\frac{2\pi}{12} = \cos\frac{4\pi}{12}$ and $\sin\frac{\pi}{12} = \cos\frac{5\pi}{12}$ ***M1A1***

$\Rightarrow \operatorname{Re} S = \operatorname{Im} S$ ***AG***

Note: Accept a geometrical method.

(ii) $\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}$ ***M1A1***

$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$ ***A1***

(iii) $\cos\frac{5\pi}{12} = \cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \cos\frac{\pi}{6}\cos\frac{\pi}{4} - \sin\frac{\pi}{6}\sin\frac{\pi}{4}$ ***(M1)***

Note: Allow alternative methods *eg* $\cos\frac{5\pi}{12} = \sin\frac{\pi}{12} = \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$.

$= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} - \frac{1}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$ ***(A1)***

$\operatorname{Re} S = \cos\frac{\pi}{12} + \cos\frac{2\pi}{12} + \cos\frac{3\pi}{12} + \cos\frac{4\pi}{12} + \cos\frac{5\pi}{12}$

$\operatorname{Re} S = \frac{\sqrt{2}+\sqrt{6}}{4} + \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} + \frac{1}{2} + \frac{\sqrt{6}-\sqrt{2}}{4}$ ***A1***

$= \frac{1}{2}\left(\sqrt{6} + 1 + \sqrt{2} + \sqrt{3}\right)$ ***A1***

$= \frac{1}{2}\left(1 + \sqrt{2}\right)\left(1 + \sqrt{3}\right)$

$S = \operatorname{Re}(S)(1+\mathrm{i})$ since $\operatorname{Re} S = \operatorname{Im} S$, ***R1***

$S = \frac{1}{2}\left(1 + \sqrt{2}\right)\left(1 + \sqrt{3}\right)(1 + \mathrm{i})$ ***AG***

[11 marks]

26 – (MAT/10_HL_Winter_2019_Q5) – *Complex Numbers*

(a) **METHOD 1**

$|z| = \sqrt[4]{4}\left(=\sqrt{2}\right)$ ***(A1)***

$\arg(z_1) = \frac{\pi}{4}$ ***(A1)***

first solution is $1+\mathrm{i}$ ***A1***

valid attempt to find all roots (De Moivre or +/– their components) ***(M1)***

other solutions are $-1+\mathrm{i}$, $-1-\mathrm{i}$, $1-\mathrm{i}$ ***A1***

[5 marks]

METHOD 2

$z^4 = -4$

$(a+\mathrm{i}b)^4 = -4$

attempt to expand and equate **both** reals and imaginaries. ***(M1)***

$a^4 + 4a^3b\mathrm{i} - 6a^2b^2 - 4ab^3\mathrm{i} + b^4 = -4$

$\left(a^4 - 6a^4 + a^4 = -4 \Rightarrow\right) a = \pm 1$ ***and*** $\left(4a^3b - 4ab^3 = 0 \Rightarrow\right) a = \pm b$ ***(A1)***

first solution is $1+\mathrm{i}$ ***A1***

valid attempt to find all roots (De Moivre or +/- their components) ***(M1)***

other solutions are $-1+\mathrm{i}$, $-1-\mathrm{i}$, $1-\mathrm{i}$ ***A1***

[5 marks]

(b) complete method to find area of 'rectangle' ***(M1)***

$=4$ ***A1***

[2 marks]

27 – (MAT/10_HL_Winter_2020_Q4) – *Complex Numbers*

substituting $z = x+\mathrm{i}y$ **and** $z^* = x-\mathrm{i}y$ ***M1***

$$\frac{2(x+\mathrm{i}y)}{3-(x-\mathrm{i}y)} = \mathrm{i}$$

$$2x + 2\mathrm{i}y = -y + \mathrm{i}(3-x)$$

equate real and imaginary: ***M1***

$y = -2x$ AND $2y = 3-x$ ***A1***

Note: If they multiply top and bottom by the conjugate, the equations $6x - 2x^2 + 2y^2 = 0$ and $6y - 4xy = (3-x)^2 + y^2$ may be seen. Allow for ***A1***.

solving simultaneously:

$x = -1, y = 2\ (z = -1+2\mathrm{i})$ ***A1A1***

[5 marks]

28 – (MAT/10_HL_Winter_2020_Q7) – *Complex Numbers*

(a) (i) $\frac{z_1}{z_2} = \cos\left(\frac{11\pi}{12} - \frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{11\pi}{12} - \frac{\pi}{6}\right)$ ***(M1)***

$= \cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}$ ***A1***

(ii) $\frac{z_2}{z_1} = \cos\frac{3\pi}{4} - \mathrm{i}\sin\frac{3\pi}{4}$ ***A1***

Note: Allow equivalent forms in part (a), *e.g.* $\mathrm{cis}\left(-\frac{3\pi}{4}\right)$.

Note: Ignore subsequent work once correct answer(s) are seen.

[3 marks]

(b) valid attempt to calculate area of their triangle ***(M1)***

(angle between OA and OB is $\frac{\pi}{2}$) $\Rightarrow$ area $\left(= \frac{1}{2} \times 1 \times 1\right) = \frac{1}{2}$ ***A1***

[2 marks]

29 – (MathAA/11_HL_Summer_2021_Q7) – *Complex Numbers*

METHOD 1

other two roots are $a-b\mathrm{i}$ and $b-a\mathrm{i}$ ***A1***

sum of roots $=-4$ and product of roots $=400$ ***A1***

attempt to set sum of four roots equal to -4 or 4 OR
attempt to set product of four roots equal to 400 ***M1***

$a+b\mathrm{i}+a-b\mathrm{i}+b+a\mathrm{i}+b-a\mathrm{i}=-4$

$2a+2b=-4\left(\Rightarrow a+b=-2\right)$ ***A1***

$\left(a+b\mathrm{i}\right)\left(a-b\mathrm{i}\right)\ \left(b+a\mathrm{i}\right)\left(b-a\mathrm{i}\right)=400$

$\left(a^2+b^2\right)^2=400$ ***A1***

$a^2+b^2=20$

attempt to solve simultaneous equations ***(M1)***

$a=2$ or $a=-4$ ***A1A1***

[8 marks]

METHOD 2

other two roots are $a-b\mathrm{i}$ and $b-a\mathrm{i}$ ***A1***

$\left(z-\left(a+b\mathrm{i}\right)\right)\left(z-\left(a-b\mathrm{i}\right)\right)\left(z-\left(b+a\mathrm{i}\right)\right)\left(z-\left(b-a\mathrm{i}\right)\right)\left(=0\right)$ ***A1***

$\left((z-a)^2+b^2\right)\left((z-b)^2+a^2\right)\left(=0\right)$

$\left(z^2-2az+a^2+b^2\right)\left(z^2-2bz+b^2+a^2\right)\left(=0\right)$ ***A1***

Attempt to equate coefficient of z^3 and constant with the given quartic equation ***M1***

$-2a-2b=4$ and $\left(a^2+b^2\right)^2=400$ ***A1***

attempt to solve simultaneous equations ***(M1)***

$a=2$ or $a=-4$ ***A1A1***

[8 marks]

30 – (MathAA/10_HL_Winter_2021_Q12) – *Complex Numbers*

(a) (i) $\left(1+e^{i\frac{\pi}{6}}-1\right)^3$

$=\left(e^{i\frac{\pi}{6}}\right)^3$ ***A1***

$=e^{i\frac{\pi}{2}}$ ***A1***

$=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$

$=i$ ***AG***

Note: Candidates who solve the equation correctly can be awarded the above two marks. The working for part (i) may be seen in part (ii).

(ii) $(z-1)^3=e^{i\left(\frac{\pi}{2}+2\pi k\right)}$ ***(M1)***

$z-1=e^{i\left(\frac{\pi}{6}+\frac{4\pi k}{6}\right)}$ ***(M1)***

$(k=1)\Rightarrow \omega_2=1+e^{i\frac{5\pi}{6}}$ ***A1***

$(k=2)\Rightarrow \omega_3=1+e^{i\frac{9\pi}{6}}$ ***A1***

[6 marks]

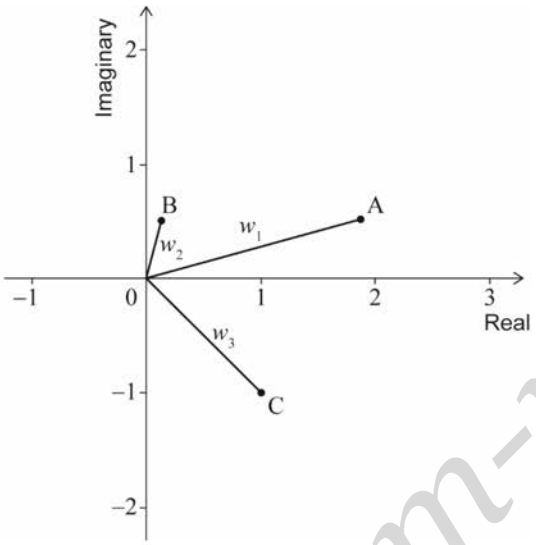
(b) **EITHER**

attempt to express $e^{i\frac{\pi}{6}}$, $e^{i\frac{5\pi}{6}}$, $e^{i\frac{9\pi}{6}}$ in Cartesian form and translate 1 unit in the positive direction of the real axis ***(M1)***

OR

attempt to express w_1, w_2 and w_3 in Cartesian form ***(M1)***

THEN



Note: To award ***A*** marks, it is not necessary to see A,B or C, the w_i , or the solid lines

A1A1A1

[4 marks]

(c) valid attempt to find $\omega_1 - \omega_3$ (or $\omega_3 - \omega_1$) ***M1***

$$\omega_1 - \omega_3 = \left(1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right) - (1 - \mathrm{i}) = \frac{\sqrt{3}}{2} + \frac{3}{2}\mathrm{i} \text{ OR } \cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{2}$$

valid attempt to find $\left|\frac{\sqrt{3}}{2} + \frac{3}{2}\mathrm{i}\right|$ ***M1***

$$= \sqrt{\frac{3}{4} + \frac{9}{4}}$$

$\mathrm{AC} = \sqrt{3}$ ***A1***

[3 marks]

(d) **METHOD 1**

$(z-1)^3 = \mathrm{i}z^3 \Rightarrow \left(\frac{z-1}{z}\right)^3 = \mathrm{i}$ ***M1***

$\left(\frac{z-1}{z}\right)^3 = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$ ***A1***

$\frac{\alpha-1}{\alpha} = \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ ***A1***

Note: This step to change from z to α may occur at any point in MS.

$\alpha - 1 = \alpha\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$

$\alpha - \alpha\mathrm{e}^{\mathrm{i}\frac{\pi}{6}} = 1$

$\alpha\left(1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right) = 1$

$\alpha = \frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}$ ***AG***

METHOD 2

$(z-1)^3 = \mathrm{i}z^3 \Rightarrow \left(\frac{z-1}{z}\right)^3 = \mathrm{i}$ ***M1***

$\left(1-\frac{1}{z}\right)^3 = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$ ***A1***

$1-\frac{1}{z} = \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ ***A1***

Note: This step to change from z to α may occur at any point in MS.

$1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}} = \frac{1}{\alpha}$

$\alpha = \frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}$ ***AG***

METHOD 3

LHS= $(z-1)^3 = \left(\frac{1}{1-e^{i\frac{\pi}{6}}} - 1\right)^3$

$= \left(\frac{e^{i\frac{\pi}{6}}}{1-e^{i\frac{\pi}{6}}}\right)^3$

$= \frac{i}{\left(1-e^{i\frac{\pi}{6}}\right)^3} \left(= \frac{i}{\frac{5}{2} - \frac{3\sqrt{3}}{2} + i\left(\frac{3\sqrt{3}}{2} - \frac{5}{2}\right)}\right)$ ***M1A1***

Note: Award ***M1*** for applying de Moivre's theorem (may be seen in modulus- argument form.)

RHS= $iz^3 = i\left(\frac{1}{1-e^{i\frac{\pi}{6}}}\right)^3$

$= \frac{i}{\left(1-e^{i\frac{\pi}{6}}\right)^3}$ ***A1***

$(z-1)^3 = iz^3$ ***AG***

METHOD 4

$(z-1)^3 = \mathrm{i}z^3$

$z^3 - 3z^2 + 3z - 1 = \mathrm{i}z^3$

$(1-\mathrm{i})z^3 - 3z^2 + 3z - 1 = 0$ ***(M1)***

$(1-\mathrm{i})\left(\frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}\right)^3 - 3\left(\frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}\right)^2 + 3\left(\frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}\right) - 1$

$= (1-\mathrm{i}) - 3\left(1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right) + 3\left(1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right)^2 - \left(1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right)^3$ ***(A1)***

$= (1-\mathrm{i}) - 3\left(1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right) + 3\left(1-2\mathrm{e}^{\mathrm{i}\frac{\pi}{6}} + \mathrm{e}^{\mathrm{i}\frac{\pi}{3}}\right) - \left(1-3\mathrm{e}^{\mathrm{i}\frac{\pi}{6}} + 3\mathrm{e}^{\mathrm{i}\frac{\pi}{3}} - \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\right)$ ***A1***

$= 0$ ***AG***

Note: If the candidate does not interpret their conclusion, award ***(M1)(A1)A0*** as appropriate.

[3 marks]

(e) **METHOD 1**

$$\frac{1}{1-e^{i\frac{\pi}{6}}}=\frac{1}{1-\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)}$$ ***M1***

$$=\frac{2}{2-\sqrt{3}-i}$$ ***A1***

attempt to use conjugate to rationalise ***M1***

$$=\frac{4-2\sqrt{3}+2i}{\left(2-\sqrt{3}\right)^2+1}$$ ***A1***

$$=\frac{4-2\sqrt{3}+2i}{8-4\sqrt{3}}$$ ***A1***

$$=\frac{1}{2}+\frac{1}{4-2\sqrt{3}}i$$

$$\Rightarrow \mathrm{Re}(\alpha)=\frac{1}{2}$$ ***A1***

Note: Their final imaginary part does not have to be correct in order for the final three ***A*** marks to be awarded

[6 marks]

METHOD 2

$$\frac{1}{1-e^{i\frac{\pi}{6}}}=\frac{1}{1-\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)}$$ ***M1***

attempt to use conjugate to rationalise ***M1***

$$=\frac{1}{\left(1-\cos\frac{\pi}{6}\right)-i\sin\frac{\pi}{6}}\times\frac{\left(1-\cos\frac{\pi}{6}\right)+i\sin\frac{\pi}{6}}{\left(1-\cos\frac{\pi}{6}\right)+i\sin\frac{\pi}{6}}$$ ***A1***

$$=\frac{\left(1-\cos\frac{\pi}{6}\right)+i\sin\frac{\pi}{6}}{\left(1-\cos\frac{\pi}{6}\right)^2+\sin^2\frac{\pi}{6}}$$ ***A1***

$$=\frac{\left(1-\cos\frac{\pi}{6}\right)+i\sin\frac{\pi}{6}}{1-2\cos\frac{\pi}{6}+\cos^2\frac{\pi}{6}+\sin^2\frac{\pi}{6}}$$

$$=\frac{\left(1-\cos\frac{\pi}{6}\right)+i\sin\frac{\pi}{6}}{2-2\cos\frac{\pi}{6}}$$ ***A1***

$$=\frac{1}{2}+\frac{i\sin\frac{\pi}{6}}{2-2\cos\frac{\pi}{6}}$$

$$\Rightarrow \mathrm{Re}(\alpha)=\frac{1}{2}$$ ***A1***

Note: Their final imaginary part does not have to be correct in order for the final three ***A*** marks to be awarded

[6 marks]

METHOD 3

attempt to multiply through by $-\dfrac{e^{i\frac{\pi}{12}}}{e^{i\frac{\pi}{12}}}$ ***M1***

$$\frac{1}{1-e^{i\frac{\pi}{6}}}=-\frac{e^{i\frac{\pi}{12}}}{e^{i\frac{\pi}{12}}-e^{-i\frac{\pi}{12}}}$$ ***A1***

attempting to re-write in r-cis form ***M1***

$$=-\frac{\cos\left(-\frac{\pi}{12}\right)+i\sin\left(-\frac{\pi}{12}\right)}{\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}-\left(\cos\left(-\frac{\pi}{12}\right)+i\sin\left(-\frac{\pi}{12}\right)\right)}$$ ***A1***

$$=-\frac{\cos\frac{\pi}{12}-i\sin\frac{\pi}{12}}{2i\sin\frac{\pi}{12}}$$ ***A1***

$$=\frac{1}{2}-\frac{1}{2i}\cot\frac{\pi}{12}\ \left(=\frac{1}{2}+\frac{1}{2}i\cot\frac{\pi}{12}\right)$$

$$\Rightarrow \mathrm{Re}(\alpha)=\frac{1}{2}$$ ***A1***

[6 marks]

METHOD 4

attempt to multiply through by $\dfrac{1-e^{i\frac{\pi}{6}}}{1-e^{i\frac{\pi}{6}}}$ ***M1***

$$\frac{1}{1-e^{i\frac{\pi}{6}}}=\frac{1-e^{i\frac{\pi}{6}}}{1-e^{i\frac{\pi}{6}}-e^{i\frac{\pi}{6}}+1}$$ ***A1***

attempting to re-write in r-cis form ***M1***

$$=\frac{1-\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}}{2-2\cos\frac{\pi}{6}}$$ ***A1***

attempt to re-write in Cartesian form ***M1***

$$=\frac{1-\frac{\sqrt{3}}{2}-\frac{1}{2}i}{2-\sqrt{3}}\left(=\frac{2-\sqrt{3}}{2}+i\frac{1}{2-\sqrt{3}}\right)$$

$$\Rightarrow \mathrm{Re}(\alpha)=\frac{1}{2}$$ ***A1***

Note: Their final imaginary part does not have to be correct in order for the final ***A*** mark to be awarded

[6 marks]

Total [22 marks]

31 - (MathAA/11_HL_Summer_2022_Q9) - *Complex Numbers*

(a) $z_1 z_2 = (1+b\mathrm{i})\left((1-b^2)-(2b)\mathrm{i}\right)$

$= (1-b^2-2\mathrm{i}^2b^2)+\mathrm{i}(-2b+b-b^3)$ ***M1***

$= (1+b^2)+\mathrm{i}(-b-b^3)$ ***A1A1***

Note: Award ***A1*** for $1+b^2$ and ***A1*** for $-b\mathrm{i}-b^3\mathrm{i}$.

[3 marks]

(b) $\arg(z_1 z_2) = \arctan\left(\dfrac{-b-b^3}{1+b^2}\right) = \dfrac{\pi}{4}$ ***(M1)***

EITHER

$\arctan(-b) = \dfrac{\pi}{4}$ (since $1+b^2 \neq 0$, for $b \in \mathbb{R}$) ***A1***

OR

$-b-b^3 = 1+b^2$ (or equivalent) ***A1***

THEN

$b = -1$ ***A1***

[3 marks]

32 – (MathAA/12_HL_Summer_2022_Q12) – *Complex Numbers*

(a) $z_2^* = r_2 e^{-i\theta}$ ***(A1)***

$z_1 z_2^* = r_1 e^{i\alpha} r_2 e^{-i\theta}$ ***A1***

$z_1 z_2^* = r_1 r_2 e^{i(\alpha-\theta)}$ ***AG***

Note: Accept working in modulus-argument form

[2 marks]

(b) $\mathrm{Re}\left(z_1 z_2^*\right) = r_1 r_2 \cos(\alpha-\theta) \quad (=0)$ ***A1***

$\alpha - \theta = \arccos 0 \ \left(r_1, r_2 > 0\right)$

$\alpha - \theta = \frac{\pi}{2}$ (as $0 < \alpha - \theta < \pi$) ***A1***

so Z_1OZ_2 is a right-angled triangle ***AG***

[2 marks]

(c) (i) **EITHER**

$\frac{z_1}{z_2}\left(= \frac{r_1}{r_2} e^{i(\alpha-\theta)}\right) = e^{i\frac{\pi}{3}}$ (since $r_1 = r_2$) ***(M1)***

OR

$z_1 = r_2 e^{i\left(\theta+\frac{\pi}{3}\right)} \left(= r_2 e^{i\theta} e^{i\frac{\pi}{3}}\right)$ ***(M1)***

THEN

$z_1 = z_2 e^{i\frac{\pi}{3}}$ ***A1***

Note: Accept working in either modulus-argument form to obtain $z_1 = z_2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ or in Cartesian form to obtain $z_1 = z_2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$.

(ii) substitutes $z_1 = z_2 e^{i\frac{\pi}{3}}$ into $z_1^2 + z_2^2$ ***M1***

$z_1^2 + z_2^2 = z_2^2 e^{i\frac{2\pi}{3}} + z_2^2 \left(= z_2^2 \left(e^{i\frac{2\pi}{3}} + 1 \right) \right)$ ***A1***

EITHER

$e^{i\frac{2\pi}{3}} + 1 = e^{i\frac{\pi}{3}}$ ***A1***

OR

$z_2^2 \left(e^{i\frac{2\pi}{3}} + 1 \right) = z_2^2 \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i + 1 \right)$

$= z_2^2 \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i \right)$ ***A1***

THEN

$z_1^2 + z_2^2 = z_2^2 e^{i\frac{\pi}{3}}$

$= z_2 \left(z_2 e^{i\frac{\pi}{3}} \right)$ and $z_2 e^{i\frac{\pi}{3}} = z_1$ ***A1***

so $z_1^2 + z_2^2 = z_1 z_2$ ***AG***

Note: For candidates who work on the LHS and RHS separately to show equality, award ***M1A1*** for $z_1^2 + z_2^2 = z_2^2 e^{i\frac{2\pi}{3}} + z_2^2 \left(= z_2^2 \left(e^{i\frac{2\pi}{3}} + 1 \right) \right)$, ***A1*** for $z_1 z_2 = z_2^2 e^{i\frac{\pi}{3}}$ and ***A1*** for $e^{i\frac{2\pi}{3}} + 1 = e^{i\frac{\pi}{3}}$. Accept working in either modulus-argument form or in Cartesian form.

[6 marks]

(d) **METHOD 1**

$z_1 + z_2 = -a$ and $z_1 z_2 = b$ ***(A1)***

$a^2 = z_1^2 + z_2^2 + 2z_1 z_2$ ***A1***

$a^2 = 2z_1 z_2 + z_1 z_2 \left(= 3z_1 z_2\right)$ ***A1***

substitutes $b = z_1 z_2$ into their expression ***M1***

$a^2 = 2b + b$ OR $a^2 = 3b$ ***A1***

Note: If $z_1 + z_2 = -a$ is not clearly recognized, award maximum ***(A0)A1A1M1A0.***

so $a^2 - 3b = 0$ ***AG***

METHOD 2

$z_1 + z_2 = -a$ and $z_1 z_2 = b$ ***(A1)***

$\left(z_1 + z_2\right)^2 = z_1^2 + z_2^2 + 2z_1 z_2$ ***A1***

$\left(z_1 + z_2\right)^2 = 2z_1 z_2 + z_1 z_2 \left(= 3z_1 z_2\right)$ ***A1***

substitutes $b = z_1 z_2$ and $z_1 + z_2 = -a$ into their expression ***M1***

$a^2 = 2b + b$ OR $a^2 = 3b$ ***A1***

Note: If $z_1 + z_2 = -a$ is not clearly recognized, award maximum ***(A0)A1A1M1A0.***

so $a^2 - 3b = 0$ ***AG***

[5 marks]

(e) $a^2 - 3 \times 12 = 0$

$a = \pm 6 \ \left(\Rightarrow z^2 \pm 6z + 12 = 0\right)$ ***A1***

for $a = -6$:

$z_1 = 3 + \sqrt{3}\mathrm{i},\ z_2 = 3 - \sqrt{3}\mathrm{i}$ and $\alpha - \theta = -\frac{5\pi}{3}$ which does not satisfy $0 < \alpha - \theta < \pi$ ***R1***

for $a = 6$:

$z_1 = -3 - \sqrt{3}\mathrm{i},\ z_2 = -3 + \sqrt{3}\mathrm{i}$ and $\alpha - \theta = \frac{\pi}{3}$ ***A1***

so (for $0 < \alpha - \theta < \pi$), only one equilateral triangle can be formed from point O and the two roots of this equation ***AG***

[3 marks]

33 - (MathAA/10_HL_Winter_2022_Q5) - *Complex Numbers*

(a) product of roots $=80$ ***(A1)***

$3-\mathrm{i}$ is a root ***(A1)***

attempt to set up an equation involving the product of their four roots and ± 80 ***(M1)***

$(3+\mathrm{i})(3-\mathrm{i})\alpha^3 = 80 \Rightarrow 10\alpha^3 = 80$

$\alpha = 2$ ***A1***

[4 marks]

(b) **METHOD 1**

sum of roots $=-p$ ***(A1)***

$-p = 3+\mathrm{i}+3-\mathrm{i}+2+4$ ***(M1)***

Note: Accept $p = 3+\mathrm{i}+3-\mathrm{i}+2+4$ for ***(M1)***

$p=-12$ ***A1***

METHOD 2

$(z-(3+\mathrm{i}))(z-(3-\mathrm{i}))(z-2)(z-4)$ ***(M1)***

$((z-3)-i)((z-3)+i)(z-2)(z-4)$ ***(A1)***

$(z^2-6z+10)(z^2-6z+8) = z^4-12z^3+....$

$p=-12$ ***A1***

[3 marks]

34 – (MathAA/10_HL_Winter_2022_Q12) – *Complex Numbers*

(a) (i) $z_0 = 1 + \mathrm{i}$ ***(A1)***

$\arg(z_0) = \arctan(1) = \frac{\pi}{4} = 45°$ ***A1***

Note: Accept any of these three forms, including an answer marked on an Argand diagram.

(ii) $\arg(z_n) = \arctan\left(\frac{1}{n^2 + n + 1}\right)$ ***A1***

[3 marks]

(b) (i) attempt to use the compound angle formula for tan ***M1***

$$\tan(\arctan(a) + \arctan(b)) = \frac{\tan(\arctan(a)) + \tan(\arctan(b))}{1 - \tan(\arctan(a))\tan(\arctan(b))}$$

$= \frac{a+b}{1-ab}$ ***A1***

$\Rightarrow \arctan(a) + \arctan(b) = \arctan\left(\frac{a+b}{1-ab}\right)$ ***AG***

(ii) **METHOD 1**

$\arg(w_1) = \arg(z_0 z_1) = \arg(z_0) + \arg(z_1)$ ***M1***

$= \arctan(1) + \arctan\left(\frac{1}{3}\right)$ ***(A1)***

$= \arctan\left(\frac{1 + \frac{1}{3}}{1 - \frac{1}{3}}\right)$ ***A1***

$= \arctan(2)$ ***AG***

METHOD 2

$w_1 = z_0 z_1 = (1+i)(3+i)$ ***(M1)***

$= 2+4i$ ***A1***

$\arg(w_1) = \arctan\left(\frac{4}{2}\right)$ or labelled Argand diagram ***A1***

$= \arctan(2)$ ***AG***

[5 marks]

(c) let $n = 0$

$$\text{LHS} = \arg(w_0) = \arg(z_0) = \arctan(1)\left(= \frac{\pi}{4}\right)$$

$$\text{RHS} = \arctan(1)\left(= \frac{\pi}{4}\right) \text{ so LHS = RHS}$$ ***R1***

Note: Award ***R0*** for not starting at $n = 0$, for example by referring to the result in (b) (ii) for $n = 1$. Award subsequent marks.

assume true for $n = k$, (so $\arg(w_k) = \arctan(k+1)$) ***M1***

Note: Do not award ***M1*** for statements such as "let $n = k$" or "$n = k$ is true". Subsequent marks can still be awarded.

$$\arg(w_{k+1})$$

$$= \arg(w_k z_{k+1})\left(= \arg(w_k) + \arg(z_{k+1})\right)$$ ***(M1)***

$$= \arctan(k+1) + \arctan\left(\frac{1}{(k+1)^2 + (k+1) + 1}\right)$$ ***A1***

$$= \arctan\left(\frac{(k+1) + \left(\frac{1}{(k+1)^2 + (k+1) + 1}\right)}{1 - (k+1)\left(\frac{1}{(k+1)^2 + (k+1) + 1}\right)}\right)$$ ***M1***

$$= \arctan\left(\frac{(k+1) + \left(\frac{1}{k^2 + 3k + 3}\right)}{1 - (k+1)\left(\frac{1}{k^2 + 3k + 3}\right)}\right)$$ ***(A1)***

$$= \arctan\left(\frac{(k+1)(k^2 + 3k + 3) + 1}{(k^2 + 3k + 3) - (k+1)}\right)$$

$= \arctan\left(\dfrac{k^3+4k^2+6k+4}{k^2+2k+2}\right)$ ***A1***

$= \arctan\left(\dfrac{(k+2)(k^2+2k+2)}{k^2+2k+2}\right)$ ***A1***

$= \arctan(k+2)\left(= \arctan\left((k+1)+1\right)\right)$ ***A1***

since true for $n=0$, and true for $n=k+1$ if true for $n=k$, the statement is true for all $n \in \mathbb{N}$ by mathematical induction ***R1***

Note: To obtain the final ***R1,*** four of the previous marks must have been awarded.

[10 marks]

1 - (MAT/10_HL_Winter_2014_Q10) - *Permutation - Combination*

(a) use of the addition principle with 3 terms
to obtain ${}^4C_3 + {}^5C_3 + {}^6C_3$ (= 4+ 10 + 20)
number of possible selections is 34

(b) **EITHER**

recognition of three cases: (2 odd and 2 even or 1 odd and 3 even or 0 odd and 4 even)
$\left({}^5C_2 \times {}^4C_2\right) + \left({}^5C_1 \times {}^4C_3\right) + \left({}^5C_0 \times {}^4C_4\right)$ (= 60 + 20 + 1)

OR

recognition to subtract the sum of 4 odd and 3 odd and 1 even from the total

${}^9C_4 - {}^5C_4 - \left({}^5C_3 \times {}^4C_1\right)$ $(=126-5-40)$

THEN

number of possible selections is 81

2 – (MAT/10_HL_Winter_2018_Q2) – *Permutation - Combination*

(a) **METHOD 1**

$\binom{8}{4}$ ***(A1)***

$= \frac{8!}{4!4!} = \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} = 7 \times 2 \times 5$ ***(M1)***

$= 70$ ***A1***

METHOD 2

recognition that they need to count the teams with 0 boys, 1 boy… 4 boys ***M1***

$1 + \binom{4}{1} \times \binom{4}{3} + \binom{4}{2} \times \binom{4}{2} + \binom{4}{1} \times \binom{4}{3} + 1$

$= 1 + (4 \times 4) + (6 \times 6) + (4 \times 4) + 1$ ***(A1)***

$= 70$ ***A1***

[3 marks]

(b) **EITHER**

recognition that the answer is the total number of teams minus the number of teams with all girls or all boys ***(M1)***

$70 - 2$

OR

recognition that the answer is the total of the number of teams with 1 boy, 2 boys, 3 boys ***(M1)***

$\binom{4}{1} \times \binom{4}{3} + \binom{4}{2} \times \binom{4}{2} + \binom{4}{1} \times \binom{4}{3} = (4 \times 4) + (6 \times 6) + (4 \times 4)$

THEN

$= 68$ ***A1***

[2 marks]

3 – (MathAA/11_HL_Summer_2021_Q9) – *Permutation - Combination*

(a) **METHOD 1**

B has one less pen to select ***(M1)***

EITHER

A and B can be placed in 6×5 ways ***(A1)***

C, D, E have 6 choices each ***(A1)***

OR

A (or B), C, D, E have 6 choices each ***(A1)***

B (or A) has only 5 choices ***(A1)***

THEN

$5\times6^4\left(=6480\right)$ ***A1***

METHOD 2

total number of ways $=6^5$ ***(A1)***

number of ways with Amber and Brownie together $=6^4$ ***(A1)***

attempt to subtract (may be seen in words) ***(M1)***

6^5-6^4

$=5\times6^4\left(=6480\right)$ ***A1***

[4 marks]

(b) **METHOD 1**

total number of ways $=6!\left(=720\right)$ ***(A1)***

number of ways with Amber and Brownie sharing a boundary

$=2\times7\times4!\left(=336\right)$ ***(A1)***

attempt to subtract (may be seen in words) ***(M1)***

$720-336=384$ ***A1***

METHOD 2

case 1: number of ways of placing A in corner pen

$3\times4\times3\times2\times1$

Four corners total no of ways is $4\times\left(3\times4\times3\times2\times1\right)=12\times4!\left(=288\right)$ ***(A1)***

case 2: number of ways of placing A in the middle pen

$2\times4\times3\times2\times1$

two middle pens so $2\times\left(2\times4\times3\times2\times1\right)=4\times4!\left(=96\right)$ ***(A1)***

attempt to add (may be seen in words) ***(M1)***

total no of ways $=288+96$

$=16\times4!\left(=384\right)$ ***A1***

[4 marks]

4 – (MathAA/10_HL_Winter_2022_Q11) – *Permutation - Combination, Remainder & Factor Theorem*

(a) (i) 5^3 ***(A1)***

$= 125$ ***A1***

(ii) ${}^5P_3 = 5 \times 4 \times 3$ ***(A1)***

$= 60$ ***A1***

[4 marks]

(b) (i) **METHOD 1**

$x^2 + 3x + 2 = (x+1)(x+2)$ ***(A1)***

correct use of factor theorem for at least one of their factors ***(M1)***

$P(-1) = 0$ or $P(-2) = 0$

attempt to find two equations in a, b and c ***(M1)***

$(-1)^3 + a(-1)^2 + b(-1) + c = 0 (\rightarrow -1 + a - b + c = 0)$

$(-2)^3 + a(-2)^2 + b(-2) + c = 0$

$-8 + 4a - 2b + c = 0$ and $-1 \div a - b + c = 0$ ***A1***

attempt to combine their two equations in $-8 + 4a - 2b + c = 0$ to eliminate c ***(M1)***

$b = 3a - 7$ ***A1***

Note: Award at most ***A1M1M1A0M1A0*** for $b = -3a - 7$ from $P(1) = P(2) = 0$

METHOD 2

$P(x) = x^3 + ax^2 + bx + c = (x^2 + 3x + 2)(x + d)$ ***(M1)***

$= x^3 + (3 + d)x^2 + (2 + 3d)x + 2d$ ***(A1)***

attempt to compare coefficients of x^2 and x ***(M1)***

$a = 3 + d$ and $b = 2 + 3d$ ***A1***

attempt to eliminate d ***(M1)***

$\Rightarrow b = 3a - 7$ ***A1***

METHOD 3

attempt to divide $x^3 + ax^2 + bx + c$ by $x^2 + 3x + 2$ ***M1***

$\frac{x^3 + ax^2 + bx + c}{x^2 + 3x + 2} = (x + a - 3) + \frac{(-3a + b + 7)x + (c - 2a + 6)}{x^2 + 3x + 2}$ ***A1A1A1***

Note: Award ***A1*** *for* $x + a - 3$, ***A1*** *for* $(-3a + b + 7)x$ and ***A1*** *for* $c - 2a + 6$

recognition that, if $(x^2 + 3x + 2)$ is a factor of $P(x)$, then $-3a + b + 7 = 0$ ***(M1)***

leading to $b = 3a - 7$ ***A1***

METHOD 4

$x^2+3x+2=(x+1)(x+2)$ ***(A1)***

attempt to use Vieta's formulae for a cubic with roots −1, −2 and "p" ***(M1)***

$(-1)+(-2)+p=-a(\Rightarrow p=3-a)$ ***A1***

$(-1)(-2)+(-1)p+(-2)p=b$ ***A1***

Attempt to eliminate "p" ***(M1)***

$2-(3-a)-2(3-a)=b$

$b=3a-7$ ***A1***

Note: Award at most ***A1M1A0A0M1A0*** for $b=-3a-7$ from roots 1, 2 and "p"

(ii) **METHOD 1**

$a=1,2,5$ lead to invalid values for b ***R1***

$a=3, b=2 \Rightarrow c=0$ so not possible ***R1***

so $a=4, b=5, c=2$ is the only solution ***AG***

METHOD 2

$c=2a-6$ ***R1***

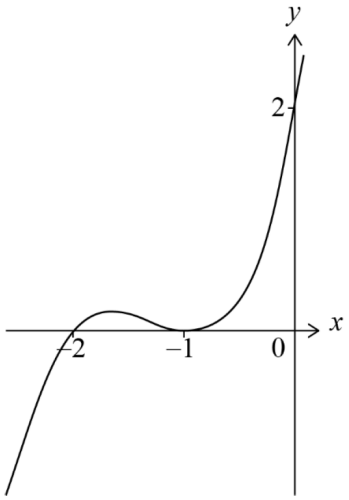
correctly argues $a=4$ is the only possibility ***R1***

so $a=4, b=5, c=2$ is the only solution ***AG***

(iii) $x^3+4x^2+5x+2=(x^2+3x+2)(x+1)$

$=(x+2)(x+1)(x+1)$ ***A1***

(iv)



positive cubic shape with y-intercept at $(0, 2)$ ***A1***

x-intercept at $(-2, 0)$ and local maximum point anywhere between $x = -2$ and $x = -1$ ***A1***

local minimum point at $(-1, 0)$ ***A1***

Note: Accept answers from an approach based on calculus.

[12 marks]

1 – (MAT/12_HL_Summer_2012_Q4) – *Binomial Theorem*

(a) $\left(x-\frac{2}{x}\right)^4 = x^4 + 4x^3\left(-\frac{2}{x}\right) + 6x^2\left(-\frac{2}{x}\right)^2 + 4x\left(-\frac{2}{x}\right)^3 + \left(-\frac{2}{x}\right)^4$

Note: Award ***(A1)*** for 3 or 4 correct terms.

Note: Accept combinatorial expressions, *e.g.* $\binom{4}{2}$ for 6.

$= x^4 - 8x^2 + 24 - \frac{32}{x^2} + \frac{16}{x^4}$

(b) constant term from expansion of $(2x^2+1)\left(x-\frac{2}{x}\right)^4 = -64 + 24 = -40$

2 – (MAT/10_HL_Winter_2012_Q2) – *Binomial Theorem*

$$\left(\frac{x}{y}-\frac{y}{x}\right)^4 = \left(\frac{x}{y}\right)^4 + 4\left(\frac{x}{y}\right)^3\left(-\frac{y}{x}\right) + 6\left(\frac{x}{y}\right)^2\left(-\frac{y}{x}\right)^2 + 4\left(\frac{x}{y}\right)\left(-\frac{y}{x}\right)^3 + \left(-\frac{y}{x}\right)^4$$

Note: Award ***M1*** for attempt to expand and ***A1*** for correct unsimplified expansion.

$$= \frac{x^4}{y^4} - 4\frac{x^2}{y^2} + 6 - 4\frac{y^2}{x^2} + \frac{y^4}{x^4} \qquad \left(= \frac{x^8 - 4x^6y^2 + 6x^4y^4 - 4x^2y^6 + y^8}{x^4y^4}\right)$$

3 – (MAT/12_HL_Summer_2013_Q3) – *Binomial Theorem*

clear attempt at binomial expansion for exponent 5

$$2^5 + 5\times 2^4 \times (-3x) + \frac{5\times 4}{2}\times 2^3 \times (-3x)^2 + \frac{5\times 4\times 3}{6}\times 2^2\times(-3x)^3$$
$$+ \frac{5\times 4\times 3\times 2}{24}\times 2\times(-3x)^4 + (-3x)^5$$

Note: Only award ***M1*** if binomial coefficients are seen.

$= 32 - 240x + 720x^2 - 1080x^3 + 810x^4 - 243x^5$

4 – (MAT/11_HL_Summer_2013_Q13) – *Probability, Binomial Theorem*

(a) $B\left(6, \frac{2}{3}\right)$

$p(4) = \binom{6}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2$

$\binom{6}{4} = 15$

$= 15 \times \frac{2^4}{3^6} = \frac{80}{243}$

(b) (i) 2 outcomes for each of the 6 games or $2^6 = 64$

(ii) $(1+x)^6 = \binom{6}{0} + \binom{6}{1}x + \binom{6}{2}x^2 + \binom{6}{3}x^3 + \binom{6}{4}x^4 + \binom{6}{5}x^5 + \binom{6}{6}x^6$

Note: Accept ${}^nC_r$ notation or $1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$

setting $x = 1$ in both sides of the expression

Note: Do not award ***R1*** if the right hand side is not in the correct form.

$64 = \binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}$

(iii) the total number of outcomes = number of ways Alfred can win no games, plus the number of ways he can win one game *etc*.

(c) (i) Let $\mathrm{P}(x, y)$ be the probability that Alfred wins x games on the first day and y on the second.

$\mathrm{P}(4, 2) = \binom{6}{4} \times \left(\frac{2}{3}\right)^4 \times \left(\frac{1}{3}\right)^2 \times \binom{6}{2} \times \left(\frac{2}{3}\right)^2 \times \left(\frac{1}{3}\right)^4$

$\binom{6}{2}^2\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^6$ or $\binom{6}{4}^2\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^6$

$r = 2$ or 4, $s = t = 6$

(ii) $\mathrm{P}(\text{Total}=6)=$

$\mathrm{P}(0,6)+\mathrm{P}(1,5)+\mathrm{P}(2,4)+\mathrm{P}(3,3)+\mathrm{P}(4,2)+\mathrm{P}(5,1)+\mathrm{P}(6,0)$

$$=\binom{6}{0}^2\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^6+\binom{6}{1}^2\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^6+\ldots+\binom{6}{6}^2\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^6$$

$$=\frac{2^6}{3^{12}}\left(\binom{6}{0}^2+\binom{6}{1}^2+\binom{6}{2}^2+\binom{6}{3}^2+\binom{6}{4}^2+\binom{6}{5}^2+\binom{6}{6}^2\right)$$

Note: Accept any valid sum of 7 probabilities.

(iii) use of $\binom{6}{i}=\binom{6}{6-i}$

(can be used either here or in (c)(ii))

$$\mathrm{P}(\text{wins 6 out of 12})=\binom{12}{6}\times\left(\frac{2}{3}\right)^6\times\left(\frac{1}{3}\right)^6=\frac{2^6}{3^{12}}\binom{12}{6}$$

$$\frac{2^6}{3^{12}}\left(\binom{6}{0}^2+\binom{6}{1}^2+\binom{6}{2}^2+\binom{6}{3}^2+\binom{6}{4}^2+\binom{6}{5}^2+\binom{6}{6}^2\right)=\frac{2^6}{3^{12}}\binom{12}{6}$$

therefore $\binom{6}{0}^2+\binom{6}{1}^2+\binom{6}{2}^2+\binom{6}{3}^2+\binom{6}{4}^2+\binom{6}{5}^2+\binom{6}{6}^2=\binom{12}{6}$

(d) (i) $\mathrm{E}(A)=\sum_{r=0}^{n}r\binom{n}{r}\left(\frac{2}{3}\right)^r\left(\frac{1}{3}\right)^{n-r}=\sum_{r=0}^{n}r\binom{n}{r}\frac{2^r}{3^n}$

$(a=2,\ b=3)$

Note: ***M0A0*** for $a=2$, $b=3$ without any method.

(ii) $n(1+x)^{n-1}=\sum_{r=1}^{n}\binom{n}{r}rx^{r-1}$

(sigma notation not necessary)

(if sigma notation used also allow lower limit to be $r=0$)

let $x=2$

$$n3^{n-1}=\sum_{r=1}^{n}\binom{n}{r}r2^{r-1}$$

multiply by 2 and divide by 3^n

$$\frac{2n}{3}=\sum_{r=1}^{n}\binom{n}{r}r\frac{2^r}{3^n}\quad\left(=\sum_{r=0}^{n}\binom{n}{r}\frac{2^r}{3^n}\right)$$

5 – (MAT/12_HL_Summer_2015_Q2) – *Binomial Theorem*

$(3-x)^4 = 1.3^4 + 4.3^3(-x) + 6.3^2(-x)^2 + 4.3(-x)^3 + 1(-x)^4$ or equivalent

$= 81 - 108x + 54x^2 - 12x^3 + x^4$

6 – (MAT/11_HL_Summer_2015_Q4) – *Binomial Theorem, Differentiation*

(a) $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

(b) $f'(x) = \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h}$

$= \lim_{h\to 0}\frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$

$= \lim_{h\to 0}\left(3x^2 + 3xh + h^2\right)$

$= 3x^2$

7 – (MAT/10_HL_Winter_2015_Q3) – *Binomial Theorem*

(a) $(2+x)^4 = 2^4 + 4\cdot 2^3 x + 6\cdot 2^2 x^2 + 4\cdot 2x^3 + x^4$ ***M1(A1)***

Note: Award ***M1*** for an expansion, by whatever method, giving five terms in any order.

$= 16 + 32x + 24x^2 + 8x^3 + x^4$ ***A1***

Note: Award ***M1A1A0*** for correct expansion not given in ascending powers of x.

(b) let $x = 0.1$ (in the binomial expansion) ***(M1)***

$2.1^4 = 16 + 3.2 + 0.24 + 0.008 + 0.0001$ ***(A1)***

$= 19.4481$ ***A1***

8 – (MAT/12_HL_Summer_2016_Q6) – *Binomial Theorem*

(a) $1,\ nx,\ \frac{n(n-1)}{2}x^2,\ \frac{n(n-1)(n-2)}{6}x^3$

Note: Award ***A1*** for the first two terms and ***A1*** for the next two terms.

Note: Accept $\binom{n}{r}$ notation.

Note: Allow the terms seen in the context of an arithmetic sum.

Note: Allow unsimplified terms, *eg*, those including powers of 1 if seen.

(b) (i) **EITHER**

using $u_3 - u_2 = u_4 - u_3$

$\frac{n(n-1)}{2} - n = \frac{n(n-1)(n-2)}{6} - \frac{n(n-1)}{2}$

attempting to remove denominators and expanding (or vice versa)

$3n^2 - 9n = n^3 - 6n^2 + 5n$ (or equivalent, *eg*, $6n^2 - 12n = n^3 - 3n^2 + 2n$)

OR

using $u_2 + u_4 = 2u_3$

$n + \frac{n(n-1)(n-2)}{6} = n(n-1)$

attempting to remove denominators and expanding (or vice versa)

$6n + n^3 - 3n^2 + 2n = 6n^2 - 6n$ (or equivalent)

THEN

$n^3 - 9n^2 + 14n = 0$

(ii) $n(n-2)(n-7) = 0$ or $(n-2)(n-7) = 0$

$n = 7$ only (as $n \geq 3$)

9 – (MAT/12_HL_Summer_2017_Q1) – *Binomial Theorem*

attempt at binomial expansion, relevant row of Pascal's triangle or use of general term with binomial coefficient must be seen ***(M1)***

term independent of x is $\binom{10}{4}\left(2x^2\right)^6\left(\frac{1}{2x^3}\right)^4$ (or equivalent) ***(A1)(A1)(A1)***

Notes: x's may be omitted.
Also accept $\binom{10}{6}$ or 210.

$= 840$ ***A1***

[5 marks]

10 – (MAT/10_HL_Winter_2017_Q4) – *Binomial Theorem*

each term is of the form $\binom{7}{r}\left(x^2\right)^{7-r}\left(\frac{-2}{x}\right)^r$ ***(M1)***

$= \binom{7}{r}x^{14-2r}(-2)^r x^{-r}$

so $14 - 3r = 8$ ***(A1)***

$r = 2$

so require $\binom{7}{2}\left(x^2\right)^5\left(\frac{-2}{x}\right)^2$ (or simply $\binom{7}{2}(-2)^2$) ***A1***

$= 21 \times 4$

$= 84$ ***A1***

Note: Candidates who attempt a full expansion, including the correct term, may only be awarded ***M1A0A0A0.***

[4 marks]

11 – (MAT/11_HL_Summer_2019_Q2) – *Binomial Theorem*

attempt at binomial expansion ***M1***

$1+\binom{11}{1}(-2x)+\binom{11}{2}(-2x)^2+\ldots$ ***(A1)***

$\binom{11}{2} = 55$

$1-22x+220x^2$ ***A1A1***

Note: ***A1*** for first two terms, ***A1*** for final term.

Note: Award ***M1(A1)A0A0*** for $(-2x)^{11}+\binom{11}{10}(-2x)^{10}+\binom{11}{9}(-2x)^9+\ldots,$

12 – (MathAA/12_HL_Summer_2021_Q3) – *Binomial Theorem*

EITHER

attempt to use the binomial expansion of $(x+k)^7$ ***(M1)***

${}^7C_0x^7k^0 + {}^7C_1x^6k^1 + {}^7C_2x^5k^2 + \ldots$ (or ${}^7C_0k^7x^0 + {}^7C_1k^5x^1 + {}^7C_2k^5x^2 + \ldots$)

identifying the correct term ${}^7C_2x^5k^2$ (or ${}^7C_5k^2x^5$) ***(A1)***

OR

attempt to use the general term ${}^7C_rx^rk^{7-r}$ (or ${}^7C_rk^rx^{7-r}$) ***(M1)***

$r=2$ (or $r=5$) ***(A1)***

THEN

${}^7C_2 = 21$ (or ${}^7C_5 = 21$) (seen anywhere) ***(A1)***

$21x^5k^2 = 63x^5$ ($21k^2 = 63$, $k^2 = 3$) ***A1***

$k = \pm\sqrt{3}$ ***A1***

Note: If working shown, award ***M1A1A1A1A0*** for $k=\sqrt{3}$.

[5 marks]

13 – (MathAA/10_HL_Winter_2021_Q9) – *Binomial Theorem*

(a) attempt to expand binomial with negative fractional power ***(M1)***

$$\frac{1}{\sqrt{1+ax}} = (1+ax)^{-\frac{1}{2}} = 1 - \frac{ax}{2} + \frac{3a^2x^2}{8} + \ldots$$ ***A1***

$$\sqrt{1-x} = (1-x)^{\frac{1}{2}} = 1 - \frac{x}{2} - \frac{x^2}{8} + \ldots$$ ***A1***

$$\frac{1}{\sqrt{1+ax}} - \sqrt{1-x} = \frac{(1-a)}{2}x + \left(\frac{3a^2+1}{8}\right)x^2 + \ldots$$

attempt to equate coefficients of x or x^2 ***(M1)***

x : $\frac{1-a}{2} = 4b$; x^2 : $\frac{3a^2+1}{8} = b$

attempt to solve simultaneously ***(M1)***

$a = -\frac{1}{3}, b = \frac{1}{6}$ ***A1***

[6 marks]

(b) $|x| < 1$ ***A1***

[1 mark]

Total [7 marks]

14 – (MathAA/11_HL_Summer_2022_Q6) – *Binomial Theorem*

EITHER

attempt to obtain the general term of the expansion

$T_{r+1} = {}^nC_r\left(8x^3\right)^{n-r}\left(-\frac{1}{2x}\right)^r$ OR $T_{r+1} = {}^nC_{n-r}\left(8x^3\right)^r\left(-\frac{1}{2x}\right)^{n-r}$ ***(M1)***

OR

recognize power of x starts at $3n$ and goes down by 4 each time ***(M1)***

THEN

recognizing the constant term when the power of x is zero (or equivalent) ***(M1)***

$r = \frac{3n}{4}$ or $n = \frac{4}{3}r$ or $3n - 4r = 0$ OR $3r - (n - r) = 0$ (or equivalent) ***A1***

r is a multiple of 3 $\left(r = 3, 6, 9, ...\right)$ or one correct value of n (seen anywhere) ***(A1)***

$n = 4k, k \in \mathbb{Z}^+$ ***A1***

Note: Accept n is a (positive) multiple of 4 or $n = 4, 8, 12, ...$

Do not accept $n = 4, 8, 12$

Note: Award full marks for a correct answer using trial and error approach showing $n = 4, 8, 12, ...$ and for recognizing that this pattern continues.

[5 marks]

15 – (MathAA/12_HL_Summer_2022_Q5) – *Binomial Theorem*

(a) **EITHER**

recognises the required term (or coefficient) in the expansion ***(M1)***

$bx^5 = {}^7C_2 x^5 1^2$ OR $b = {}^7C_2$ OR ${}^7C_5$

$b = \frac{7!}{2!5!} \left(= \frac{7!}{2!(7-2)!} \right)$

correct working ***A1***

$\frac{7\times6\times5\times4\times3\times2\times1}{2\times1\times5\times4\times3\times2\times1}$ OR $\frac{7\times6}{2!}$ OR $\frac{42}{2}$

OR

lists terms from row 7 of Pascal's triangle ***(M1)***

$1, 7, 21, \ldots$ ***A1***

THEN

$b = 21$ ***AG***

[2 marks]

(b) $a = 7$ ***(A1)***

correct equation ***A1***

$21x^5 = \frac{ax^6 + 35x^4}{2}$ OR $21x^5 = \frac{7x^6 + 35x^4}{2}$

correct quadratic equation ***A1***

$7x^2 - 42x + 35 = 0$ OR $x^2 - 6x + 5 = 0$ (or equivalent)

valid attempt to solve their quadratic ***(M1)***

$(x-1)(x-5) = 0$ OR $x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(5)}}{2(1)}$

$x = 1,\ x = 5$ ***A1***

Note: Award final ***A0*** for obtaining $x = 0,\ x = 1,\ x = 5$.

[5 marks]

1 - (MAT/12_HL_Summer_2012_Q1) - *Remainder & Factor Theorem*

let $f(x)=2x^3+kx^2+6x+32$

let $g(x)=x^4-6x^2-k^2x+9$

$f(-1)=-2+k-6+32(=24+k)$

$g(-1)=1-6+k^2+9(=4+k^2)$

$\Rightarrow 24+k=4+k^2$

$\Rightarrow k^2-k-20=0$

$\Rightarrow (k-5)(k+4)=0$

$\Rightarrow k=5,-4$

2 - (MAT/10_HL_Winter_2013_Q1) - *Remainder & Factor Theorem*

$f(-2)=0 \ \ (\Rightarrow -24+4p-2q-2=0)$ ***M1***

$f(-1)=4 \ \ (\Rightarrow -3+p-q-2=4)$ ***M1***

Note: In each case award the ***M*** marks if correct substitution attempted and right-hand side correct.

attempt to solve simultaneously ($2p-q=13$, $p-q=9$) ***M1***

$p=4$ ***A1***

$q=-5$ ***A1***

3 - (MAT/11_HL_Summer_2014_Q1) - *Remainder & Factor Theorem*

$P(2)=24+2a+b=2,\ P(-1)=-3-a+b=5$

$(2a+b=-22, -a+b=8)$

Note: Award ***M1*** for substitution of 2 or -1 and equating to remainder, ***A1*** for each correct equation.

attempt to solve simultaneously

$a=-10$, $b=-2$

4 – (MAT/10_HL_Winter_2017_Q3) – *Remainder & Factor Theorem*

(a) $q(4) = 0$ ***(M1)***

$192 - 176 + 4k + 8 = 0 \ (24 + 4k = 0)$ ***A1***

$k = -6$ ***A1***

[3 marks]

(b) $3x^3 - 11x^2 - 6x + 8 = (x - 4)\left(3x^2 + px - 2\right)$

equate coefficients of x^2: ***(M1)***

$-12 + p = -11$

$p = 1$

$(x - 4)\left(3x^2 + x - 2\right)$ ***(A1)***

$(x - 4)(3x - 2)(x + 1)$ ***A1***

Note: Allow part (b) marks if any of this work is seen in part (a).

Note: Allow equivalent methods (*eg*, synthetic division) for the ***M*** marks in each part.

[3 marks]

5 – (MAT/11_HL_Summer_2018_Q1) – *Remainder & Factor Theorem*

attempt to substitute $x = -1$ or $x = 2$ or to divide polynomials ***(M1)***

$1 - p - q + 5 = 7$, $16 + 8p + 2q + 5 = 1$ or equivalent ***A1A1***

attempt to solve their two equations ***M1***

$p = -3$, $q = 2$ ***A1***

[5 marks]

6 – (MAT/11_HL_Summer_2019_Q10) – *Remainder & Factor Theorem, Probability*

(a) (i) $p(2) = 8 - 12 + 16 - 24$ ***(M1)***

Note: Award ***M1*** for a valid attempt at remainder theorem or polynomial division.

$= -12$ ***A1***

remainder $= -12$

(ii) $p(3) = 27 - 27 + 24 - 24 = 0$ ***A1***

remainder $= 0$

[3 marks]

(b) $x = 3$ (is a zero) ***A1***

Note: Can be seen anywhere.

EITHER

factorise to get $(x-3)\left(x^2+8\right)$ ***(M1)A1***

$x^2 + 8 \neq 0$ (for $x \in \mathbb{R}$) (or equivalent statement) ***R1***

Note: Award ***R1*** if correct two complex roots are given.

OR

$p'(x) = 3x^2 - 6x + 8$ ***A1***

attempting to show $p'(x) \neq 0$ ***M1***

eg discriminant $= 36 - 96 < 0$, completing the square

no turning points ***R1***

THEN

only one real zero (as the curve is continuous) ***AG***

[4 marks]

(c) new graph is $y = p(2x)$ ***(M1)***

stretch parallel to the x-axis (with $x = 0$ invariant), scale factor 0.5 ***A1***

[2 marks]

Note: Accept "horizontal" instead of "parallel to the x-axis".

(d) $\frac{6\lambda^3 e^{-\lambda}}{6} = \frac{3\lambda^2 e^{-\lambda}}{2} - 2\lambda e^{-\lambda} + 3e^{-\lambda}$ ***M1A1***

Note: Allow factorials in the denominator for ***A1***.

$2\lambda^3 - 3\lambda^2 + 4\lambda - 6 = 0$ ***A1***

Note: Accept any correct cubic equation without factorials and $e^{-\lambda}$.

EITHER

$4\left(2\lambda^3 - 3\lambda^2 + 4\lambda - 6\right) = 8\lambda^3 - 12\lambda^2 + 16\lambda - 24 = 0$ ***(M1)***

$2\lambda = 3$ ***(A1)***

OR

$(2\lambda - 3)\left(\lambda^2 + 2\right) = 0$ ***(M1)(A1)***

THEN

$\lambda = 1.5$ ***A1***

[6 marks]

7 – (MathAA/10_HL_Winter_2022_Q11) – *Permutation - Combination, Remainder & Factor Theorem*

(a) (i) 5^3 ***(A1)***

$= 125$ ***A1***

(ii) ${}^5P_3 = 5 \times 4 \times 3$ ***(A1)***

$= 60$ ***A1***

[4 marks]

(b) (i) **METHOD 1**

$x^2 + 3x + 2 = (x+1)(x+2)$ ***(A1)***

correct use of factor theorem for at least one of their factors ***(M1)***

$P(-1) = 0$ or $P(-2) = 0$

attempt to find two equations in a, b and c ***(M1)***

$(-1)^3 + a(-1)^2 + b(-1) + c = 0 (\rightarrow -1 + a - b + c = 0)$

$(-2)^3 + a(-2)^2 + b(-2) + c = 0$

$-8 + 4a - 2b + c = 0$ and $-1 \div a - b + c = 0$ ***A1***

attempt to combine their two equations in $-8 + 4a - 2b + c = 0$ to eliminate c ***(M1)***

$b = 3a - 7$ ***A1***

Note: Award at most ***A1M1M1A0M1A0*** for $b = -3a - 7$ from $P(1) = P(2) = 0$

METHOD 2

$P(x) = x^3 + ax^2 + bx + c = (x^2 + 3x + 2)(x + d)$ ***(M1)***

$= x^3 + (3 + d)x^2 + (2 + 3d)x + 2d$ ***(A1)***

attempt to compare coefficients of x^2 and x ***(M1)***

$a = 3 + d$ and $b = 2 + 3d$ ***A1***

attempt to eliminate d ***(M1)***

$\Rightarrow b = 3a - 7$ ***A1***

METHOD 3

attempt to divide $x^3 + ax^2 + bx + c$ by $x^2 + 3x + 2$ ***M1***

$\frac{x^3 + ax^2 + bx + c}{x^2 + 3x + 2} = (x + a - 3) + \frac{(-3a + b + 7)x + (c - 2a + 6)}{x^2 + 3x + 2}$ ***A1A1A1***

Note: Award ***A1*** *for* $x + a - 3$, ***A1*** *for* $(-3a + b + 7)x$ and ***A1*** *for* $c - 2a + 6$

recognition that, if $(x^2 + 3x + 2)$ is a factor of $P(x)$, then $-3a + b + 7 = 0$ ***(M1)***

leading to $b = 3a - 7$ ***A1***

METHOD 4

$x^2+3x+2=(x+1)(x+2)$ ***(A1)***

attempt to use Vieta's formulae for a cubic with roots −1, −2 and "p" ***(M1)***

$(-1)+(-2)+p=-a(\Rightarrow p=3-a)$ ***A1***

$(-1)(-2)+(-1)p+(-2)p=b$ ***A1***

Attempt to eliminate "p" ***(M1)***

$2-(3-a)-2(3-a)=b$

$b=3a-7$ ***A1***

Note: Award at most ***A1M1A0A0M1A0*** for $b=-3a-7$ from roots 1, 2 and "p"

(ii) **METHOD 1**

$a=1,2,5$ lead to invalid values for b ***R1***

$a=3, b=2 \Rightarrow c=0$ so not possible ***R1***

so $a=4, b=5, c=2$ is the only solution ***AG***

METHOD 2

$c=2a-6$ ***R1***

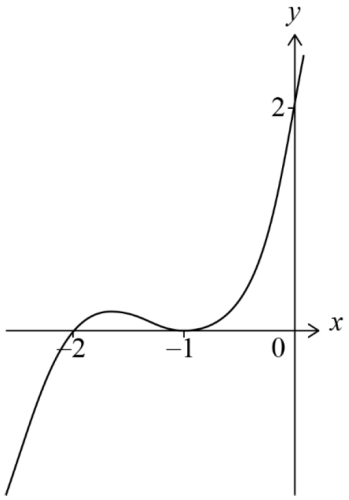
correctly argues $a=4$ is the only possibility ***R1***

so $a=4, b=5, c=2$ is the only solution ***AG***

(iii) $x^3+4x^2+5x+2=(x^2+3x+2)(x+1)$

$=(x+2)(x+1)(x+1)$ ***A1***

(iv)



positive cubic shape with y-intercept at $(0, 2)$ ***A1***

x-intercept at $(-2, 0)$ and local maximum point anywhere between $x = -2$ and $x = -1$ ***A1***

local minimum point at $(-1, 0)$ ***A1***

Note: Accept answers from an approach based on calculus.

[12 marks]

1 - (MAT/11_HL_Summer_2012_Q11) - *Mathematical Induction*

(a) if $n = 1$

$X^1 = U^{-1}A^1U$ which is given, so true for $n = 1$

Assume true for $n = k$

$X^k = U^{-1}A^kU$

Note: Only award ***M1*** if the word "true" or equivalent appears.

if $n = k + 1$

$$\begin{aligned} X^{k+1} &= XX^k \\ &= U^{-1}AUU^{-1}A^kU \\ &= U^{-1}AIA^kU = U^{-1}AA^kU \\ &= U^{-1}A^{k+1}U \end{aligned}$$

As true for $n = 1$, and true for $n = k \Rightarrow$ true for $n = k + 1$, then by the principle of mathematical induction the statement is true for all $n(\in \mathbb{Z}^+)$

(b) (i) **METHOD 1**

$$\boldsymbol{AU} = \boldsymbol{UD} \Rightarrow \boldsymbol{D} = \boldsymbol{U}^{-1}\boldsymbol{AU}$$

$$\boldsymbol{U}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}$$

$$\boldsymbol{D} = \boldsymbol{U}^{-1}\boldsymbol{AU} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

METHOD 2

$$\begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} 3 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3a+c & 3b+d \\ a+c & b+d \end{pmatrix}$$

solving simultaneously

$a = 1,\ c = 0,\ b = 0,\ d = -1$

$$\boldsymbol{D} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(ii) $\boldsymbol{D}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(iii) $\boldsymbol{D}^{2n} = \boldsymbol{U}^{-1}\boldsymbol{A}^{2n}\boldsymbol{U} \Rightarrow \boldsymbol{A}^{2n} = \boldsymbol{UD}^{2n}\boldsymbol{U}^{-1}$

$$= \boldsymbol{UIU}^{-1}$$

$$= \boldsymbol{UU}^{-1} = \boldsymbol{I}$$

(iv) $\boldsymbol{A}^{2n} = \boldsymbol{I} \Rightarrow \boldsymbol{A}^n\boldsymbol{A}^n = \boldsymbol{I}$

$\Rightarrow (\boldsymbol{A}^n)^{-1} = \boldsymbol{A}^n$

2 – (MAT/12_HL_Summer_2012_Q13) – *Differentiation, Mathematical Induction*

(a) let $f(x)=\dfrac{1}{2x+1}$ and using the result $f'(x)=\lim\limits_{h\to 0}\left(\dfrac{f(x+h)-f(x)}{h}\right)$

$$f'(x)=\lim_{h\to 0}\left(\frac{\dfrac{1}{2(x+h)+1}-\dfrac{1}{2x+1}}{h}\right)$$

$$\Rightarrow f'(x)=\lim_{h\to 0}\left(\frac{[2x+1]-[2(x+h)+1]}{h[2(x+h)+1][2x+1]}\right)$$

$$\Rightarrow f'(x)=\lim_{h\to 0}\left(\frac{-2}{[2(x+h)+1][2x+1]}\right)$$

$$\Rightarrow f'(x)=\frac{-2}{(2x+1)^2}$$

(b) let $y=\dfrac{1}{2x+1}$

we want to prove that $\dfrac{\mathrm{d}^n y}{\mathrm{d}x^n}=(-1)^n\dfrac{2^n n!}{(2x+1)^{n+1}}$

let $n=1\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}x}=(-1)^1\dfrac{2^1 1!}{(2x+1)^{1+1}}$

$\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{-2}{(2x+1)^2}$ which is the same result as part (a)

hence the result is true for $n=1$

assume the result is true for $n=k$: $\dfrac{\mathrm{d}^k y}{\mathrm{d}x^k}=(-1)^k\dfrac{2^k k!}{(2x+1)^{k+1}}$

$$\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}}=\frac{\mathrm{d}}{\mathrm{d}x}\left[(-1)^k\frac{2^k k!}{(2x+1)^{k+1}}\right]$$

$$\Rightarrow\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}}=\frac{\mathrm{d}}{\mathrm{d}x}\left[(-1)^k 2^k k!(2x+1)^{-k-1}\right]$$

$$\Rightarrow\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}}=(-1)^k 2^k k!(-k-1)(2x+1)^{-k-2}\times 2$$

$$\Rightarrow\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}}=(-1)^{k+1}2^{k+1}(k+1)!(2x+1)^{-k-2}$$

$$\Rightarrow\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}}=(-1)^{k+1}\frac{2^{k+1}(k+1)!}{(2x+1)^{k+2}}$$

hence if the result is true for $n=k$, it is true for $n=k+1$

since the result is true for $n=1$, the result is proved by mathematical induction

3 – (MAT/10_HL_Winter_2012_Q12) – *Functions - Roots, Mathematical Induction*

(a) $(f \circ f)(x) = f\left(\frac{x}{2-x}\right) = \frac{\frac{x}{2-x}}{2-\frac{x}{2-x}}$

$(f \circ f)(x) = \frac{x}{4-3x}$

(b) $P(n): \underbrace{(f \circ f \circ \ldots \circ f)}_{n \text{ times}}(x) = F_n(x)$

$P(1):\ f(x) = F_1(x)$

$LHS = f(x) = \frac{x}{2-x}$ and $RHS = F_1(x) = \frac{x}{2^1-(2^1-1)x} = \frac{x}{2-x}$

$\therefore P(1)$ true

assume that $P(k)$ is true, *i.e.*, $\underbrace{(f \circ f \circ \ldots \circ f)}_{\text{k times}}(x) = F_k(x)$

consider $P(k+1)$

EITHER

$\underbrace{(f \circ f \circ \ldots \circ f)}_{\text{k+1 times}}(x) = \left(f \circ \underbrace{f \circ f \circ \ldots \circ f}_{\text{k times}}\right)(x) = f\left(F_k(x)\right)$

$= f\left(\frac{x}{2^k-(2^k-1)x}\right) = \frac{\frac{x}{2^k-(2^k-1)x}}{2-\frac{x}{2^k-(2^k-1)x}}$

$= \frac{x}{2\left(2^k-(2^k-1)x\right)-x} = \frac{x}{2^{k+1}-(2^{k+1}-2)x-x}$

OR

$\underbrace{(f \circ f \circ \ldots \circ f)}_{\text{k+1 times}}(x) = \left(f \circ \underbrace{f \circ f \circ \ldots \circ f}_{\text{k times}}\right)(x) = F_k(f(x))$

$= F_k\left(\frac{x}{2-x}\right) = \frac{\frac{x}{2-x}}{2^k-(2^k-1)\frac{x}{2-x}}$

$= \frac{x}{2^{k+1}-2^k x-2^k x+x}$

THEN

$= \frac{x}{2^{k+1}-(2^{k+1}-1)x} = F_{k+1}(x)$

$P(k)$ true implies $P(k+1)$ true, $P(1)$ true so $P(n)$ true for all $n \in \mathbb{Z}^+$

(c) **METHOD 1**

$$x = \frac{y}{2^n - (2^n - 1)y} \Rightarrow 2^n x - (2^n - 1)xy = y$$

$$\Rightarrow 2^n x = \left((2^n - 1)x + 1\right)y \Rightarrow y = \frac{2^n x}{(2^n - 1)x + 1}$$

$$F_n^{-1}(x) = \frac{2^n x}{(2^n - 1)x + 1}$$

$$F_n^{-1}(x) = \frac{x}{\frac{2^n - 1}{2^n}x + \frac{1}{2^n}}$$

$$F_n^{-1}(x) = \frac{x}{(1 - 2^{-n})x + 2^{-n}}$$

$$F_n^{-1}(x) = \frac{x}{2^{-n} - (2^{-n} - 1)x}$$

METHOD 2

attempt $F_{-n}\left(F_n(x)\right)$

$$= F_{-n}\left(\frac{x}{2^n - (2^n - 1)x}\right) = \frac{\frac{x}{2^n - (2^n - 1)x}}{2^{-n} - (2^{-n} - 1)\frac{x}{2^n - (2^n - 1)x}}$$

$$= \frac{x}{2^{-n}(2^n - (2^n - 1)x) - (2^{-n} - 1)x}$$

Note: Award *A1* marks for numerators and denominators.

$$= \frac{x}{1} = x$$

METHOD 3

attempt $F_n\left(F_{-n}(x)\right)$

$$= F_n\left(\frac{x}{2^{-n} - (2^{-n} - 1)x}\right) = \frac{\frac{x}{2^{-n} - (2^{-n} - 1)x}}{2^n - (2^n - 1)\frac{x}{2^{-n} - (2^{-n} - 1)x}}$$

$$= \frac{x}{2^n\left(2^{-n} - (2^{-n} - 1)x\right) - (2^n - 1)x}$$

Note: Award *A1* marks for numerators and denominators.

$$= \frac{x}{1} = x$$

(d) (i) $F_n(0)=0,\ F_n(1)=1$

(ii) **METHOD 1**

$2^n-(2^n-1)x-1=(2^n-1)(1-x)$

>0 if $0<x<1$ and $n\in\mathbb{Z}^+$

so $2^n-(2^n-1)x>1$ and $F_n(x)=\dfrac{x}{2^n-(2^n-1)x}<\dfrac{x}{1}\ (<x)$

$F_n(x)=\dfrac{x}{2^n-(2^n-1)x}<x$ for $0<x<1$ and $n\in\mathbb{Z}^+$

METHOD 2

$\dfrac{x}{2^n-(2^n-1)x}<x \Leftrightarrow 2^n-(2^n-1)x>1$

$\Leftrightarrow (2^n-1)x<2^n-1$

$\Leftrightarrow x<\dfrac{2^n-1}{2^n-1}=1$ true in the interval $]0,1[$

(iii) $B_n=2\left(A_n-\dfrac{1}{2}\right)\ (=2A_n-1)$

4 – (MAT/10_HL_Winter_2013_Q6) – *Mathematical Induction*

$n=1$: $1^3+11=12$

$=3\times 4$ **or** a multiple of 3

assume the proposition is true for $n=k$ (*ie* $k^3+11k=3\text{m}$)

Note: Do not award ***M1*** for statements with "Let $n=k$".

consider $n=k+1$: $(k+1)^3+11(k+1)$

$=k^3+3k^2+3k+1+11k+11$

$=k^3+11k+(3k^2+3k+12)$

$=3(m+k^2+k+4)$

Note: Accept $k^3+11k+3(k^2+k+4)$ or statement that $k^3+11k+(3k^2+3k+12)$ is a multiple of 3.

true for $n=1$, and $n=k$ true $\Rightarrow n=k+1$ true

hence true for all $n\in\mathbb{Z}^+$

5 - (MAT/10_HL_Winter_2014_Q8) - *Mathematical Induction*

let $\mathrm{P}(n)$ be the proposition that $(2n)! \geq 2^n (n!)^2$, $n \in \mathbb{Z}^+$
consider $\mathrm{P}(1)$:
$2! = 2$ and $2^1 (1!)^2 = 2$ so $\mathrm{P}(1)$ is true
assume $\mathrm{P}(k)$ is true *ie* $(2k)! \geq 2^k (k!)^2$, $k \in \mathbb{Z}^+$

Note: Do not award ***M1*** for statements such as "let $n = k$".

consider $\mathrm{P}(k+1)$:

$$\big(2(k+1)\big)! = (2k+2)(2k+1)(2k)!$$
$$\big(2(k+1)\big)! \geq (2k+2)(2k+1)(k!)^2 2^k$$

$$= 2(k+1)(2k+1)(k!)^2 2^k$$
$$> 2^{k+1}(k+1)(k+1)(k!)^2 \text{ since } 2k+1 > k+1$$
$$= 2^{k+1}\big((k+1)!\big)^2$$

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and $\mathrm{P}(1)$ is true, so $\mathrm{P}(n)$ is true for $n \in \mathbb{Z}^+$

6 – (MAT/12_HL_Summer_2015_Q13) – *Mathematical Induction*

(a) $\frac{1}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{\sqrt{n}+\sqrt{n+1}} \times \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}$

$= \frac{\sqrt{n+1}-\sqrt{n}}{(n+1)-n}$

$= \sqrt{n+1}-\sqrt{n}$

(b) $\sqrt{2}-1 = \frac{1}{\sqrt{2}+\sqrt{1}}$

$< \frac{1}{\sqrt{2}}$

(c) consider the case $n=2$: required to prove that $1+\frac{1}{\sqrt{2}} > \sqrt{2}$

from part (b) $\frac{1}{\sqrt{2}} > \sqrt{2}-1$

hence $1+\frac{1}{\sqrt{2}} > \sqrt{2}$ is true for $n=2$

now assume true for $n=k$: $\sum_{r=1}^{r=k} \frac{1}{\sqrt{r}} > \sqrt{k}$

$\frac{1}{\sqrt{1}}+\ldots+\frac{1}{\sqrt{k}} > \sqrt{k}$

attempt to prove true for $n=k+1$: $\frac{1}{\sqrt{1}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} > \sqrt{k+1}$

from assumption, we have that $\frac{1}{\sqrt{1}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} > \sqrt{k}+\frac{1}{\sqrt{k+1}}$

so attempt to show that $\sqrt{k}+\frac{1}{\sqrt{k+1}} > \sqrt{k+1}$

EITHER

$$\frac{1}{\sqrt{k+1}} > \sqrt{k+1} - \sqrt{k}$$

$$\frac{1}{\sqrt{k+1}} > \frac{1}{\sqrt{k}+\sqrt{k+1}}$$, (from part a), which is true

OR

$$\sqrt{k} + \frac{1}{\sqrt{k+1}} = \frac{\sqrt{k+1}\sqrt{k}+1}{\sqrt{k+1}}.$$

$$> \frac{\sqrt{k}\sqrt{k}+1}{\sqrt{k+1}} = \sqrt{k+1}$$

THEN

so true for $n = 2$ and $n = k$ true $\Rightarrow n = k+1$ true. Hence true for all $n \geq 2$

7 - (MAT/10_HL_Winter_2015_Q8) - *Mathematical Induction*

(a) $\sin\left(\theta + \frac{\pi}{2}\right) = \sin\theta \cos\frac{\pi}{2} + \cos\theta \sin\frac{\pi}{2}$

$= \cos\theta$

Note: Accept a transformation/graphical based approach.

(b) consider $n = 1$, $f'(x) = a\cos(ax)$

since $\sin\left(ax + \frac{\pi}{2}\right) = \cos ax$ then the proposition is true for $n = 1$

assume that the proposition is true for $n = k$ so $f^{(k)}(x) = a^k \sin\left(ax + \frac{k\pi}{2}\right)$

$$f^{(k+1)}(x) = \frac{\mathrm{d}\left(f^{(k)}(x)\right)}{\mathrm{d}x} \left(= a\left(a^k \cos\left(ax + \frac{k\pi}{2}\right)\right)\right)$$

$= a^{k+1} \sin\left(ax + \frac{k\pi}{2} + \frac{\pi}{2}\right)$ (using part (a))

$$= a^{k+1} \sin\left(ax + \frac{(k+1)\pi}{2}\right)$$

given that the proposition is true for $n = k$ then we have shown that the proposition is true for $n = k+1$. Since we have shown that the proposition is true for $n = 1$ then the proposition is true for all $n \in \mathbb{Z}^+$

8 – (MAT/12_HL_Summer_2016_Q8) – *Mathematical Induction*

let $\mathrm{P}(n)$ be the proposition that $n\left(n^2+5\right)$ is divisible by 6 for $n\in\mathbb{Z}^+$

consider $\mathrm{P}(1)$:

when $n=1$, $n\left(n^2+5\right)=1\times\left(1^2+5\right)=6$ and so $\mathrm{P}(1)$ is true

assume $\mathrm{P}(k)$ is true *ie*, $k\left(k^2+5\right)=6m$ where $k,\ m\in\mathbb{Z}^+$

Note: Do not award ***M1*** for statements such as "let $n=k$".

consider $\mathrm{P}(k+1)$:

$(k+1)\left((k+1)^2+5\right)$

$=(k+1)\left(k^2+2k+6\right)$

$=k^3+3k^2+8k+6$

$=\left(k^3+5k\right)+\left(3k^2+3k+6\right)$

$=k\left(k^2+5\right)+3k(k+1)+6$

$k(k+1)$ is even hence all three terms are divisible by 6

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and $\mathrm{P}(1)$ is true, so $\mathrm{P}(n)$ is true for $n\in\mathbb{Z}^+$

9 – (MAT/11_HL_Summer_2017_Q8) – *Mathematical Induction*

let $P(n)$ be the proposition that $4^n+15n-1$ is divisible by 9

showing true for $n=1$ ***A1***

ie for $n=1$, $4^1+15\times1-1=18$

which is divisible by 9, therefore $P(1)$ is true

assume $P(k)$ is true so $4^k+15k-1=9A,\ \left(A\in\mathbb{Z}^+\right)$ ***M1***

Note: Only award ***M1*** if "truth assumed" or equivalent.

consider $4^{k+1}+15(k+1)-1$

$=4\times4^k+15k+14$

$=4(9A-15k+1)+15k+14$ ***M1***

$=4\times9A-45k+18$ ***A1***

$=9(4A-5k+2)$ which is divisible by 9 ***R1***

Note: Award ***R1*** for either the expression or the statement above.

since $P(1)$ is true and $P(k)$ true implies $P(k+1)$ is true, therefore (by the principle of mathematical induction) $P(n)$ is true for $n\in\mathbb{Z}^+$ ***R1***

Note: Only award the final ***R1*** if the 2 ***M1***s have been awarded.

[6 marks]

10 – (MAT/12_HL_Summer_2017_Q8) – *Mathematical Induction*

$$\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{n-1}{2}=\binom{n}{3}$$

show true for $n=3$ ***(M1)***

LHS $=\binom{2}{2}=1$ RHS $=\binom{3}{3}=1$ ***A1***

hence true for $n=3$

assume true for $n=k$: $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{k-1}{2}=\binom{k}{3}$ ***M1***

consider for $n=k+1$: $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{k-1}{2}+\binom{k}{2}$ ***(M1)***

$=\binom{k}{3}+\binom{k}{2}$ ***A1***

$=\dfrac{k!}{(k-3)!3!}+\dfrac{k!}{(k-2)!2!}\left(=\dfrac{k!}{3!}\left[\dfrac{1}{(k-3)!}+\dfrac{3}{(k-2)!}\right]\right)$ or any correct expression with a visible common factor ***(A1)***

$=\dfrac{k!}{3!}\left[\dfrac{k-2+3}{(k-2)!}\right]$ or any correct expression with a common denominator ***(A1)***

$=\dfrac{k!}{3!}\left[\dfrac{k+1}{(k-2)!}\right]$

Note: At least one of the above three lines or equivalent must be seen.

$=\dfrac{(k+1)!}{3!(k-2)!}$ or equivalent ***A1***

$=\binom{k+1}{3}$

Result is true for $k=3$. If result is true for k it is true for $k+1$. Hence result is true for all $k\geq 3$. Hence proved by induction. ***R1***

Note: In order to award the ***R1*** at least ***[5 marks]*** must have been awarded.

[9 marks]

11 – (MAT/10_HL_Winter_2017_Q11) – *Mathematical Induction*

(a) even function ***A1***

since $\cos kx = \cos(-kx)$ **and** $f_n(x)$ is a product of even functions ***R1***

OR

even function ***A1***

since $(\cos 2x)(\cos 4x)\ldots = (\cos(-2x))(\cos(-4x))\ldots$ ***R1***

Note: Do not award ***A0R1***.

[2 marks]

(b) consider the case $n = 1$

$\frac{\sin 4x}{2\sin 2x} = \frac{2\sin 2x\cos 2x}{2\sin 2x} = \cos 2x$ ***M1***

hence true for $n = 1$ ***R1***

assume true for $n = k$, *ie*, $(\cos 2x)(\cos 4x)\ldots(\cos 2^k x) = \frac{\sin 2^{k+1}x}{2^k \sin 2x}$ ***M1***

Note: Do not award ***M1*** for "let $n = k$" or "assume $n = k$" or equivalent.

consider $n = k + 1$:

$f_{k+1}(x) = f_k(x)(\cos 2^{k+1}x)$ ***(M1)***

$= \frac{\sin 2^{k+1}x}{2^k \sin 2x}\cos 2^{k+1}x$ ***A1***

$= \frac{2\sin 2^{k+1}x\cos 2^{k+1}x}{2^{k+1}\sin 2x}$ ***A1***

$= \frac{\sin 2^{k+2}x}{2^{k+1}\sin 2x}$ ***A1***

so $n = 1$ true and $n = k$ true $\Rightarrow$ $n = k + 1$ true. Hence true for all $n \in \mathbb{Z}^+$ ***R1***

Note: To obtain the final ***R1***, all the previous ***M*** marks must have been awarded.

[8 marks]

(c) attempt to use $f' = \frac{vu' - uv'}{v^2}$ (or correct product rule) ***M1***

$$f_n'(x) = \frac{\left(2^n \sin 2x\right)\left(2^{n+1} \cos 2^{n+1} x\right) - \left(\sin 2^{n+1} x\right)\left(2^{n+1} \cos 2x\right)}{\left(2^n \sin 2x\right)^2}$$ ***A1A1***

Note: Award ***A1*** for correct numerator and ***A1*** for correct denominator.

[3 marks]

(d) $$f_n'\left(\frac{\pi}{4}\right) = \frac{\left(2^n \sin \frac{\pi}{2}\right)\left(2^{n+1} \cos 2^{n+1} \frac{\pi}{4}\right) - \left(\sin 2^{n+1} \frac{\pi}{4}\right)\left(2^{n+1} \cos \frac{\pi}{2}\right)}{\left(2^n \sin \frac{\pi}{2}\right)^2}$$ ***(M1)(A1)***

$$f_n'\left(\frac{\pi}{4}\right) = \frac{\left(2^n\right)\left(2^{n+1} \cos 2^{n+1} \frac{\pi}{4}\right)}{\left(2^n\right)^2}$$ ***(A1)***

$$= 2\cos 2^{n+1} \frac{\pi}{4} \left(= 2\cos 2^{n-1} \pi\right)$$ ***A1***

$$f_n'\left(\frac{\pi}{4}\right) = 2$$ ***A1***

$$f_n\left(\frac{\pi}{4}\right) = 0$$ ***A1***

Note: This ***A*** mark is independent from the previous marks.

$$y = 2\left(x - \frac{\pi}{4}\right)$$ ***M1A1***

$$4x - 2y - \pi = 0$$ ***AG***

[8 marks]

12 – (MAT/11_HL_Summer_2018_Q6) – *Mathematical Induction*

if $n = 1$

$\text{LHS} = 1;\ \text{RHS} = 4 - \frac{3}{2^0} = 4 - 3 = 1$ ***M1***

hence true for $n = 1$

assume true for $n = k$ ***M1***

Note: Assumption of truth must be present. Following marks are not dependent on the first two ***M1*** marks.

so $1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + k\left(\frac{1}{2}\right)^{k-1} = 4 - \frac{k+2}{2^{k-1}}$

if $n = k+1$

$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + k\left(\frac{1}{2}\right)^{k-1} + (k+1)\left(\frac{1}{2}\right)^k$

$= 4 - \frac{k+2}{2^{k-1}} + (k+1)\left(\frac{1}{2}\right)^k$ ***M1A1***

finding a common denominator for the two fractions ***M1***

$= 4 - \frac{2(k+2)}{2^k} + \frac{k+1}{2^k}$

$= 4 - \frac{2(k+2)-(k+1)}{2^k} = 4 - \frac{k+3}{2^k}\left(= 4 - \frac{(k+1)+2}{2^{(k+1)-1}}\right)$ ***A1***

hence if true for $n = k$ then also true for $n = k+1$, as true for $n = 1$, so true (for all $n \in \mathbb{Z}^+$) ***R1***

Note: Award the final ***R1*** only if the first four marks have been awarded.

[7 marks]

13 - (MAT/10_HL_Winter_2018_Q6) - *Mathematical Induction*

consider $n = 1$. $1(1!) = 1$ and $2! - 1 = 1$ therefore true for $n = 1$ ***R1***

Note: There must be evidence that $n = 1$ has been substituted into both expressions, or an expression such LHS=RHS=1 is used. "therefore true for $n = 1$" or an equivalent statement must be seen.

assume true for $n = k$, (so that $\sum_{r=1}^{k} r(r!) = (k+1)! - 1$) ***M1***

Note: Assumption of truth must be present.

consider $n = k+1$

$\sum_{r=1}^{k+1} r(r!) = \sum_{r=1}^{k} r(r!) + (k+1)(k+1)!$ ***(M1)***

$= (k+1)! - 1 + (k+1)(k+1)!$ ***A1***

$= (k+2)(k+1)! - 1$ ***M1***

Note: ***M1*** is for factorising $(k+1)!$

$= (k+2)! - 1$

$= \left((k+1)+1\right)! - 1$

so if true for $n = k$, then also true for $n = k+1$, and as true for $n = 1$ then true for all $n \left(\in \mathbb{Z}^+\right)$ ***R1***

Note: Only award final ***R1*** if all three method marks have been awarded. Award ***R0*** if the proof is developed from both LHS and RHS.

14 – (MAT/10_HL_Winter_2019_Q6) – *Mathematical Induction*

$f'(x) = e^{2x} + 2xe^{2x}$ ***A1***

Note: This must be obtained from the candidate differentiating $f(x)$.

$= \left(2^1 x + 1 \times 2^{1-1}\right) e^{2x}$ ***A1***

(hence true for $n = 1$)

assume true for $n = k$: ***M1***

$f^{(k)}(x) = \left(2^k x + k2^{k-1}\right) e^{2x}$

Note: Award ***M1*** if truth is assumed. Do not allow "let $n = k$".

consider $n = k + 1$:

$f^{(k+1)}(x) = \frac{d}{dx}\left(\left(2^k x + k2^{k-1}\right) e^{2x}\right)$

attempt to differentiate $f^{(k)}(x)$ ***M1***

$f^{(k+1)}(x) = 2^k e^{2x} + 2\left(2^k x + k2^{k-1}\right) e^{2x}$ ***A1***

$f^{(k+1)}(x) = \left(2^k + 2^{k+1} x + k2^k\right) e^{2x}$

$f^{(k+1)}(x) = \left(2^{k+1} x + (k+1)2^k\right) e^{2x}$ ***A1***

$= \left(2^{k+1} x + (k+1)2^{(k+1)-1}\right) e^{2x}$

True for $n = 1$ and $n = k$ true implies true for $n = k + 1$.

Therefore the statement is true for all $n\left(\in \mathbb{Z}^+\right)$ ***R1***

Note: Do not award final ***R1*** if the two previous ***M1*s** are not awarded.
Allow full marks for candidates who use the base case $n = 0$.

[7 marks]

15 – (MathAA/12_HL_Summer_2021_Q12) – *Trigonometry, Mathematical Induction*

(a) **EITHER**

horizontal stretch/scaling with scale factor $\frac{1}{2}$

Note: Do not allow 'shrink' or 'compression'

followed by a horizontal translation/shift $\frac{1}{2}$ units to the left ***A2***

Note: Do not allow 'move'

OR

horizontal translation/shift 1 unit to the left

followed by horizontal stretch/scaling with scale factor $\frac{1}{2}$ ***A2***

THEN

vertical translation/shift up by $\frac{\pi}{4}$ (or translation through $\begin{pmatrix} 0 \\ \frac{\pi}{4} \end{pmatrix}$) ***A1***

(may be seen anywhere)

[3 marks]

(b) let $\alpha = \arctan p$ and $\beta = \arctan q$ ***M1***

$p = \tan\alpha$ and $q = \tan\beta$ ***(A1)***

$\tan(\alpha+\beta) = \frac{p+q}{1-pq}$ ***A1***

$\alpha + \beta = \arctan\left(\frac{p+q}{1-pq}\right)$ ***A1***

so $\arctan p + \arctan q \equiv \arctan\left(\frac{p+q}{1-pq}\right)$ where $p, q > 0$ and $pq < 1$ ***AG***

[4 marks]

(c) **METHOD 1**

$\frac{\pi}{4} = \arctan 1$ (or equivalent) ***A1***

$\arctan\left(\frac{x}{x+1}\right) + \arctan 1 = \arctan\left(\frac{\frac{x}{x+1}+1}{1-\frac{x}{x+1}(1)}\right)$ ***A1***

$= \arctan\left(\frac{\frac{x+x+1}{x+1}}{\frac{x+1-x}{x+1}}\right)$ ***A1***

$= \arctan(2x+1)$ ***AG***

[3 marks]

METHOD 2

$\tan\frac{\pi}{4} = 1$ (or equivalent) ***A1***

Consider $\arctan(2x+1) - \arctan\left(\frac{x}{x+1}\right) = \frac{\pi}{4}$

$\tan\left(\arctan(2x+1) - \arctan\left(\frac{x}{x+1}\right)\right)$

$= \arctan\left(\frac{2x+1-\frac{x}{x+1}}{1+\frac{x(2x+1)}{x+1}}\right)$ ***A1***

$= \arctan\left(\frac{(2x+1)(x+1)-x}{x+1+x(2x+1)}\right)$ ***A1***

$= \arctan 1$ ***AG***

[3 marks]

METHOD 3

$\tan\left(\arctan\left(2x+1\right)\right)=\tan\left(\arctan\left(\frac{x}{x+1}\right)+\frac{\pi}{4}\right)$

$\tan\frac{\pi}{4}=1$ (or equivalent) ***A1***

LHS = $2x+1$ ***A1***

RHS = $\dfrac{\frac{x}{x+1}+1}{1-\frac{x}{x+1}}\left(=2x+1\right)$ ***A1***

[3 marks]

(d) let $\mathrm{P}(n)$ be the proposition that $\sum_{r=1}^{n}\arctan\left(\frac{1}{2r^2}\right)=\arctan\left(\frac{n}{n+1}\right)$ for $n\in\mathbb{Z}^+$

consider $\mathrm{P}(1)$:

when $n=1$, $\sum_{r=1}^{1}\arctan\left(\frac{1}{2r^2}\right)=\arctan\left(\frac{1}{2}\right)=\mathrm{RHS}$ and so $\mathrm{P}(1)$ is true ***R1***

assume $\mathrm{P}(k)$ is true, ie. $\sum_{r=1}^{k}\arctan\left(\frac{1}{2r^2}\right)=\arctan\left(\frac{k}{k+1}\right)$ $\left(k\in\mathbb{Z}^+\right)$ ***M1***

Note: Award ***M0*** for statements such as "let $n=k$".
Note: Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

consider $\mathrm{P}(k+1)$:

$$\sum_{r=1}^{k+1}\arctan\left(\frac{1}{2r^2}\right)=\sum_{r=1}^{k}\arctan\left(\frac{1}{2r^2}\right)+\arctan\left(\frac{1}{2(k+1)^2}\right)$$ ***(M1)***

$$=\arctan\left(\frac{k}{k+1}\right)+\arctan\left(\frac{1}{2(k+1)^2}\right)$$ ***A1***

$$=\arctan\left(\frac{\frac{k}{k+1}+\frac{1}{2(k+1)^2}}{1-\left(\frac{k}{k+1}\right)\left(\frac{1}{2(k+1)^2}\right)}\right)$$ ***M1***

$$=\arctan\left(\frac{(k+1)(2k^2+2k+1)}{2(k+1)^3-k}\right)$$ ***A1***

Note: Award ***A1*** for correct numerator, with $(k+1)$ factored. Denominator does not need to be simplified

$$=\arctan\left(\frac{(k+1)(2k^2+2k+1)}{2k^3+6k^2+5k+2}\right)$$ ***A1***

Note: Award ***A1*** for denominator correctly expanded. Numerator does not need to be simplified. These two ***A*** marks may be awarded in any order

$$=\arctan\left(\frac{(k+1)(2k^2+2k+1)}{(k+2)(2k^2+2k+1)}\right)=\arctan\left(\frac{k+1}{k+2}\right)$$ ***A1***

Note: The word 'arctan' must be present to be able to award the last three A marks

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and $\mathrm{P}(1)$ is true, so

$\mathrm{P}(n)$ is true for $n \in \mathbb{Z}^+$ ***R1***

Note: Award the final ***R1*** mark provided at least four of the previous marks have been awarded.

Note: To award the final ***R1***, the truth of $\mathrm{P}(k)$ must be mentioned. '$\mathrm{P}(k)$ implies $\mathrm{P}(k+1)$' is insufficient to award the mark.

[9 marks]

16 – (MathAA/11_HL_Summer_2021_Q12) – *Differentiation, Mathematical Induction*

(a) attempt to use the chain rule ***M1***

$f'(x)=\frac{1}{2}(1+x)^{-\frac{1}{2}}$ ***A1***

$f''(x)=-\frac{1}{4}(1+x)^{-\frac{3}{2}}$ ***A1***

$=-\frac{1}{4\sqrt{(1+x)^3}}$ ***AG***

Note: Award ***M1A0A0*** for $f'(x)=\frac{1}{\sqrt{1+x}}$ or equivalent seen

[3 marks]

(b) let $n=2$

$f''(x)=\left(-\frac{1}{4\sqrt{(1+x)^3}}=\right)\left(-\frac{1}{4}\right)^1\frac{1!}{0!}(1+x)^{\frac{1}{2}-2}$ ***R1***

Note: Award ***R0*** for not starting at $n=2$. Award subsequent marks as appropriate.

assume true for $n=k$, (so $f^{(k)}(x)=\left(-\frac{1}{4}\right)^{k-1}\frac{(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k}$) ***M1***

Note: Do not award ***M1*** for statements such as "let $n=k$" or "$n=k$ is true". Subsequent marks can still be awarded.

consider $n=k+1$

$\text{LHS}=f^{(k+1)}(x)=\frac{d\left(f^{(k)}(x)\right)}{dx}$ ***M1***

$=\left(-\frac{1}{4}\right)^{k-1}\frac{(2k-3)!}{(k-2)!}\left(\frac{1}{2}-k\right)(1+x)^{\frac{1}{2}-k-1}$ (or equivalent) ***A1***

EITHER

$\text{RHS}=f^{(k+1)}(x)=\left(-\frac{1}{4}\right)^{k}\frac{(2k-1)!}{(k-1)!}(1+x)^{\frac{1}{2}-k-1}$ (or equivalent) ***A1***

$=\left(-\frac{1}{4}\right)^{k}\frac{(2k-1)(2k-2)(2k-3)!}{(k-1)(k-2)!}(1+x)^{\frac{1}{2}-k-1}$ ***A1***

$$=\left(-\frac{1}{4}\right)^{(k+1)-1}\frac{\left(2(k+1)-3\right)!}{((k+1)-2)!}(1+x)^{\frac{1}{2}-(k+1)}=\text{RHS}$$

THEN
since true for $n=2$, and true for $n=k+1$ if true for $n=k$, the statement is true for all $n\in\mathbb{Z}, n\geq 2$ by mathematical induction ***R1***

Note: To obtain the final ***R1,*** at least four of the previous marks must have been awarded.

[9 marks]

(c) **METHOD 1**

$h(x)=\sqrt{1+x}\,\mathrm{e}^{mx}$

using product rule to find $h'(x)$ ***(M1)***

$h'(x)=\sqrt{1+x}\,m\mathrm{e}^{mx}+\frac{1}{2\sqrt{1+x}}\mathrm{e}^{mx}$ ***A1***

$h''(x)=m\left(\sqrt{1+x}\,m\mathrm{e}^{mx}+\frac{1}{2\sqrt{1+x}}\mathrm{e}^{mx}\right)+\frac{1}{2\sqrt{1+x}}m\mathrm{e}^{mx}-\frac{1}{4\sqrt{(1+x)^3}}\mathrm{e}^{mx}$ ***A1***

substituting $x=0$ into $h''(x)$ ***M1***

$h''(0)=m^2+\frac{1}{2}m+\frac{1}{2}m-\frac{1}{4}\left(=m^2+m-\frac{1}{4}\right)$ ***A1***

$h(x)=h(0)+xh'(0)+\frac{x^2}{2!}h''(0)+\ldots$

equating x^2 coefficient to $\frac{7}{4}$ ***M1***

$\frac{h''(0)}{2!}=\frac{7}{4}\quad\left(\Rightarrow h''(0)=\frac{7}{2}\right)$

$4m^2+4m-15=0$ ***A1***

$(2m+5)(2m-3)=0$

$m=-\frac{5}{2}$ or $m=\frac{3}{2}$ ***A1***

[8 marks]

METHOD 2

EITHER

attempt to find $f(0)$, $f'(0)$, $f''(0)$ ***(M1)***

$f(x)=(1+x)^{\frac{1}{2}}$ $\quad f(0)=1$

$f'(x)=\frac{1}{2}(1+x)^{-\frac{1}{2}}$ $\quad f'(0)=\frac{1}{2}$

$f''(x)=-\frac{1}{4}(1+x)^{-\frac{3}{2}}$ $\quad f''(0)=-\frac{1}{4}$

$f(x)=1+\frac{1}{2}x-\frac{1}{8}x^2+\ldots$ ***A1***

OR

attempt to apply binomial theorem for rational exponents ***(M1)***

$$f(x)=(1+x)^{\frac{1}{2}}=1+\frac{1}{2}x+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2+\ldots$$

$f(x)=1+\frac{1}{2}x-\frac{1}{8}x^2+\ldots$ ***A1***

THEN

$g(x)=1+mx+\frac{m^2}{2}x^2+\ldots$ ***(A1)***

$h(x)=\left(1+\frac{1}{2}x-\frac{1}{8}x^2+\ldots\right)\left(1+mx+\frac{m^2}{2}x^2+\ldots\right)$ ***(M1)***

coefficient of x^2 is $\frac{m^2}{2}+\frac{m}{2}-\frac{1}{8}$ ***A1***

attempt to set equal to $\frac{7}{4}$ and solve ***M1***

$\frac{m^2}{2}+\frac{m}{2}-\frac{1}{8}=\frac{7}{4}$

$4m^2+4m-15=0$ ***A1***

$(2m+5)(2m-3)=0$

$m=-\frac{5}{2}$ or $m=\frac{3}{2}$ ***A1***

[8 marks]

METHOD 3

$g'(x) = me^{mx}$ and $g''(x) = m^2e^{mx}$ ***(A1)***

$h(x) = h(0) + xh'(0) + \frac{x^2}{2!}h''(0) + ...$

equating x^2 coefficient to $\frac{7}{4}$ ***M1***

$\frac{h''(0)}{2!} = \frac{7}{4}$ $\left(\Rightarrow h''(0) = \frac{7}{2}\right)$

using product rule to find $h'(x)$ and $h''(x)$ ***(M1)***

$h'(x) = f(x)g'(x) + f'(x)g(x)$

$h''(x) = f(x)g''(x) + 2f'(x)g'(x) + f''(x)g(x)$ ***A1***

substituting $x = 0$ into $h''(x)$ ***M1***

$h''(0) = f(0)g''(0) + 2g'(0)f'(0) + g(0)f''(0)$

$= 1 \times m^2 + 2m \times \frac{1}{2} + 1 \times \left(-\frac{1}{4}\right)$ $\left(= m^2 + m - \frac{1}{4}\right)$ ***A1***

$4m^2 + 4m - 15 = 0$ ***A1***

$(2m+5)(2m-3) = 0$

$m = -\frac{5}{2}$ or $m = \frac{3}{2}$ ***A1***

[8 marks]

17 – (MathAA/10_HL_Winter_2021_Q11) – *Mathematical Induction*

(a) For $n=1$

LHS: $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\mathrm{e}^x\right)=x^2\mathrm{e}^x+2x\mathrm{e}^x\left(=\mathrm{e}^x\left(x^2+2x\right)\right)$ ***A1***

RHS: $\left(x^2+2(1)x+1(1-1)\right)\mathrm{e}^x\left(=\mathrm{e}^x\left(x^2+2x\right)\right)$ ***A1***

so true for $n=1$

now assume true for $n=k$; i.e. $\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(x^2\mathrm{e}^x\right)=\left[x^2+2kx+k(k-1)\right]\mathrm{e}^x$ ***M1***

Note: Do not award ***M1*** for statements such as "let $n=k$". Subsequent marks can still be awarded.

attempt to differentiate the RHS ***M1***

$\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}\left(x^2\mathrm{e}^x\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\left[x^2+2kx+k(k-1)\right]\mathrm{e}^x\right)$

$=(2x+2k)\mathrm{e}^x+\left(x^2+2kx+k(k-1)\right)\mathrm{e}^x$ ***A1***

$=\left[x^2+2(k+1)x+k(k+1)\right]\mathrm{e}^x$ ***A1***

so true for $n=k$ implies true for $n=k+1$

therefore $n=1$ true and $n=k$ true $\Rightarrow$ $n=k+1$ true

therefore, true for all $n\in\mathbb{Z}^+$ ***R1***

Note: Award **R1** only if three of the previous four marks have been awarded

[7 marks]

(b) **METHOD 1**

attempt to use $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2\mathrm{e}^x\right)=\left[x^2+2nx+n(n-1)\right]\mathrm{e}^x$ ***(M1)***

Note: For $x=0$, $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2\mathrm{e}^x\right)_{|x=0}=n(n-1)$ may be seen.

$f(0)=0, f'(0)=0, f''(0)=2, f'''(0)=6, f^{(4)}(0)=12$

use of $f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{(4)}(0)+\ldots$ ***(M1)***

$\Rightarrow f(x)\approx x^2+x^3+\frac{1}{2}x^4$ ***A1***

[3 marks]

METHOD 2

'$x^2\times$ Maclaurin series of e^x' ***(M1)***

$x^2\left(1+x+\frac{x^2}{2!}+\ldots\right)$ ***(A1)***

$\Rightarrow f(x)\approx x^2+x^3+\frac{1}{2}x^4$ ***A1***

[3 marks]

(c) **METHOD 1**

attempt to substitute $x^2e^x \approx x^2 + x^3 + \frac{1}{2}x^4$ into $\frac{\left(x^2e^x - x^2\right)^3}{x^9}$ ***M1***

$$\frac{\left(x^2e^x - x^2\right)^3}{x^9} \approx \frac{\left(x^2 + x^3 + \frac{1}{2}x^4(+\ldots) - x^2\right)^3}{x^9}$$ ***(A1)***

EITHER

$$= \frac{\left(x^3 + \frac{1}{2}x^4 + \ldots\right)^3}{x^9}$$ ***A1***

$$= \frac{x^9(+\text{higher order terms})}{x^9}$$

OR

$$\left(\frac{x^3 + \frac{1}{2}x^4(+\ldots)}{x^3}\right)^3$$ ***A1***

$$\left(1 + \frac{1}{2}x(+\ldots)\right)^3$$

THEN

$= 1$ (+ higher order terms)

So $\lim_{x \to 0}\left[\frac{\left(x^2e^x - x^2\right)^3}{x^9}\right] = 1$ ***A1***

[4 marks]

METHOD 2

$$\lim_{x\to 0}\left[\frac{\left(x^2e^x - x^2\right)^3}{x^9}\right] = \lim_{x\to 0}\left(\frac{x^2e^x - x^2}{x^3}\right)^3$$ ***M1***

$$= \lim_{x\to 0}\left(\frac{e^x - 1}{x}\right)^3$$ ***(A1)***

attempt to use L'Hôpital's rule ***M1***

$$= \lim_{x\to 0}\left(\frac{e^x - 0}{1}\right)^3$$

$$= \left[\lim_{x\to 0} e^x\right]^3$$

$$= 1$$ ***A1***

[4 marks]

Total [14 marks]

1 – (MAT/11_HL_Summer_2015_Q1) – *Radian*

(a) **METHOD 1**

area $= \pi 2^2 - \frac{1}{2} 2^2 \theta (= 3\pi)$

Note: Award ***M1*** for using area formula.

$\Rightarrow 2\theta = \pi \Rightarrow \theta = \frac{\pi}{2}$

Note: Degrees loses final A1

METHOD 2

let $x = 2\pi - \theta$

area $= \frac{1}{2} 2^2 x (= 3\pi)$

$\Rightarrow x = \frac{3}{2}\pi$

$\Rightarrow \theta = \frac{\pi}{2}$

METHOD 3

Area of circle is 4π

Shaded area is $\frac{3}{4}$ of the circle

$\Rightarrow \theta = \frac{\pi}{2}$

(b) arc length $= 2\frac{3\pi}{2}$

perimeter $= 2\frac{3\pi}{2} + 2 \times 2$

$= 3\pi + 4$

2 – (MAT/10_HL_Winter_2015_Q1) – *Radian*

$$\text{arc length } = \frac{2}{x} = rx \left(\Rightarrow r = \frac{2}{x^2} \right)$$

$$16 = \frac{1}{2}\left(\frac{2}{x^2}\right)^2 x \left(\Rightarrow \frac{2}{x^3} = 16 \right)$$

Note: Award M1s for attempts at the use of arc-length and sector-area formulae.

$$x = \frac{1}{2}$$

$$\text{arc length } = 4(\text{cm})$$

3 – (MAT/11_HL_Summer_2019_Q3) – *Radian*

$A = P$

use of the correct formula for area and arc length ***(M1)***

perimeter is $r\theta + 2r$ ***(A1)***

Note: ***A1*** independent of previous ***M1***.

$\frac{1}{2}r^2(1) = r(1) + 2r$ ***A1***

$r^2 - 6r = 0$

$r = 6$ (as $r > 0$) ***A1***

Note: Do not award final ***A1*** if $r = 0$ is included.

1 - (MAT/11_HL_Summer_2012_Q5) - *Trigonometry*

(a) $\cos x = 0,\ \sin x = 0$

$$x = \frac{n\pi}{2}, n \in \mathbb{Z}$$

(b) **EITHER**

$$\frac{\sin 3x \cos x - \cos 3x \sin x}{\sin x \cos x}$$

$$= \frac{\sin(3x - x)}{\frac{1}{2}\sin 2x}$$

$$= 2$$

OR

$$\frac{\sin 2x \cos x + \cos 2x \sin x}{\sin x} - \frac{\cos 2x \cos x - \sin 2x \sin x}{\cos x}$$

$$= \frac{2\sin x \cos^2 x + 2\cos^2 x \sin x - \sin x}{\sin x} - \frac{2\cos^3 x - \cos x - 2\sin^2 x \cos x}{\cos x}$$

$$= 4\cos^2 x - 1 - 2\cos^2 x + 1 + 2\sin^2 x$$

$$= 2\cos^2 x + 2\sin^2 x$$

$$= 2$$

2 – (MAT/12_HL_Summer_2012_Q9) – *Trigonometry*

METHOD 1

$$\frac{\cos A+\sin A}{\cos A-\sin A}=\sec 2A+\tan 2A$$

consider right hand side

$$\sec 2A+\tan 2A=\frac{1}{\cos 2A}+\frac{\sin 2A}{\cos 2A}$$

$$=\frac{\cos^2 A+2\sin A\cos A+\sin^2 A}{\cos^2 A-\sin^2 A}$$

Note: Award ***A1*** for recognizing the need for single angles and ***A1*** for recognizing $\cos^2 A+\sin^2 A=1$.

$$=\frac{(\cos A+\sin A)^2}{(\cos A+\sin A)(\cos A-\sin A)}$$

$$=\frac{\cos A+\sin A}{\cos A-\sin A}$$

METHOD 2

$$\frac{\cos A+\sin A}{\cos A-\sin A}=\frac{(\cos A+\sin A)^2}{(\cos A-\sin A)(\cos A+\sin A)}$$

$$=\frac{\cos^2 A+2\sin A\cos A+\sin^2 A}{\cos^2 A-\sin^2 A}$$

Note: Award ***A1*** for correct numerator and ***A1*** for correct denominator.

$$=\frac{1+\sin 2A}{\cos 2A}$$

$$=\sec 2A+\tan 2A$$

3 – (MAT/11_HL_Summer_2012_Q10) – *Trigonometry*

(a) $\cos\hat{A} = \dfrac{\text{BA}}{\sqrt{2}}$

$\sin\hat{A} = \dfrac{\text{BC}}{\sqrt{2}}$

$\cos\hat{A} - \sin\hat{A} = \dfrac{\text{BA} - \text{BC}}{\sqrt{2}}$

$= \dfrac{1}{\sqrt{2}}$

(b) $\cos^2\hat{A} - 2\cos\hat{A}\sin\hat{A} + \sin^2\hat{A} = \dfrac{1}{2}$

$1 - 2\sin\hat{A}\cos\hat{A} = \dfrac{1}{2}$

$\sin 2\hat{A} = \dfrac{1}{2}$

$2\hat{A} = 30^\circ$

angles in the triangle are 15° and 75°

Note: Accept answers in radians.

(c) $\text{BC}^2 + (\text{BC}+1)^2 = 2$

$2\text{BC}^2 + 2\text{BC} - 1 = 0$

$\text{BC} = \dfrac{-2+\sqrt{12}}{4}\left(= \dfrac{\sqrt{3}-1}{2}\right)$

$\sin\hat{A} = \dfrac{\text{BC}}{\sqrt{2}} = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

$= \dfrac{\sqrt{6}-\sqrt{2}}{4}$

(d) **EITHER**

$h = \text{AB}\sin \hat{A}$

$= (\text{BC}+1)\sin \hat{A}$

$= \frac{\sqrt{3}+1}{2} \times \frac{\sqrt{6}-\sqrt{2}}{4} = \frac{\sqrt{2}}{4}$

OR

$\frac{1}{2}AB.BC = \frac{1}{2}AC.h$

$\frac{\sqrt{3}-1}{2} \cdot \frac{\sqrt{3}+1}{2} = \sqrt{2}h$

$\frac{2}{4} = \sqrt{2}h$

$h = \frac{1}{2\sqrt{2}}$

4 – (MAT/10_HL_Winter_2012_Q1) – *Trigonometry*

$\sin\alpha = \sqrt{1-\left(-\frac{3}{4}\right)^2} = \frac{\sqrt{7}}{4}$

attempt to use double angle formula

$\sin 2\alpha = 2\frac{\sqrt{7}}{4}\left(-\frac{3}{4}\right) = -\frac{3\sqrt{7}}{8}$

5 – (MAT/10_HL_Winter_2012_Q7) – *Trigonometry*

(a) attempt to apply cosine rule

$4^2 = 6^2 + \text{QR}^2 - 2\cdot\text{QR}\cdot 6\cos 30^\circ$ $\left(\textbf{or } \text{QR}^2 - 6\sqrt{3}\ \text{QR} + 20 = 0\right)$

$\text{QR} = 3\sqrt{3}+\sqrt{7}$ or $\text{QR} = 3\sqrt{3}-\sqrt{7}$

(b) **METHOD 1**

$k \geq 6$

$k = 6\sin 30^\circ = 3$

Note: The ***M1*** in (b) is for recognizing the right-angled triangle case.

METHOD 2

$k \geq 6$

use of discriminant: $108 - 4\left(36 - k^2\right) = 0$

$k = 3$

6 – (MAT/12_HL_Summer_2013_Q10) – *Trigonometry*

(a) attempt at use of $\tan(A+B)=\dfrac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$

$$\frac{1}{p}=\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}}\quad\left(=\frac{1}{3}\right)$$

$p=3$

Note: the value of p needs to be stated for the final mark.

(b) $\tan\left(\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{1}{5}\right)+\arctan\left(\frac{1}{8}\right)\right)=\dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}=1$

$$\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{1}{5}\right)+\arctan\left(\frac{1}{8}\right)=\frac{\pi}{4}$$

7 – (MAT/11_HL_Summer_2013_Q11) – *Trigonometry*

(a) (i) $\cos\left(\frac{\pi}{6}\right)\cos x - \sin\left(\frac{\pi}{6}\right)\sin x$

$\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x$

(ii) $\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x = \frac{1}{2}$

$\cos\left(\frac{\pi}{6} + x\right) = \frac{1}{2}$

$\frac{\pi}{6} + x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$

$x = \frac{\pi}{6}$ or $\frac{3\pi}{2}$

(b) (i) **METHOD 1**

substitute $x = 1$

$p(1) = 0$

hence $x = 1$ is a zero

METHOD 2

Correct result when dividing by $x - 1$

Statement that remainder is zero

hence $x = 1$ is a zero

(ii) $x = 1$ is a root

valid method, for example, dividing or comparing coefficients (may be seen in (b)(i))

other roots are -1 and $\frac{1}{2}$

(iii) $2\sin\theta\cos\theta\cos\theta + \sin^2\theta$

$2\sin\theta\left(1 - \sin^2\theta\right) + \sin^2\theta$

$\left(2\sin\theta - 2\sin^3\theta + \sin^2\theta\right)$

(iv) $2\sin\theta - 2\sin^3\theta + \sin^2\theta = 1$

$\sin\theta = -1, \frac{1}{2}, 1$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{2}$

8 – (MAT/10_HL_Winter_2013_Q8) – *Trigonometry*

(a) $\sin(x+y)\sin(x-y)$

$$= (\sin x\cos y + \cos x\sin y)(\sin x\cos y - \cos x\sin y)$$
$$= \sin^2 x\cos^2 y + \sin x\sin y\cos x\cos y - \sin x\sin y\cos x\cos y - \cos^2 x\sin^2 y$$
$$= \sin^2 x\cos^2 y - \cos^2 x\sin^2 y$$
$$= \sin^2 x\left(1-\sin^2 y\right) - \sin^2 y\left(1-\sin^2 x\right)$$
$$= \sin^2 x - \sin^2 x\sin^2 y - \sin^2 y + \sin^2 x\sin^2 y$$
$$= \sin^2 x - \sin^2 y$$

(b) $f(x) = \sin^2 x - \frac{1}{4}$

range is $f \in \left[-\frac{1}{4}, \frac{3}{4}\right]$

Note: Award ***A1*** for each end point. Condone incorrect brackets.

(c) $g(x) = \dfrac{1}{\sin^2 x - \frac{1}{4}}$

range is $g \in]-\infty, -4] \cup \left[\frac{4}{3}, \infty\right[$

9 – (MAT/10_HL_Winter_2013_Q12) – *Complex Numbers, Trigonometry, Integration*

(a) $z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$
$= \cos n\theta + \cos n\theta + i\sin n\theta - i\sin n\theta$
$= 2\cos n\theta$

(b) $\left(z + z^{-1}\right)^4 = z^4 + 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z^2}\right) + 4z\left(\frac{1}{z^3}\right) + \frac{1}{z^4}$

Note: Accept $\left(z + z^{-1}\right)^4 = 16\cos^4\theta$.

(c) **METHOD 1**

$\left(z + z^{-1}\right)^4 = \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$

$(2\cos\theta)^4 = 2\cos 4\theta + 8\cos 2\theta + 6$

Note: Award ***A1*** for RHS, ***A1*** for LHS independent of the ***M1***.

$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

$\left(\text{or } p = \frac{1}{8},\ q = \frac{1}{2},\ r = \frac{3}{8}\right)$

METHOD 2

$\cos^4\theta = \left(\frac{\cos 2\theta + 1}{2}\right)^2$

$= \frac{1}{4}(\cos^2 2\theta + 2\cos 2\theta + 1)$

$= \frac{1}{4}\left(\frac{\cos 4\theta + 1}{2} + 2\cos 2\theta + 1\right)$

$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

$\left(\text{or } p = \frac{1}{8},\ q = \frac{1}{2},\ r = \frac{3}{8}\right)$

(d) $\left(z+z^{-1}\right)^6 = z^6 + 6z^5\left(\frac{1}{z}\right) + 15z^4\left(\frac{1}{z^2}\right) + 20z^3\left(\frac{1}{z^3}\right) + 15z^2\left(\frac{1}{z^4}\right) + 6z\left(\frac{1}{z^5}\right) + \frac{1}{z^6}$

$\left(z+z^{-1}\right)^6 = \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$

$(2\cos\theta)^6 = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$

Note: Award ***A1*** for RHS, ***A1*** for LHS, independent of the ***M1***.

$\cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$

Note: Accept a purely trigonometric solution as for (c).

(e) $\int_0^{\frac{\pi}{2}} \cos^6\theta \, d\theta = \int_0^{\frac{\pi}{2}} \left(\frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}\right) d\theta$

$= \left[\frac{1}{192}\sin 6\theta + \frac{3}{64}\sin 4\theta + \frac{15}{64}\sin 2\theta + \frac{5}{16}\theta\right]_0^{\frac{\pi}{2}}$

$= \frac{5\pi}{32}$

(f) $V = \pi\int_0^{\frac{\pi}{2}} \sin^2 x \cos^4 x \, dx$

$= \pi\int_0^{\frac{\pi}{2}} \cos^4 x \, dx - \pi\int_0^{\frac{\pi}{2}} \cos^6 x \, dx$

$\int_0^{\frac{\pi}{2}} \cos^4 x \, dx = \frac{3\pi}{16}$

$V = \frac{3\pi^2}{16} - \frac{5\pi^2}{32} = \frac{\pi^2}{32}$

(g) (i) constant term $= \binom{2k}{k} = \frac{(2k)!}{k!k!} = \frac{(2k)!}{(k!)^2}$ (accept C_k^{2k})

(ii) $2^{2k}\int_0^{\frac{\pi}{2}} \cos^{2k}\theta \mathrm{d}\theta = \frac{(2k)!}{(k!)^2}\frac{\pi}{2}$

$\int_0^{\frac{\pi}{2}} \cos^{2k}\theta \mathrm{d}\theta = \frac{(2k)!\pi}{2^{2k+1}(k!)^2}$ $\left(\text{or } \frac{\binom{2k}{k}\pi}{2^{2k+1}}\right)$

10 – (MAT/11_HL_Summer_2014_Q5) – *Trigonometry*

(a) $\cos x = 2\cos^2\frac{1}{2}x - 1$

$\cos\frac{1}{2}x = \pm\sqrt{\frac{1+\cos x}{2}}$

positive as $0 \le x \le \pi$

$\cos\frac{1}{2}x = \sqrt{\frac{1+\cos x}{2}}$

(b) $\cos 2\theta = 1 - 2\sin^2\theta$

$\sin\frac{1}{2}x = \sqrt{\frac{1-\cos x}{2}}$

(c) $\sqrt{2}\int_0^{\frac{\pi}{2}} \cos\frac{1}{2}x + \sin\frac{1}{2}x \mathrm{d}x$

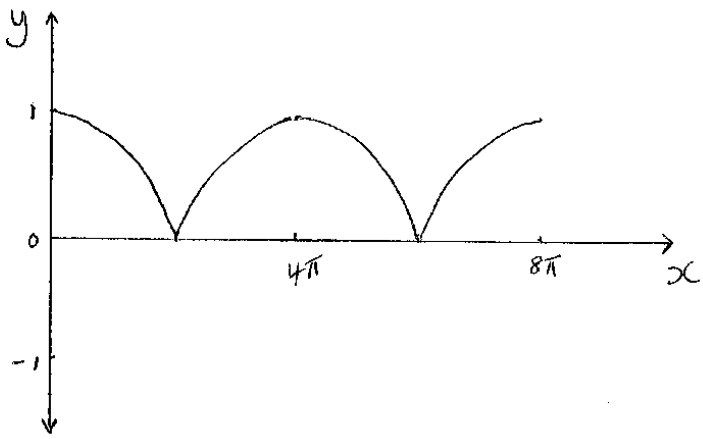
$= \sqrt{2}\left[2\sin\frac{1}{2}x - 2\cos\frac{1}{2}x\right]_0^{\frac{\pi}{2}}$

$= \sqrt{2}(0) - \sqrt{2}(0-2)$

$= 2\sqrt{2}$

11 – (MAT/12_HL_Summer_2014_Q5) – *Graphs, Trigonometry*

(a)



(b) $\left|\cos\left(\frac{x}{4}\right)\right| = \frac{1}{2}$

$x = \frac{4\pi}{3}$

attempting to find any other solutions

Note: Award ***(M1)*** if at least one of the other solutions is correct (in radians or degrees) or clear use of symmetry is seen.

$x = 8\pi - \frac{4\pi}{3} = \frac{20\pi}{3}$

$x = 4\pi - \frac{4\pi}{3} = \frac{8\pi}{3}$

$x = 4\pi + \frac{4\pi}{3} = \frac{16\pi}{3}$

12 - (MAT/11_HL_Summer_2014_Q7) - *Trigonometry*

METHOD 1

$AD^2 = 2^2 + 3^2 - 2\times2\times3\times\cos 60^\circ$

$\left(\text{or } AD^2 = 1^2 + 3^2 - 2\times1\times3\times\cos 60^\circ\right)$

Note: ***M1*** for use of cosine rule with 60° angle.

$AD^2 = 7$

$\cos D\hat{A}C = \dfrac{9+7-4}{2\times3\times\sqrt{7}}$

Note: ***M1*** for use of cosine rule involving $D\hat{A}C$.

$= \dfrac{2}{\sqrt{7}}$

METHOD 2

let point E be the foot of the perpendicular from D to AC

$EC = 1$ (by similar triangles, or triangle properties)

(or $AE = 2$)

$DE = \sqrt{3}$ and $AD = \sqrt{7}$ (by Pythagoras)

$\cos D\hat{A}C = \dfrac{2}{\sqrt{7}}$

13 – (MAT/12_HL_Summer_2014_Q9) – *Sequences - Series, Trigonometry*

(a) $\sin x$, $\sin 2x$ and $4\sin x\cos^2 x$

$$r = \frac{2\sin x\cos x}{\sin x} = 2\cos x$$

Note: Accept $\frac{\sin 2x}{\sin x}$.

(b) **EITHER**

$$|r| < 1 \Rightarrow |2\cos x| < 1$$

OR

$$-1 < r < 1 \Rightarrow -1 < 2\cos x < 1$$

THEN

$$0 < \cos x < \frac{1}{2} \text{ for } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$-\frac{\pi}{2} < x < -\frac{\pi}{3} \text{ or } \frac{\pi}{3} < x < \frac{\pi}{2}$$

(c) $S_\infty = \frac{\sin x}{1 - 2\cos x}$

$$S_\infty = \frac{\sin\left(\arccos\left(\frac{1}{4}\right)\right)}{1 - 2\cos\left(\arccos\left(\frac{1}{4}\right)\right)}$$

$$= \frac{\frac{\sqrt{15}}{4}}{\frac{1}{2}}$$

Note: Award *A1* for correct numerator and *A1* for correct denominator.

$$= \frac{\sqrt{15}}{2}$$

14 - (MAT/11_HL_Summer_2014_Q10) - *Trigonometry*

$\sin^2 x + \cos^2 x + 2\sin x\cos x = \frac{4}{9}$

using $\sin^2 x + \cos^2 x = 1$

$2\sin x\cos x = -\frac{5}{9}$

using $2\sin x\cos x = \sin 2x$

$\sin 2x = -\frac{5}{9}$

$\cos 4x = 1 - 2\sin^2 2x$

Note: Award this ***M1*** for decomposition of $\cos 4x$ using double angle formula anywhere in the solution.

$= 1 - 2\times\frac{25}{81}$

$= \frac{31}{81}$

15 – (MAT/10_HL_Winter_2014_Q13) – *Complex Numbers, Trigonometry, Integration*

(a) (i) METHOD 1

$$(1+\mathrm{i}\tan\theta)^n+(1-\mathrm{i}\tan\theta)^n=(1+\mathrm{i}\frac{\sin\theta}{\cos\theta})^n+(1-\mathrm{i}\frac{\sin\theta}{\cos\theta})^n$$

$$=\left(\frac{\cos\theta+i\sin\theta}{\cos\theta}\right)^n+\left(\frac{\cos\theta-i\sin\theta}{\cos\theta}\right)^n$$

by de Moivre's theorem

$$\left(\frac{\cos\theta+i\sin\theta}{\cos\theta}\right)^n=\frac{\cos n\theta+i\sin n\theta}{\cos^n\theta}$$

recognition that $\cos\theta-i\sin\theta$ is the complex conjugate of $\cos\theta+i\sin\theta$

use of the fact that the operation of complex conjugation commutes with the operation of raising to an integer power:

$$\left(\frac{\cos\theta-i\sin\theta}{\cos\theta}\right)^n=\frac{\cos n\theta-i\sin n\theta}{\cos^n\theta}$$

$$(1+\mathrm{i}\tan\theta)^n+(1-\mathrm{i}\tan\theta)^n=\frac{2\cos n\theta}{\cos^n\theta}$$

METHOD 2

$$(1+\mathrm{i}\tan\theta)^n+(1-\mathrm{i}\tan\theta)^n=(1+\mathrm{i}\tan\theta)^n+(1+i\tan(-\theta))^n$$

$$=\frac{(\cos\theta+i\sin\theta)^n}{\cos^n\theta}+\frac{\left(\cos(-\theta)+i\sin(-\theta)\right)^n}{\cos^n\theta}$$

Note: Award ***M1*** for converting to cosine and sine terms.

use of de Moivre's theorem

$$=\frac{1}{\cos^n\theta}\left(\cos n\theta+\mathrm{i}\sin n\theta+\cos(-n\theta)+\mathrm{i}\sin(-n\theta)\right)$$

$$=\frac{2\cos n\theta}{\cos^n\theta}\text{ as }\cos(-n\theta)=\cos n\theta\text{ and }\sin(-n\theta)=-\sin n\theta$$

(ii) $\left(1+\mathrm{i}\tan\frac{3\pi}{8}\right)^4+\left(1-\mathrm{i}\tan\frac{3\pi}{8}\right)^4=\dfrac{2\cos\left(4\times\frac{3\pi}{8}\right)}{\cos^4\frac{3\pi}{8}}$

$=\dfrac{2\cos\frac{3\pi}{2}}{\cos^4\frac{3\pi}{8}}$

$=0$ as $\cos\frac{3\pi}{2}=0$

Note: The above working could involve theta and the solution of $\cos(4\theta)=0$.

so $\mathrm{i}\tan\frac{3\pi}{8}$ is a root of the equation

(iii) either $-\mathrm{i}\tan\frac{3\pi}{8}$ or $-\mathrm{i}\tan\frac{\pi}{8}$ or $\mathrm{i}\tan\frac{\pi}{8}$

Note: Accept $\mathrm{i}\tan\frac{5\pi}{8}$ or $\mathrm{i}\tan\frac{7\pi}{8}$.
Accept $-\left(1+\sqrt{2}\right)\mathrm{i}$ or $\left(1-\sqrt{2}\right)\mathrm{i}$ or $\left(-1+\sqrt{2}\right)\mathrm{i}$.

(b) (i) $\tan\frac{\pi}{4}=\dfrac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$

$\tan^2\frac{\pi}{8}+2\tan\frac{\pi}{8}-1=0$

let $t=\tan\frac{\pi}{8}$

attempting to solve $t^2+2t-1=0$ for t

$t=-1\pm\sqrt{2}$

$\frac{\pi}{8}$ is a first quadrant angle and tan is positive in this quadrant, so

$\tan\frac{\pi}{8}>0$

$\tan\frac{\pi}{8}=\sqrt{2}-1$

(ii) $\cos 4x = 2\cos^2 2x - 1$

$$= 2\left(2\cos^2 x - 1\right)^2 - 1$$

$$= 2\left(4\cos^4 x - 4\cos^2 x + 1\right) - 1$$

$$= 8\cos^4 x - 8\cos^2 x + 1$$

Note: Accept equivalent complex number derivation.

(iii) $\int_0^{\frac{\pi}{8}} \frac{2\cos 4x}{\cos^2 x}\,\mathrm{d}x = 2\int_0^{\frac{\pi}{8}} \frac{8\cos^4 x - 8\cos^2 x + 1}{\cos^2 x}\,\mathrm{d}x$

$$= 2\int_0^{\frac{\pi}{8}} 8\cos^2 x - 8 + \sec^2 x\,\mathrm{d}x$$

Note: The ***M1*** is for an integrand involving no fractions.

use of $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$

$$= 2\int_0^{\frac{\pi}{8}} 4\cos 2x - 4 + \sec^2 x\,\mathrm{d}x$$

$$= \left[4\sin 2x - 8x + 2\tan x\right]_0^{\frac{\pi}{8}}$$

$$= 4\sqrt{2} - \pi - 2 \text{ (or equivalent)}$$

16 - (MAT/12_HL_Summer_2015_Q3) - *Trigonometry*

$\tan x + \tan 2x = 0$

$\tan x + \frac{2\tan x}{1 - \tan^2 x} = 0$

$\tan x - \tan^3 x + 2\tan x = 0$

$\tan x\left(3 - \tan^2 x\right) = 0$

$\tan x = 0 \Rightarrow x = 0,\ x = 180^\circ$

Note: If $x = 360^\circ$ seen anywhere award ***A0***

$\tan x = \sqrt{3} \Rightarrow x = 60^\circ,\ 240^\circ$

$\tan x = -\sqrt{3} \Rightarrow x = 120^\circ,\ 300^\circ$

17 – (MAT/12_HL_Summer_2015_Q6) – *Trigonometry*

(a) any attempt to use sine rule

$$\frac{\text{AB}}{\sin\frac{\pi}{3}} = \frac{\sqrt{3}}{\sin\left(\frac{2\pi}{3} - \theta\right)}$$

$$= \frac{\sqrt{3}}{\sin\frac{2\pi}{3}\cos\theta - \cos\frac{2\pi}{3}\sin\theta}$$

Note: Condone use of degrees.

$$= \frac{\sqrt{3}}{\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta}$$

$$\frac{\text{AB}}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta}$$

$$\therefore \text{AB} = \frac{3}{\sqrt{3}\cos\theta + \sin\theta}$$

(b) **METHOD 1**

$$(\text{AB})' = \frac{-3\left(-\sqrt{3}\sin\theta + \cos\theta\right)}{\left(\sqrt{3}\cos\theta + \sin\theta\right)^2}$$

setting $(\text{AB})' = 0$

$$\tan\theta = \frac{1}{\sqrt{3}}$$

$$\theta = \frac{\pi}{6}$$

METHOD 2

$$\mathrm{AB} = \frac{\sqrt{3}\sin\frac{\pi}{3}}{\sin\left(\frac{2\pi}{3} - \theta\right)}$$

AB minimum when $\sin\left(\frac{2\pi}{3} - \theta\right)$ is maximum

$$\sin\left(\frac{2\pi}{3} - \theta\right) = 1$$

$$\frac{2\pi}{3} - \theta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{6}$$

METHOD 3

shortest distance from B to AC is perpendicular to AC

$$\theta = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$$

18 - (MAT/10_HL_Winter_2015_Q9) - *Trigonometry*

$(\sin 2x - \sin x) - (\cos 2x - \cos x) = 1$

attempt to use both double-angle formulae, in whatever form

$(2\sin x\cos x - \sin x) - (2\cos^2 x - 1 - \cos x) = 1$

or $(2\sin x\cos x - \sin x) - \left(2\cos^2 x - \cos x\right) = 0$ for example

Note: Allow any rearrangement of the above equations.

$\sin x(2\cos x - 1) - \cos x(2\cos x - 1) = 0$

$(\sin x - \cos x)(2\cos x - 1) = 0$

$\tan x = 1$ and $\cos x = \frac{1}{2}$

Note: These ***A*** marks are dependent on the ***M*** mark awarded for factorisation.

$x = -\frac{3\pi}{4}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{4}$

19 – (MAT/11_HL_Summer_2016_Q3) – *Trigonometry*

(a) $a = 1.5 \;\; d = 2$

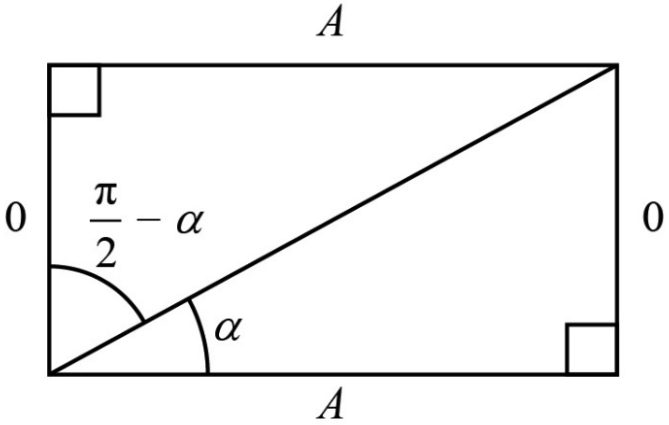
(b) $b = \dfrac{2\pi}{2} = \pi$

(c) attempt to solve an appropriate equation or apply a horizontal translation
$c = 1.5$

20 – (MAT/12_HL_Summer_2016_Q3) – *Trigonometry, Integration*

(a) **EITHER**

use of a diagram and trig ratios
eg,



$\tan\alpha = \frac{O}{A} \Rightarrow \cot\alpha = \frac{A}{O}$

from diagram, $\tan\left(\frac{\pi}{2} - \alpha\right) = \frac{A}{O}$

OR

use of $\tan\left(\frac{\pi}{2} - \alpha\right) = \frac{\sin\left(\frac{\pi}{2} - \alpha\right)}{\cos\left(\frac{\pi}{2} - \alpha\right)} = \frac{\cos\alpha}{\sin\alpha}$

THEN

$\cot\alpha = \tan\left(\frac{\pi}{2} - \alpha\right)$

(b) $\int_{\tan\alpha}^{\cot\alpha} \frac{1}{1+x^2}\,dx = [\arctan x]_{\tan\alpha}^{\cot\alpha}$

Note: Limits (or absence of such) may be ignored at this stage.

$= \arctan(\cot\alpha) - \arctan(\tan\alpha)$

$= \frac{\pi}{2} - \alpha - \alpha$

$= \frac{\pi}{2} - 2\alpha$

21 – (MAT/11_HL_Summer_2016_Q5) – *Trigonometry*

(a) $\left(1-\sqrt{3}\right)^2 = 4 - 2\sqrt{3}$

Note: Award ***A0*** for $1-2\sqrt{3}+3$.

(b) $\cos(60^\circ - 45^\circ) = \cos(60^\circ)\cos(45^\circ) + \sin(60^\circ)\sin(45^\circ)$

$= \frac{1}{2}\times\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}\left(\text{or } \frac{1}{2}\times\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}\right)$

$= \frac{\sqrt{2}+\sqrt{6}}{4}\left(\text{or } \frac{1+\sqrt{3}}{2\sqrt{2}}\right)$

(c) $BC^2 = 2 + 4 - 2\times\sqrt{2}\times 2\cos(15^\circ)$

$= 6 - \sqrt{2}\left(\sqrt{2}+\sqrt{6}\right)$

$= 4 - \sqrt{12}\ \left(= 4 - 2\sqrt{3}\right)$

$BC = \pm\left(1-\sqrt{3}\right)$

$BC = -1 + \sqrt{3}$

22 – (MAT/12_HL_Summer_2016_Q9) – *Trigonometry*

(a) **EITHER**

$$\text{LHS} = \frac{\sqrt{3}-1}{\frac{\sqrt{6}-\sqrt{2}}{4}} + \frac{\sqrt{3}+1}{\frac{\sqrt{6}+\sqrt{2}}{4}}$$

$$= \frac{\sqrt{3}-1}{\frac{\sqrt{3}-1}{2\sqrt{2}}} + \frac{\sqrt{3}+1}{\frac{\sqrt{3}+1}{2\sqrt{2}}}$$

$$= 2\sqrt{2} + 2\sqrt{2}$$

$\text{LHS} = 4\sqrt{2} \Rightarrow x = \frac{\pi}{12}$ is a solution

OR

$$\text{LHS} = \frac{\sqrt{3}-1}{\frac{\sqrt{6}-\sqrt{2}}{4}} + \frac{\sqrt{3}+1}{\frac{\sqrt{6}+\sqrt{2}}{4}}$$

$$= \frac{\left(\sqrt{3}-1\right)\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)+\left(\sqrt{3}+1\right)\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)}{\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)}$$

$= 2\sqrt{18} - 2\sqrt{2}$ (or equivalent)

$\text{LHS} = 4\sqrt{2} \Rightarrow x = \frac{\pi}{12}$ is a solution

(b) $$\frac{\sqrt{2}}{4}\left(\frac{\sqrt{3}-1}{\sin x} + \frac{\sqrt{3}+1}{\cos x}\right) = 2 \Rightarrow \frac{\sin\frac{\pi}{12}}{\sin x} + \frac{\cos\frac{\pi}{12}}{\cos x} = 2$$

$$\frac{\sin\frac{\pi}{12}\cos x + \cos\frac{\pi}{12}\sin x}{\sin x\cos x} = 2$$

$$\sin\frac{\pi}{12}\cos x + \cos\frac{\pi}{12}\sin x = 2\sin x\cos x$$

$$\sin\left(\frac{\pi}{12} + x\right) = \sin 2x$$

$\frac{\pi}{12} + x = \pi - 2x$ or $\pi - \left(\frac{\pi}{12} + x\right) = 2x$

$$x = \frac{11\pi}{36}$$

23 – (MAT/10_HL_Winter_2016_Q13) – *Trigonometry*

(a) $\sin\frac{\pi}{4}+\sin\frac{3\pi}{4}+\sin\frac{5\pi}{4}+\sin\frac{7\pi}{4}+\sin\frac{9\pi}{4}=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{2}$ ***(M1)A1***

Note: Award ***M1*** for 5 equal terms with + or – signs.

[2 marks]

(b) $\frac{1-\cos 2x}{2\sin x}\equiv\frac{1-\left(1-2\sin^2 x\right)}{2\sin x}$ ***M1***

$\equiv\frac{2\sin^2 x}{2\sin x}$ ***A1***

$\equiv\sin x$ ***AG***

[2 marks]

(c) let $\mathrm{P}(n): \sin x + \sin 3x + \ldots + \sin(2n-1)x \equiv \dfrac{1-\cos 2nx}{2\sin x}$

if $n = 1$

$\mathrm{P}(1)$: $\dfrac{1-\cos 2x}{2\sin x} \equiv \sin x$ which is true (as proved in part (b)) ***R1***

assume $\mathrm{P}(k)$ true, $\sin x + \sin 3x + \ldots + \sin(2k-1)x \equiv \dfrac{1-\cos 2kx}{2\sin x}$ ***M1***

Notes: Only award ***M1*** if the words "assume" and "true" appear. Do not award ***M1*** for "let $n = k$" only. Subsequent marks are independent of this ***M1***.

consider $\mathrm{P}(k+1)$:

$\mathrm{P}(k+1): \sin x + \sin 3x + \ldots + \sin(2k-1)x + \sin(2k+1)x \equiv \dfrac{1-\cos 2(k+1)x}{2\sin x}$

$LHS = \sin x + \sin 3x + \ldots + \sin(2k-1)x + \sin(2k+1)x$ ***M1***

$\equiv \dfrac{1-\cos 2kx}{2\sin x} + \sin(2k+1)x$ ***A1***

$\equiv \dfrac{1-\cos 2kx + 2\sin x \sin(2k+1)x}{2\sin x}$

$\equiv \dfrac{1-\cos 2kx + 2\sin x \cos x \sin 2kx + 2\sin^2 x \cos 2kx}{2\sin x}$ ***M1***

$\equiv \dfrac{1-\left(\left(1-2\sin^2 x\right)\cos 2kx - \sin 2x \sin 2kx\right)}{2\sin x}$ ***M1***

$\equiv \dfrac{1-\left(\cos 2x \cos 2kx - \sin 2x \sin 2kx\right)}{2\sin x}$ ***A1***

$\equiv \dfrac{1-\cos\left(2kx+2x\right)}{2\sin x}$ ***A1***

$\equiv \dfrac{1-\cos 2(k+1)x}{2\sin x}$

so if true for $n = k$, then also true for $n = k+1$

as true for $n = 1$ then true for all $n \in \mathbb{Z}^+$ ***R1***

Note: Accept answers using transformation formula for product of sines if steps are shown clearly.

Note: Award ***R1*** only if candidate is awarded at least 5 marks in the previous steps.

[9 marks]

(d) **EITHER**

$\sin x + \sin 3x = \cos x \Rightarrow \dfrac{1-\cos 4x}{2\sin x} = \cos x$ ***M1***

$\Rightarrow 1 - \cos 4x = 2\sin x \cos x$, $(\sin x \neq 0)$ ***A1***

$\Rightarrow 1 - \left(1 - 2\sin^2 2x\right) = \sin 2x$ ***M1***

$\Rightarrow \sin 2x(2\sin 2x - 1) = 0$ ***M1***

$\Rightarrow \sin 2x = 0$ or $\sin 2x = \dfrac{1}{2}$ ***A1***

$2x = \pi$, $2x = \dfrac{\pi}{6}$ and $2x = \dfrac{5\pi}{6}$

OR

$\sin x + \sin 3x = \cos x \Rightarrow 2\sin 2x \cos x = \cos x$ ***M1A1***

$\Rightarrow \left(2\sin 2x - 1\right)\cos x = 0$, $(\sin x \neq 0)$ ***M1A1***

$\Rightarrow \sin 2x = \dfrac{1}{2}$ or $\cos x = 0$ ***A1***

$2x = \dfrac{\pi}{6}$, $2x = \dfrac{5\pi}{6}$ and $x = \dfrac{\pi}{2}$

THEN

$\therefore\ x = \dfrac{\pi}{2}$, $x = \dfrac{\pi}{12}$ and $x = \dfrac{5\pi}{12}$ ***A1***

Note: Do not award the final ***A1*** if extra solutions are seen.

[6 marks]

24 – (MAT/11_HL_Summer_2017_Q3) – *Trigonometry*

METHOD 1

use of $\sec^2 x = \tan^2 x + 1$ ***M1***

$\tan^2 x + 2\tan x + 1 = 0$

$(\tan x + 1)^2 = 0$ ***(M1)***

$\tan x = -1$ ***A1***

$x = \frac{3\pi}{4}, \frac{7\pi}{4}$ ***A1A1***

METHOD 2

$\frac{1}{\cos^2 x} + \frac{2\sin x}{\cos x} = 0$ ***M1***

$1 + 2\sin x\cos x = 0$

$\sin 2x = -1$ ***M1A1***

$2x = \frac{3\pi}{2}, \frac{7\pi}{2}$

$x = \frac{3\pi}{4}, \frac{7\pi}{4}$ ***A1A1***

Note: Award ***A1A0*** if extra solutions given or if solutions given in degrees (or both).

[5 marks]

25 – (MAT/12_HL_Summer_2017_Q11) – *Trigonometry, Complex Numbers*

(a) $2\sin(x+60^\circ)=\cos(x+30^\circ)$

$2(\sin x\cos 60^\circ+\cos x\sin 60^\circ)=\cos x\cos 30^\circ-\sin x\sin 30^\circ$ ***(M1)(A1)***

$2\sin x\times\frac{1}{2}+2\cos x\times\frac{\sqrt{3}}{2}=\cos x\times\frac{\sqrt{3}}{2}-\sin x\times\frac{1}{2}$ ***A1***

$\Rightarrow\frac{3}{2}\sin x=-\frac{\sqrt{3}}{2}\cos x$

$\Rightarrow\tan x=-\frac{1}{\sqrt{3}}$ ***M1***

$\Rightarrow x=150^\circ$ ***A1***

[5 marks]

(b) **EITHER**

choosing two appropriate angles, for example 60° and 45° ***M1***

$\sin 105^\circ=\sin 60^\circ\cos 45^\circ+\cos 60^\circ\sin 45^\circ$ and

$\cos 105^\circ=\cos 60^\circ\cos 45^\circ-\sin 60^\circ\sin 45^\circ$ ***(A1)***

$\sin 105^\circ+\cos 105^\circ=\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}+\frac{1}{2}\times\frac{1}{\sqrt{2}}+\frac{1}{2}\times\frac{1}{\sqrt{2}}-\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}$ ***A1***

$=\frac{1}{\sqrt{2}}$ ***AG***

OR

attempt to square the expression ***M1***

$\left(\sin 105^\circ+\cos 105^\circ\right)^2=\sin^2 105^\circ+2\sin 105^\circ\cos 105^\circ+\cos^2 105^\circ$

$\left(\sin 105^\circ+\cos 105^\circ\right)^2=1+\sin 210^\circ$ ***A1***

$=\frac{1}{2}$ ***A1***

$\sin 105^\circ+\cos 105^\circ=\frac{1}{\sqrt{2}}$ ***AG***

[3 marks]

(c) (i) **EITHER**

$z = (1 - \cos 2\theta) - \mathrm{i}\sin 2\theta$

$|z| = \sqrt{(1-\cos 2\theta)^2 + (\sin 2\theta)^2}$ ***M1***

$|z| = \sqrt{1 - 2\cos 2\theta + \cos^2 2\theta + \sin^2 2\theta}$ ***A1***

$= \sqrt{2}\sqrt{(1-\cos 2\theta)}$ ***A1***

$= \sqrt{2\left(2\sin^2\theta\right)}$

$= 2\sin\theta$ ***A1***

let $\arg(z) = \alpha$

$\tan\alpha = -\dfrac{\sin 2\theta}{1-\cos 2\theta}$ ***M1***

$= \dfrac{-2\sin\theta\cos\theta}{2\sin^2\theta}$ ***(A1)***

$= -\cot\theta$ ***A1***

$\arg(z) = \alpha = -\arctan\left(\tan(\frac{\pi}{2} - \theta)\right)$ ***A1***

$= \theta - \dfrac{\pi}{2}$ ***A1***

OR

$z = (1 - \cos 2\theta) - \mathrm{i}\sin 2\theta$

$= 2\sin^2\theta - 2i\sin\theta\cos\theta$ ***M1A1***

$= 2\sin\theta\left(\sin\theta - i\cos\theta\right)$ ***(A1)***

$= -2i\sin\theta\left(\cos\theta + i\sin\theta\right)$ ***M1A1***

$= 2\sin\theta\left(\cos\left(\theta - \dfrac{\pi}{2}\right) + i\sin\left(\theta - \dfrac{\pi}{2}\right)\right)$ ***M1A1***

$|z| = 2\sin\theta$ ***A1***

$\arg(z) = \theta - \dfrac{\pi}{2}$ ***A1***

(ii) attempt to apply De Moivre's theorem ***M1***

$$(1-\cos 2\theta - \mathrm{i}\sin 2\theta)^{\frac{1}{3}} = 2^{\frac{1}{3}}(\sin\theta)^{\frac{1}{3}}\left[\cos\left(\frac{\theta-\frac{\pi}{2}+2n\pi}{3}\right)+\mathrm{i}\sin\left(\frac{\theta-\frac{\pi}{2}+2n\pi}{3}\right)\right]$$

A1A1A1

Note: ***A1*** for modulus, ***A1*** for dividing argument of z by 3 and ***A1*** for $2n\pi$.

Hence cube roots are the above expression when $n=-1,\ 0,\ 1$.
Equivalent forms are acceptable. ***A1***

[14 marks]

26 – (MAT/11_HL_Summer_2018_Q8) – *Trigonometry*

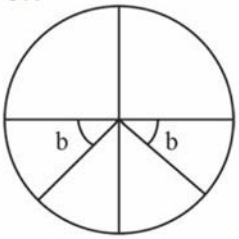
$\sin 2x = -\sin b$

EITHER

$\sin 2x = \sin(-b)$ or $\sin 2x = \sin(\pi + b)$ or $\sin 2x = \sin(2\pi - b)\ldots$ ***(M1)(A1)***

Note: Award ***M1*** for any one of the above, ***A1*** for having final two.

OR



(M1)(A1)

Note: Award ***M1*** for one of the angles shown with b clearly labelled, ***A1*** for both angles shown. Do not award ***A1*** if an angle is shown in the second quadrant and subsequent ***A1*** marks not awarded.

THEN

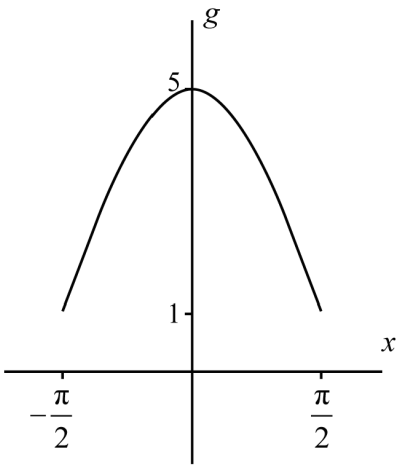
$2x = \pi + b$ or $2x = 2\pi - b$ ***(A1)(A1)***

$x = \frac{\pi}{2} + \frac{b}{2},\ x = \pi - \frac{b}{2}$ ***A1***

[5 marks]

27 – (MAT/10_HL_Winter_2018_Q3) – *Trigonometry*

(a)



concave down and symmetrical over correct domain ***A1***

indication of maximum and minimum values of the function (correct range) ***A1A1***

[3 marks]

(b) $a = 0$ ***A1***

Note: Award ***A1*** for $a = 0$ only if consistent with their graph.

[1 mark]

(c) (i) $1 \le x \le 5$ ***A1***

Note: Allow FT from their graph.

(ii) $y = 4\cos x + 1$

$x = 4\cos y + 1$

$\frac{x-1}{4} = \cos y$ ***(M1)***

$\Rightarrow y = \arccos\left(\frac{x-1}{4}\right)$

$\Rightarrow g^{-1}(x) = \arccos\left(\frac{x-1}{4}\right)$ ***A1***

[3 marks]

28 – (MAT/11_HL_Summer_2019_Q4) – *Trigonometry*

(a) **EITHER**

$\frac{5\sqrt{15}}{2} = \frac{1}{2} \times 4 \times 5 \sin\theta$ ***A1***

OR

height of triangle is $\frac{5\sqrt{15}}{4}$ if using 4 as the base or $\sqrt{15}$ if using 5 as the base ***A1***

THEN

$\sin\theta = \frac{\sqrt{15}}{4}$ ***AG***

[1 mark]

(b) let the third side be x

$x^2 = 4^2 + 5^2 - 2 \times 4 \times 5 \times \cos\theta$ ***M1***

valid attempt to find $\cos\theta$ ***(M1)***

Note: Do not accept writing $\cos\left(\arcsin\left(\frac{\sqrt{15}}{4}\right)\right)$ as a valid method.

$\cos\theta = \pm\sqrt{1 - \frac{15}{16}}$

$= \frac{1}{4}, -\frac{1}{4}$ ***A1A1***

$x^2 = 16 + 25 - 2 \times 4 \times 5 \times \pm\frac{1}{4}$

$x = \sqrt{31}$ or $\sqrt{51}$ ***A1A1***

[6 marks]

29 – (MAT/11_HL_Summer_2019_Q9) – *Trigonometry, Integration*

(a) $(\sin x+\cos x)^2=\sin^2 x+2\sin x\cos x+\cos^2 x$ ***M1A1***

Note: Do not award the ***M1*** for just $\sin^2 x+\cos^2 x$.

Note: Do not award ***A1*** if correct expression is followed by incorrect working.

$=1+\sin 2x$ ***AG***

[2 marks]

(b) $\sec 2x+\tan 2x=\dfrac{1}{\cos 2x}+\dfrac{\sin 2x}{\cos 2x}$ ***M1***

Note: ***M1*** is for an attempt to change both terms into sine and cosine forms (with the same argument) or both terms into functions of $\tan x$.

$=\dfrac{1+\sin 2x}{\cos 2x}$

$=\dfrac{(\sin x+\cos x)^2}{\cos^2 x-\sin^2 x}$ ***A1A1***

Note: Award ***A1*** for numerator, ***A1*** for denominator.

$=\dfrac{(\sin x+\cos x)^2}{(\cos x-\sin x)(\cos x+\sin x)}$ ***M1***

$=\dfrac{\cos x+\sin x}{\cos x-\sin x}$ ***AG***

Note: Apply MS in reverse if candidates have worked from RHS to LHS.

Note: Alternative method using $\tan 2x$ and $\sec 2x$ in terms of $\tan x$.

[4 marks]

(c) **METHOD 1**

$\int_0^{\frac{\pi}{6}}\left(\dfrac{\cos x+\sin x}{\cos x-\sin x}\right)\mathrm{d}x$ ***A1***

Note: Award ***A1*** for correct expression with or without limits.

EITHER

$=\left[-\ln(\cos x-\sin x)\right]_0^{\frac{\pi}{6}}$ or $\left[\ln(\cos x-\sin x)\right]_{\frac{\pi}{6}}^{0}$ ***(M1)A1A1***

Note: Award ***M1*** for integration by inspection or substitution, ***A1*** for $\ln(\cos x-\sin x)$, ***A1*** for completely correct expression including limits.

$=-\ln\left(\cos\dfrac{\pi}{6}-\sin\dfrac{\pi}{6}\right)+\ln(\cos 0-\sin 0)$ ***M1***

Note: Award ***M1*** for substitution of limits into their integral and subtraction.

$=-\ln\left(\dfrac{\sqrt{3}}{2}-\dfrac{1}{2}\right)$ ***(A1)***

OR

let $u = \cos x - \sin x$ ***M1***

$\frac{du}{dx} = -\sin x - \cos x = -(\sin x + \cos x)$

$-\int_1^{\frac{\sqrt{3}}{2}-\frac{1}{2}} \left(\frac{1}{u}\right) du$ ***A1A1***

Note: Award ***A1*** for correct limits even if seen later, ***A1*** for integral.

$= \left[-\ln u\right]_1^{\frac{\sqrt{3}}{2}-\frac{1}{2}}$ or $\left[\ln u\right]_{\frac{\sqrt{3}}{2}-\frac{1}{2}}^1$ ***A1***

$= -\ln\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)(+\ln 1)$ ***M1***

THEN

$= \ln\left(\frac{2}{\sqrt{3}-1}\right)$ ***M1***

Note: Award ***M1*** for both putting the expression over a common denominator and for correct use of law of logarithms.

$= \ln\left(1+\sqrt{3}\right)$ ***(M1)A1***

[9 marks]

30 – (MAT/10_HL_Winter_2019_Q4) – *Trigonometry*

attempt to use $\cos(2A+B) = \cos 2A \cos B - \sin 2A \sin B$ (may be seen later) ***M1***

attempt to use any double angle formulae (seen anywhere) ***M1***

attempt to find either $\sin A$ or $\cos B$ (seen anywhere) ***M1***

$\cos A = \frac{2}{3} \Rightarrow \sin A\left(=\sqrt{1-\frac{4}{9}}\right) = \frac{\sqrt{5}}{3}$ ***(A1)***

$\sin B = \frac{1}{3} \Rightarrow \cos B\left(=\sqrt{1-\frac{1}{9}} = \frac{\sqrt{8}}{3}\right) = \frac{2\sqrt{2}}{3}$ ***A1***

$\cos 2A\left(=2\cos^2 A - 1\right) = -\frac{1}{9}$ ***A1***

$\sin 2A\left(=2\sin A\cos A\right) = \frac{4\sqrt{5}}{9}$ ***A1***

So $\cos(2A+B) = \left(-\frac{1}{9}\right)\left(\frac{2\sqrt{2}}{3}\right) - \left(\frac{4\sqrt{5}}{9}\right)\left(\frac{1}{3}\right)$

$= -\frac{2\sqrt{2}}{27} - \frac{4\sqrt{5}}{27}$ ***AG***

[7 marks]

31 - (MAT/10_HL_Winter_2019_Q9) - *Trigonometry*

(a) $\cos 105 = \cos(180 - 75) = -\cos 75$ ***R1***

$= -q$ ***AG***

Note: Accept arguments using the unit circle or graphical/diagrammatical considerations.

[1 mark]

(b) $AD = CD \Rightarrow C\hat{A}D = 45$ ***A1***

valid method to find $B\hat{A}C$ ***(M1)***

for example: $BC = r \Rightarrow B\hat{C}A = 60$

$\Rightarrow B\hat{A}C = 30$ ***A1***

hence $B\hat{A}D = 45 + 30 = 75$ ***AG***

[3 marks]

(c) (i) $AB = r\sqrt{3}$, $AD(= CD) = r\sqrt{2}$ ***A1A1***

applying cosine rule ***(M1)***

$BD^2 = (r\sqrt{3})^2 + (r\sqrt{2})^2 - 2(r\sqrt{3})(r\sqrt{2})\cos 75$ ***A1***

$= 3r^2 + 2r^2 - 2r^2\sqrt{6}\cos 75$

$= 5r^2 - 2r^2 q\sqrt{6}$ ***AG***

(ii) $B\hat{C}D = 105$ ***(A1)***

attempt to use cosine rule on ΔBCD ***(M1)***

$BD^2 = r^2 + (r\sqrt{2})^2 - 2r(r\sqrt{2})\cos 105$

$= 3r^2 + 2r^2 q\sqrt{2}$ ***A1***

[7 marks]

(d) $5r^2 - 2r^2 q\sqrt{6} = 3r^2 + 2r^2 q\sqrt{2}$ ***(M1)(A1)***

$2r^2 = 2r^2 q(\sqrt{6} + \sqrt{2})$ ***A1***

Note: Award ***A1*** for any correct intermediate step seen using only two terms.

$q = \dfrac{1}{\sqrt{6} + \sqrt{2}}$ ***AG***

Note: Do not award the final ***A1*** if follow through is being applied.

[3 marks]

32 – (MAT/10_HL_Winter_2020_Q6) – *Quadratics, Trigonometry*

METHOD 1

$\sin\theta\cos\theta = \frac{c}{a}$ and $\sin\theta + \cos\theta = -\frac{b}{a}$ ***A1***

attempt to square $\sin\theta + \cos\theta$ ***M1***

$\left(\frac{b^2}{a^2} =\right)(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$ ***A1***

$\frac{b^2}{a^2}(= 1 + 2\sin\theta\cos\theta) = 1 + \frac{2c}{a}$ ***A1***

$b^2 = a^2 + 2ac$ ***AG***

[4 marks]

METHOD 2

$a\sin^2\theta + b\sin\theta + c = 0$ and $a\cos^2\theta + b\cos\theta + c = 0$ ***A1***

adding the two equations ***M1***

$a + b(\sin\theta + \cos\theta) + 2c = 0$ ***A1***

$a + b \times -\frac{b}{a} + 2c = 0$ ***A1***

$a^2 - b^2 + 2ac = 0$

$b^2 = a^2 + 2ac$ ***AG***

[4 marks]

33 – (MAT/10_HL_Winter_2020_Q8) – *Trigonometry*

(a) **METHOD 1**

attempt to replace $\tan x = \frac{\sin x}{\cos x}$ ***M1***

$$\frac{\sin x \tan x}{1-\cos x} \equiv \frac{\sin^2 x}{\cos x(1-\cos x)}$$

attempt to use $\sin^2 x + \cos^2 x = 1$ ***M1***

$$= \frac{1-\cos^2 x}{\cos x(1-\cos x)}$$

$$= \frac{(1+\cos x)(1-\cos x)}{\cos x(1-\cos x)}$$ ***A1***

$$= \frac{(1+\cos x)}{\cos x}$$

$$= 1 + \frac{1}{\cos x}$$ ***AG***

Note: Award marks in reverse for working from RHS to LHS.

[3 marks]

METHOD 2

attempt to replace $\tan x = \frac{\sin x}{\cos x}$ ***M1***

$$\frac{\sin x \tan x}{1-\cos x} \equiv \frac{\sin^2 x}{\cos x(1-\cos x)}$$

$$\equiv \frac{\sin^2 x(1+\cos x)}{\cos x(1-\cos x)(1+\cos x)} \equiv \frac{\sin^2 x(1+\cos x)}{\cos x(1-\cos^2 x)}$$

attempt to use $\sin^2 x + \cos^2 x = 1$ ***M1***

$$\equiv \frac{\sin^2 x(1+\cos x)}{\cos x \sin^2 x}$$ ***A1***

$$= \frac{(1+\cos x)}{\cos x}$$

$$= 1 + \frac{1}{\cos x}$$ ***AG***

Note: Award marks in reverse for working from RHS to LHS.

[3 marks]

(b) **METHOD 1**

consider $1+\frac{1}{\cos x}=k$, leading to $\cos x=\frac{1}{k-1}$ ***(M1)***

consider graph of $y=\frac{1}{x-1}$ or range of solutions for $y=\cos x$ ***(M1)***

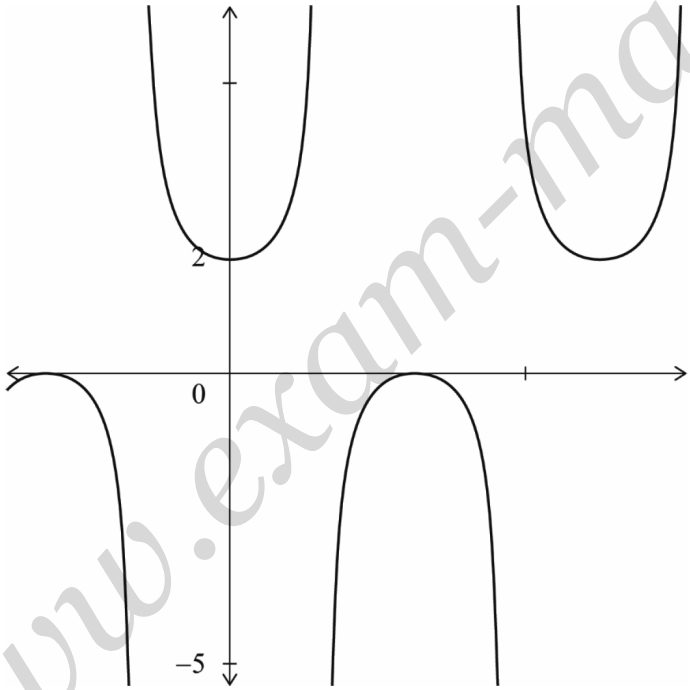
(no solutions if $y<-1$ or $y>1$ $\Rightarrow$) $0<k<2$ ***A1A1***

Note: Award ***A1*** for 0 and 2 seen as critical values, ***A1*** for correct inequalities. These may also be expressed as '$k>0$ and $k<2$'.

[4 marks]

METHOD 2

consider graph of $y=1+\sec x$ ***M1***



A1

no real solutions if $0<k<2$ ***A1A1***

Note: Award ***A1*** for 0 and 2 seen as critical values, ***A1*** for correct inequalities. These may also be expressed as '$k>0$ and $k<2$'.

[4 marks]

METHOD 3

consider $-1 \le \cos x \le 1$, ***(M1)***

$\frac{1}{\cos x} \le -1$ or $\frac{1}{\cos x} \ge 1$ ***(M1)***

$1+\frac{1}{\cos x} \le 0$ or $1+\frac{1}{\cos x} \ge 2$

no solutions if $0 < k < 2$ ***A1A1***

Note: Award ***A1*** for 0 and 2 seen as critical values, ***A1*** for correct inequalities.
These may also be expressed as '$k > 0$ and $k < 2$'.

[4 marks]

34 – (MathAA/12_HL_Summer_2021_Q2) – *Trigonometry*

(a) attempt to use $\cos^2 x = 1 - \sin^2 x$ ***M1***

$2\sin^2 x - 5\sin x + 2 = 0$ ***A1***

EITHER
attempting to factorise ***M1***
$(2\sin x - 1)(\sin x - 2)$ ***A1***

OR

attempting to use the quadratic formula ***M1***
$\sin x = \frac{5 \pm \sqrt{5^2 - 4 \times 2 \times 2}}{4} \left(= \frac{5 \pm 3}{4} \right)$ ***A1***

THEN
$\sin x = \frac{1}{2}$ ***(A1)***
$x = \frac{\pi}{6}, \frac{5\pi}{6}$ ***A1A1***

Total [7 marks]

35 – (MathAA/11_HL_Summer_2021_Q5) – *Trigonometry*

(a)

Note: Do not award the final ***A1*** for proofs which work from both sides to find a common expression other than $2\sin x\cos x - 2\sin^2 x$.

METHOD 1 (LHS to RHS)

attempt to use double angle formula for $\sin 2x$ or $\cos 2x$ ***M1***

LHS= $2\sin x\cos x + \cos 2x - 1$ OR

$\sin 2x + 1 - 2\sin^2 x - 1$ OR

$2\sin x\cos x + 1 - 2\sin^2 x - 1$

$= 2\sin x\cos x - 2\sin^2 x$ ***A1***

$\sin 2x + \cos 2x - 1 = 2\sin x(\cos x - \sin x)$ =RHS ***AG***

METHOD 2 (RHS to LHS)

RHS $= 2\sin x\cos x - 2\sin^2 x$

attempt to use double angle formula for $\sin 2x$ or $\cos 2x$ ***M1***

$= \sin 2x + 1 - 2\sin^2 x - 1$ ***A1***

$= \sin 2x + \cos 2x - 1 =$ LHS ***AG***

[2 marks]

(b) attempt to factorise ***M1***

$(\cos x - \sin x)(2\sin x + 1) = 0$ ***A1***

recognition of $\cos x = \sin x \Rightarrow \dfrac{\sin x}{\cos x} = \tan x = 1$ OR $\sin x = -\dfrac{1}{2}$ ***(M1)***

one correct reference angle seen anywhere, accept degrees ***(A1)***

$\dfrac{\pi}{4}$ OR $\dfrac{\pi}{6}$ (accept $-\dfrac{\pi}{6}, \dfrac{7\pi}{6}$)

Note*:* This ***(M1)(A1)*** is independent of the previous ***M1A1.***

$x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}, \dfrac{\pi}{4}, \dfrac{5\pi}{4}$ ***A2***

Note: Award ***A1*** for any two correct (radian) answers.
Award ***A1A0*** if additional values given with the four correct (radian) answers. Award ***A1A0*** for four correct answers given in degrees.

[6 marks]

36 – (MathAA/12_HL_Summer_2021_Q6) – *Trigonometry*

METHOD 1

attempt to use the cosine rule to find the value of x ***(M1)***

$100 = x^2 + 4x^2 - 2(x)(2x)\left(\frac{3}{4}\right)$ ***A1***

$2x^2 = 100$

$x^2 = 50$ OR $x = \sqrt{50}$ $\left(= 5\sqrt{2}\right)$ ***A1***

attempt to find $\sin\hat{C}$ (seen anywhere) ***(M1)***

$\sin^2\hat{C} + \left(\frac{3}{4}\right)^2 = 1$ or $x^2 + 3^2 = 4^2$ OR right triangle with side 3 and hypotenuse 4

$\sin\hat{C} = \frac{\sqrt{7}}{4}$ ***(A1)***

Note: The marks for finding $\sin\hat{C}$ may be awarded independently of the first three marks for finding x.

correct substitution into the area formula using their value of x (or x^2) and their value of $\sin\hat{C}$ ***(M1)***

$A = \frac{1}{2} \times 5\sqrt{2} \times 10\sqrt{2} \times \frac{\sqrt{7}}{4}$ or $A = \frac{1}{2} \times \sqrt{50} \times 2\sqrt{50} \times \frac{\sqrt{7}}{4}$

$A = \frac{25\sqrt{7}}{2}$ ***A1***

METHOD 2

attempt to find the height, h, of the triangle in terms of x ***(M1)***

$h^2+\left(\frac{3}{4}x\right)^2=x^2$ OR $h^2+\left(\frac{5}{4}x\right)^2=10^2$ OR $h=\frac{\sqrt{7}}{4}x$ ***A1***

equating their expressions for either h^2 or h ***(M1)***

$x^2-\left(\frac{3}{4}x\right)^2=10^2-\left(\frac{5}{4}x\right)^2$ OR $\sqrt{100-\frac{25}{16}x^2}=\frac{\sqrt{7}}{4}x$ (or equivalent) ***A1***

$x^2=50$ OR $x=\sqrt{50}\ \left(=5\sqrt{2}\right)$ ***A1***

correct substitution into the area formula using their value of x (or x^2) ***(M1)***

$A=\frac{1}{2}\times 2\sqrt{50}\times\frac{\sqrt{7}}{4}\sqrt{50}$ OR $A=\frac{1}{2}\left(2\times 5\sqrt{2}\right)\left(\frac{\sqrt{7}}{4}5\sqrt{2}\right)$

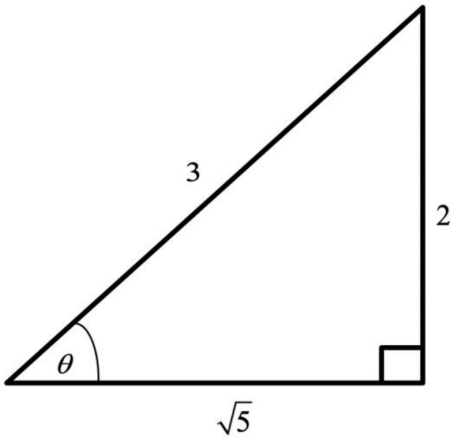
$A=\frac{25\sqrt{7}}{2}$ ***A1***

Total [7 marks]

37 – (MathAA/11_HL_Summer_2021_Q6) – *Trigonometry*

METHOD 1

attempt to use a right angled triangle ***M1***



correct placement of all three values and θ seen in the triangle ***(A1)***

$\cot\theta < 0$ (since $\operatorname{cosec}\theta > 0$ puts θ in the second quadrant) ***R1***

$\cot\theta = -\frac{\sqrt{5}}{2}$ ***A1***

Note: Award ***M1A1R0A0*** for $\cot\theta = \frac{\sqrt{5}}{2}$ seen as the final answer

The ***R1*** should be awarded independently for a negative value only given as a final answer.

[4 marks]

METHOD 2

Attempt to use $1+\cot^2\theta = \operatorname{cosec}^2\theta$ ***M1***

$1+\cot^2\theta = \frac{9}{4}$

$\cot^2\theta = \frac{5}{4}$ ***(A1)***

$\cot\theta = \pm\frac{\sqrt{5}}{2}$

$\cot\theta < 0$ (since $\operatorname{cosec}\theta > 0$ puts θ in the second quadrant) ***R1***

$\cot\theta = -\frac{\sqrt{5}}{2}$ ***A1***

Note: Award ***M1A1R0A0*** for $\cot\theta = \frac{\sqrt{5}}{2}$ seen as the final answer

The ***R1*** should be awarded independently for a negative value only given as a final answer.

METHOD 3

$\sin\theta = \frac{2}{3}$

attempt to use $\sin^2\theta + \cos^2\theta = 1$ ***M1***

$\frac{4}{9} + \cos^2\theta = 1$

$\cos^2\theta = \frac{5}{9}$ ***(A1)***

$\cos\theta = \pm\frac{\sqrt{5}}{3}$

$\cos\theta < 0$ (since $\operatorname{cosec}\theta > 0$ puts θ in the second quadrant) ***R1***

$\cos\theta = -\frac{\sqrt{5}}{3}$

$\cot\theta = -\frac{\sqrt{5}}{2}$ ***A1***

Note: Award ***M1A1R0A0*** for $\cot\theta = \frac{\sqrt{5}}{2}$ seen as the final answer

The ***R1*** should be awarded independently for a negative value only given as a final answer.

38 – (MathAA/12_HL_Summer_2021_Q10) – *Trigonometry*

(a) $6+6\cos x=0$ (or setting their $f'(x)=0$) ***(M1)***

$\cos x=-1$ (or $\sin x=0$)

$x=\pi,\ x=3\pi$ ***A1A1***

[3 marks]

(b) attempt to integrate $\int_{\pi}^{3\pi}(6+6\cos x)\mathrm{d}x$ ***(M1)***

$=\left[6x+6\sin x\right]_{\pi}^{3\pi}$ ***A1A1***

substitute their limits into their integrated expression and subtract ***(M1)***

$=(18\pi+6\sin 3\pi)-(6\pi+6\sin\pi)$

$=(6(3\pi)+0)-(6\pi+0)\ \ (=18\pi-6\pi)$ ***A1***

area $=12\pi$ ***AG***

[5 marks]

(c) attempt to substitute into the formula for surface area (including base) ***(M1)***

$\pi(2^2)+\pi(2)(l)=12\pi$ ***(A1)***

$4\pi+2\pi l=12\pi$

$2\pi l=8\pi$

$l=4$ ***A1***

[3 marks]

(d) valid attempt to find the height of the cone ***(M1)***

e.g. $2^2+h^2=(\text{their } l)^2$

$h=\sqrt{12}\ \left(=2\sqrt{3}\right)$ ***(A1)***

attempt to use $V=\frac{1}{3}\pi r^2 h$ with their values substituted ***M1***

$\left(\frac{1}{3}\pi(2^2)(\sqrt{12})\right)$

volume $=\frac{4\pi\sqrt{12}}{3}\left(=\frac{8\pi\sqrt{3}}{3}=\frac{8\pi}{\sqrt{3}}\right)$ ***A1***

[4 marks]

39 – (MathAA/12_HL_Summer_2021_Q12) – *Trigonometry, Mathematical Induction*

(a) **EITHER**

horizontal stretch/scaling with scale factor $\frac{1}{2}$

Note: Do not allow 'shrink' or 'compression'

followed by a horizontal translation/shift $\frac{1}{2}$ units to the left ***A2***

Note: Do not allow 'move'

OR

horizontal translation/shift 1 unit to the left

followed by horizontal stretch/scaling with scale factor $\frac{1}{2}$ ***A2***

THEN

vertical translation/shift up by $\frac{\pi}{4}$ (or translation through $\begin{pmatrix} 0 \\ \frac{\pi}{4} \end{pmatrix}$) ***A1***

(may be seen anywhere)

[3 marks]

(b) let $\alpha = \arctan p$ and $\beta = \arctan q$ ***M1***

$p = \tan\alpha$ and $q = \tan\beta$ ***(A1)***

$\tan(\alpha+\beta) = \frac{p+q}{1-pq}$ ***A1***

$\alpha + \beta = \arctan\left(\frac{p+q}{1-pq}\right)$ ***A1***

so $\arctan p + \arctan q \equiv \arctan\left(\frac{p+q}{1-pq}\right)$ where $p, q > 0$ and $pq < 1$ ***AG***

[4 marks]

(c) **METHOD 1**

$\frac{\pi}{4} = \arctan 1$ (or equivalent) ***A1***

$\arctan\left(\frac{x}{x+1}\right) + \arctan 1 = \arctan\left(\frac{\frac{x}{x+1}+1}{1-\frac{x}{x+1}(1)}\right)$ ***A1***

$= \arctan\left(\frac{\frac{x+x+1}{x+1}}{\frac{x+1-x}{x+1}}\right)$ ***A1***

$= \arctan(2x+1)$ ***AG***

[3 marks]

METHOD 2

$\tan\frac{\pi}{4} = 1$ (or equivalent) ***A1***

Consider $\arctan(2x+1) - \arctan\left(\frac{x}{x+1}\right) = \frac{\pi}{4}$

$\tan\left(\arctan(2x+1) - \arctan\left(\frac{x}{x+1}\right)\right)$

$= \arctan\left(\frac{2x+1-\frac{x}{x+1}}{1+\frac{x(2x+1)}{x+1}}\right)$ ***A1***

$= \arctan\left(\frac{(2x+1)(x+1)-x}{x+1+x(2x+1)}\right)$ ***A1***

$= \arctan 1$ ***AG***

[3 marks]

METHOD 3

$\tan\left(\arctan\left(2x+1\right)\right)=\tan\left(\arctan\left(\frac{x}{x+1}\right)+\frac{\pi}{4}\right)$

$\tan\frac{\pi}{4}=1$ (or equivalent) ***A1***

LHS = $2x+1$ ***A1***

RHS = $\dfrac{\frac{x}{x+1}+1}{1-\frac{x}{x+1}}\left(=2x+1\right)$ ***A1***

[3 marks]

(d) let $\mathrm{P}(n)$ be the proposition that $\sum_{r=1}^{n}\arctan\left(\frac{1}{2r^2}\right)=\arctan\left(\frac{n}{n+1}\right)$ for $n\in\mathbb{Z}^+$

consider $\mathrm{P}(1)$:

when $n=1$, $\sum_{r=1}^{1}\arctan\left(\frac{1}{2r^2}\right)=\arctan\left(\frac{1}{2}\right)=\mathrm{RHS}$ and so $\mathrm{P}(1)$ is true ***R1***

assume $\mathrm{P}(k)$ is true, ie. $\sum_{r=1}^{k}\arctan\left(\frac{1}{2r^2}\right)=\arctan\left(\frac{k}{k+1}\right)$ $\left(k\in\mathbb{Z}^+\right)$ ***M1***

Note: Award ***M0*** for statements such as "let $n=k$".
Note: Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

consider $\mathrm{P}(k+1)$:

$$\sum_{r=1}^{k+1}\arctan\left(\frac{1}{2r^2}\right)=\sum_{r=1}^{k}\arctan\left(\frac{1}{2r^2}\right)+\arctan\left(\frac{1}{2(k+1)^2}\right)$$ ***(M1)***

$$=\arctan\left(\frac{k}{k+1}\right)+\arctan\left(\frac{1}{2(k+1)^2}\right)$$ ***A1***

$$=\arctan\left(\frac{\frac{k}{k+1}+\frac{1}{2(k+1)^2}}{1-\left(\frac{k}{k+1}\right)\left(\frac{1}{2(k+1)^2}\right)}\right)$$ ***M1***

$$=\arctan\left(\frac{(k+1)(2k^2+2k+1)}{2(k+1)^3-k}\right)$$ ***A1***

Note: Award ***A1*** for correct numerator, with $(k+1)$ factored. Denominator does not need to be simplified

$$=\arctan\left(\frac{(k+1)(2k^2+2k+1)}{2k^3+6k^2+5k+2}\right)$$ ***A1***

Note: Award ***A1*** for denominator correctly expanded. Numerator does not need to be simplified. These two ***A*** marks may be awarded in any order

$$=\arctan\left(\frac{(k+1)(2k^2+2k+1)}{(k+2)(2k^2+2k+1)}\right)=\arctan\left(\frac{k+1}{k+2}\right)$$ ***A1***

Note: The word 'arctan' must be present to be able to award the last three A marks

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and $\mathrm{P}(1)$ is true, so

$\mathrm{P}(n)$ is true for $n \in \mathbb{Z}^+$ ***R1***

Note: Award the final ***R1*** mark provided at least four of the previous marks have been awarded.

Note: To award the final ***R1***, the truth of $\mathrm{P}(k)$ must be mentioned. '$\mathrm{P}(k)$ implies $\mathrm{P}(k+1)$' is insufficient to award the mark.

[9 marks]

40 – (MathAA/10_HL_Winter_2021_Q6) – *Trigonometry*

(a) **METHOD 1**

attempt to write all LHS terms over a common denominator of $x-1$ ***(M1)***

$2x-3-\frac{6}{x-1}=\frac{2x(x-1)-3(x-1)-6}{x-1}$ OR $\frac{(2x-3)(x-1)}{x-1}-\frac{6}{x-1}$

$=\frac{2x^2-2x-3x+3-6}{x-1}$ OR $\frac{2x^2-5x+3}{x-1}-\frac{6}{x-1}$ ***A1***

$=\frac{2x^2-5x-3}{x-1}$ ***AG***

[2 marks]

METHOD 2

attempt to use algebraic division on RHS ***(M1)***

correctly obtains quotient of $2x-3$ and remainder -6 ***A1***

$=2x-3-\frac{6}{x-1}$ as required. ***AG***

[2 marks]

(b) consider the equation $\frac{2\sin^2 2\theta - 5\sin 2\theta - 3}{\sin 2\theta - 1} = 0$ ***(M1)***

$\Rightarrow 2\sin^2 2\theta \cdot \cdot 5\sin 2\theta \cdot \cdot 3 = 0$

EITHER

attempt to factorise in the form $(2\sin 2\theta + a)(\sin 2\theta + b)$ ***(M1)***

Note: Accept any variable in place of $\sin 2\theta$.

$(2\sin 2\theta + 1)(\sin 2\theta - 3) = 0$

OR

attempt to substitute into quadratic formula ***(M1)***

$\sin 2\theta = \frac{5 \pm \sqrt{49}}{4}$

THEN

$\sin 2\theta = -\frac{1}{2}$ or $\sin 2\theta = 3$ ***(A1)***

Note: Award ***A1*** for $\sin 2\theta = -\frac{1}{2}$ only.

one of $\frac{7\pi}{6}$ OR $\frac{11\pi}{6}$ (accept 210 or 330) ***(A1)***

$\theta = \frac{7\pi}{12}, \frac{11\pi}{12}$ (must be in radians) ***A1***

Note: Award ***A0*** if additional answers given.

[5 marks]

Total [7 marks]

41 – (MathAA/12_HL_Summer_2022_Q4) – *Trigonometry*

determines $\frac{\pi}{4}$ (or $45°$) as the first quadrant (reference) angle ***(A1)***

attempts to solve $\frac{x}{2}+\frac{\pi}{3}=\frac{\pi}{4}$ ***(M1)***

Note: Award ***M1*** for attempting to solve $\frac{x}{2}+\frac{\pi}{3}=\frac{\pi}{4},\frac{7\pi}{4}(,...)$

$\frac{x}{2}+\frac{\pi}{3}=\frac{\pi}{4}\Rightarrow x<0$ and so $\frac{\pi}{4}$ is rejected ***(R1)***

$\frac{x}{2}+\frac{\pi}{3}=2\pi-\frac{\pi}{4}\ \left(=\frac{7\pi}{4}\right)$ ***A1***

$x=\frac{17\pi}{6}$ (must be in radians) ***A1***

[5 marks]

42 – (MathAA/10_HL_Winter_2022_Q8) – *Trigonometry, Algebra-functions - Roots*

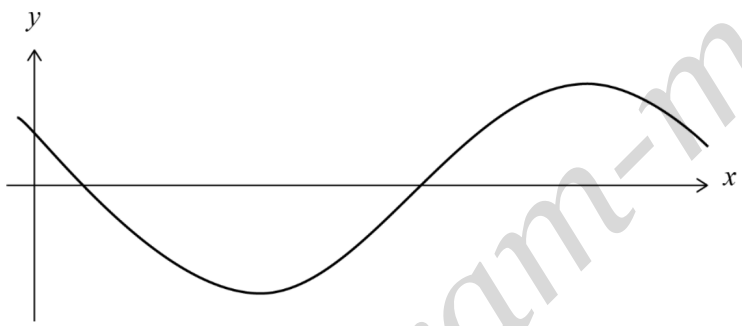
(a)

$a = \frac{\pi}{2}$ ***A2***

Note: For sinusoidal graph through the origin seen with incorrect a, or use of horizontal line test with incorrect a, award ***A1A0***

(b) $a = \pi$ ***A1***

(c)



sketch showing sinusoidal shape decreasing as it crosses the y-axis (below or above the origin) ***(A1)***

$a = k - \pi$ ***A1***

[5 marks]

1 - (MAT/12_HL_Summer_2012_Q5) - *Matrices*

(a) $4\boldsymbol{A} - 5\boldsymbol{BX} = \boldsymbol{B}$

$\Rightarrow \boldsymbol{BX} = \frac{4}{5}\boldsymbol{A} - \frac{1}{5}\boldsymbol{B}$

$\Rightarrow \boldsymbol{X} = \frac{1}{5}\boldsymbol{B}^{-1}(4\boldsymbol{A} - \boldsymbol{B}) \quad \left(= \frac{4}{5}\boldsymbol{B}^{-1}\boldsymbol{A} - \frac{1}{5}\boldsymbol{I}\right)$

(b) if $\boldsymbol{A} = 2\boldsymbol{B}$ then $\boldsymbol{B}^{-1}\boldsymbol{A} = 2\boldsymbol{I}$

$\Rightarrow \boldsymbol{X} = \frac{8}{5}\boldsymbol{I} - \frac{1}{5}\boldsymbol{I}$

$\Rightarrow \boldsymbol{X} = \frac{7}{5}\boldsymbol{I} \left(= \begin{pmatrix} \frac{7}{5} & 0 \\ 0 & \frac{7}{5} \end{pmatrix} \right)$

2 – (MAT/10_HL_Winter_2012_Q6) – *Matrices*

METHOD 1

(a) $\det\begin{pmatrix} 1 & 3 & a-1 \\ 2 & 2 & a-2 \\ 3 & 1 & a-3 \end{pmatrix}$

$= 1\big(2(a-3)-(a-2)\big) - 3\big(2(a-3)-3(a-2)\big) + (a-1)(2-6)$

(or equivalent)

$= 0$ (therefore there is no unique solution)

(b) $\left(\begin{array}{ccc|c} 1 & 3 & a-1 & 1 \\ 2 & 2 & a-2 & 1 \\ 3 & 1 & a-3 & b \end{array}\right) : \left(\begin{array}{ccc|c} 1 & 3 & a-1 & 1 \\ 0 & -4 & -a & -1 \\ 0 & -8 & -2a & b-3 \end{array}\right)$

$: \left(\begin{array}{ccc|c} 1 & 3 & a-1 & 1 \\ 0 & -4 & -a & -1 \\ 0 & 0 & 0 & b-1 \end{array}\right)$

$b = 1$

Note: Award ***M1*** for an attempt to use row operations.

METHOD 2

(a) $\left(\begin{array}{ccc|c} 1 & 3 & a-1 & 1 \\ 2 & 2 & a-2 & 1 \\ 3 & 1 & a-3 & b \end{array}\right) : \left(\begin{array}{ccc|c} 1 & 3 & a-1 & 1 \\ 0 & -4 & -a & -1 \\ 0 & -8 & -2a & b-3 \end{array}\right)$

$: \left(\begin{array}{ccc|c} 1 & 3 & a-1 & 1 \\ 0 & -4 & -a & -1 \\ 0 & 0 & 0 & b-1 \end{array}\right)$ (and 3 zeros imply no unique solution)

(b) $b = 1$

Note: Award ***A4*** only if "$b-1$" seen in (a).

3 – (MAT/10_HL_Winter_2012_Q11) – *Matrices*

(a) $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$

$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow 3A = \begin{pmatrix} 6 & 3 \\ 3 & 3 \end{pmatrix}$

$(A^2 - 3A)^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^2$

$= I$

(b) (i) $\begin{pmatrix} a & a-1 \\ b & b \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \Rightarrow \begin{cases} a + 3a - 3 = 0 \\ b + 3b = -2 \end{cases}$

$a = \frac{3}{4}$ and $b = -\frac{1}{2}$

(ii) **METHOD 1**

$A^{-1}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \Rightarrow A\begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\begin{pmatrix} a & a-1 \\ b & b \end{pmatrix}\begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow \begin{cases} -2a + 2 = 1 \\ -2b = 3 \end{cases}$

$a = \frac{1}{2}$ and $b = -\frac{3}{2}$

METHOD 2

serious attempt at finding A^{-1}

$\left(A^{-1} = \frac{1}{b}\begin{pmatrix} b & 1-a \\ -b & a \end{pmatrix} \right)$

obtain $b + 3 - 3a = 0$ and $-b + 3a = -2b$ or equivalent linear equations

$a = \frac{1}{2}$ and $b = -\frac{3}{2}$

(c) (i) **METHOD 1**

$$\underbrace{\begin{pmatrix} a & a-1 \\ b & b \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & a-1 \\ b & b \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{b}\begin{pmatrix} b & -a+1 \\ -b & a \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{-a+1}{b} \\ \frac{a}{b} \end{pmatrix}$$

METHOD 2

$ax + (a-1)y = 0$ and $bx + by = 1$

attempt to solve

obtain $\left(\frac{1-a}{b}, \frac{a}{b}\right)$

(ii) gradient of l_1 is $\frac{-a}{a-1}$ and gradient of l_2 is -1

the lines are perpendicular $\Rightarrow \frac{-a}{a-1} = 1 \Rightarrow a = \frac{1}{2}$

so they intersect at $\left(\frac{1}{2b}, \frac{1}{2b}\right)$

4 – (MAT/12_HL_Summer_2013_Q2) – *Matrices*

(a) $\det A = (5 \times 2 - 3 \times 3) = 1$

$$A^{-1} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$$

(b) $$\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ 2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \times 3 + (-3) \times 2 & 2 \times (-5) + (-3) \times (-3) \\ (-3) \times 3 + 5 \times 2 & (-3) \times (-5) + 5 \times (-3) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

5 – (MAT/11_HL_Summer_2013_Q3) – *Matrices*

METHOD 1

$$A^{-1}=\frac{1}{-1}\begin{pmatrix}1 & -2\\ -1 & 1\end{pmatrix}\left(=\begin{pmatrix}-1 & 2\\ 1 & -1\end{pmatrix}\right)$$

$$X=A^{-1}BA=\begin{pmatrix}-1 & 2\\ 1 & -1\end{pmatrix}\begin{pmatrix}4 & -3\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}$$

$$\begin{pmatrix}-2 & 5\\ 3 & -4\end{pmatrix}\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}\text{or}\begin{pmatrix}-1 & 2\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 5\\ 2 & 3\end{pmatrix}=\begin{pmatrix}3 & 1\\ -1 & 2\end{pmatrix}$$

Note: Accept the answer $\frac{1}{-1}\begin{pmatrix}-3 & -1\\ 1 & -2\end{pmatrix}$.

METHOD 2

$$X=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

$$AX=BA$$

$$\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}4 & -3\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}$$

$$\begin{pmatrix}a+2c & b+2d\\ a+c & b+d\end{pmatrix}=\begin{pmatrix}1 & 5\\ 2 & 3\end{pmatrix}$$

$a+2c=1$
$b+2d=5$
$a+c=2$
$b+d=3$
$a=3, b=1, c=-1, d=2$

METHOD 3

$$A^{-1}=\frac{1}{-1}\begin{pmatrix}1 & -2\\ -1 & 1\end{pmatrix}\left(=\begin{pmatrix}-1 & 2\\ 1 & -1\end{pmatrix}\right)$$

$$X=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

$$\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}-1 & 2\\ 1 & -1\end{pmatrix}=\begin{pmatrix}4 & -3\\ 1 & 1\end{pmatrix}$$

$a=3, b=1, c=-1, d=2$

6 - (MAT/11_HL_Summer_2013_Q6) - *Matrices*

when $n=1$, $(\boldsymbol{A}+\boldsymbol{I})^1 = 2^0(\boldsymbol{A}+\boldsymbol{I})$ (or $1\times(\boldsymbol{A}+\boldsymbol{I})$)
so true for $n=1$
assume the statement is true for $n=k$
$(\boldsymbol{A}+\boldsymbol{I})^k = 2^{k-1}(\boldsymbol{A}+\boldsymbol{I})$

Note: Award ***M1*** only if assumption of truth is clear.

$$\begin{aligned}(\boldsymbol{A}+\boldsymbol{I})^{k+1} &= (\boldsymbol{A}+\boldsymbol{I})^k(\boldsymbol{A}+\boldsymbol{I})\\ &= 2^{k-1}(\boldsymbol{A}+\boldsymbol{I})(\boldsymbol{A}+\boldsymbol{I})\\ &= 2^{k-1}(\boldsymbol{A}^2+\boldsymbol{I}\boldsymbol{A}+\boldsymbol{A}\boldsymbol{I}+\boldsymbol{I}^2)\\ &= 2^{k-1}(2\boldsymbol{A}+2\boldsymbol{I})\\ &= 2^k(\boldsymbol{A}+\boldsymbol{I})\end{aligned}$$

therefore if true for $n=k$ then true for $n=k+1$; as true for $n=1$ so true for all $n(\in\mathbb{Z}^+)$

7 - (MAT/10_HL_Winter_2013_Q4) - *Matrices*

(if A is singular then) det $A=0$ (seen anywhere)
$\det A = b(b+1)-a(a+1)$
$b+b^2-a-a^2=0$

EITHER

$b-a+(b+a)(b-a)=0$
$(b-a)(1+b+a)=0$

OR

$b-a=a^2-b^2$
$b-a=(a+b)(a-b)$ or $-(a-b)=(a+b)(a-b)$

THEN

$a+b=-1$

8 – (MAT/12_HL_Summer_2014_Q3) – *Matrices*

(a) **EITHER**

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 3 & -1 & 14 & 6 \\ 1 & 2 & 0 & -5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$ ***M1***

row of zeroes implies infinite solutions, (or equivalent). ***R1***

Note: Award ***M1*** for any attempt at row reduction.

OR

$$\begin{vmatrix} 1 & 1 & 2 \\ 3 & -1 & 14 \\ 1 & 2 & 0 \end{vmatrix} = 0$$ ***M1***

$$\begin{vmatrix} 1 & 1 & 2 \\ 3 & -1 & 14 \\ 1 & 2 & 0 \end{vmatrix} = 0 \text{ with one valid point}$$ ***R1***

OR

$$x + y + 2z = -2$$
$$3x - y + 14z = 6$$
$$x + 2y = -5 \quad \Rightarrow x = -5 - 2y$$

substitute $x = -5 - 2y$ into the first two equations:

$$-5 - 2y + y + 2z = -2$$
$$3(-5 - 2y) - y + 14z = 6$$ ***M1***
$$-y + 2z = 3$$
$$-7y + 14z = 21$$

the latter two equations are equivalent (by multiplying by 7) therefore an infinite number of solutions. ***R1***

OR

for example, $7 \times R_1 - R_2$ gives $4x + 8y = -20$ ***M1***

this equation is a multiple of the third equation, therefore an infinite number of solutions. ***R1***

(b) let $y = t$

then $x = -5 - 2t$

$z = \dfrac{t+3}{2}$

OR

let $x = t$

then $y = \dfrac{-5-t}{2}$

$z = \dfrac{1-t}{4}$

OR

let $z = t$

then $x = 1 - 4t$

$y = -3 + 2t$

OR

attempt to find cross product of two normal vectors:

$$eg: \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix} = -4i + 2j + k$$

$x = 1 - 4t$

$y = -3 + 2t$

$z = t$

(or equivalent)

9 – (MAT/12_HL_Summer_2016_Q1) – *Matrices*

EITHER

eliminating a variable, x, for example to obtain $y + 3z = -16$ and $-5y - 3z = 8$
attempting to find the value of one variable
point of intersection is $(-1,\ 2,\ -6)$

OR

attempting row reduction of relevant matrix, *eg*. $\left(\begin{array}{ccc|c} 2 & 1 & -1 & 6 \\ 1 & 3 & 1 & -1 \\ 1 & 2 & -2 & 15 \end{array}\right)$

correct matrix with two zeroes in a column, *eg*. $\left(\begin{array}{ccc|c} 2 & 1 & -1 & 6 \\ 0 & 5 & 3 & -8 \\ 0 & 1 & 3 & -16 \end{array}\right)$

further attempt at reduction
point of intersection is $(-1,\ 2,\ -6)$

10 – (MAT/10_HL_Winter_2018_Q4) – *Matrices*

(a) an attempt at a valid method *eg* by inspection or row reduction ***(M1)***

$2 \times R_2 = R_1 \Rightarrow 2a = -1$

$\Rightarrow a = -\frac{1}{2}$ ***A1***

[2 marks]

(b) using elimination or row reduction to eliminate one variable ***(M1)***
correct pair of equations in 2 variables, such as

$\left.\begin{array}{l} 5x + 10y = 25 \\ 5x + 12y = 4 \end{array}\right\}$ ***A1***

Note: Award ***A1*** for $z = 0$ and one other equation in two variables.

attempting to solve for these two variables ***(M1)***

$x = 26, y = -10.5, z = 0$ ***A1A1***

Note: Award ***A1A0*** for only two correct values, and ***A0A0*** for only one.

Note: Award marks in part (b) for equivalent steps seen in part (a).

[5 marks]

1 - (MAT/12_HL_Summer_2012_Q2) - *Vectors - Lines - Planes*

perpendicular when $\begin{pmatrix} 1 \\ 2\cos x \\ 0 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 2\sin x \\ 1 \end{pmatrix} = 0$

$\Rightarrow -1 + 4\sin x \cos x = 0$

$\Rightarrow \sin 2x = \frac{1}{2}$

$\Rightarrow 2x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\Rightarrow x = \frac{\pi}{12}, \frac{5\pi}{12}$

2 – (MAT/10_HL_Winter_2012_Q9) – *Vectors - Lines - Planes*

METHOD 1

(a) $9t_A = 7 - 4t_B$ and
$3 - 6t_A = -6 + 7t_B$

solve simultaneously

$t_A = \frac{1}{3}, t_B = 1$

Note: Only need to see one time for the ***A1***.

therefore meet at $(3, 1)$

(b) boats do not collide because the two times $\left(t_A = \frac{1}{3}, t_B = 1\right)$ are different

METHOD 2

(a) path of A is a straight line: $y = -\frac{2}{3}x + 3$

Note: Award ***M1*** for an attempt at simultaneous equations.

path of B is a straight line: $y = -\frac{7}{4}x + \frac{25}{4}$

$-\frac{2}{3}x + 3 = -\frac{7}{4}x + \frac{25}{4}$ $(\Rightarrow x = 3)$

so the common point is $(3, 1)$

(b) for boat A, $9t = 3 \Rightarrow t = \frac{1}{3}$ and for boat B, $7 - 4t = 3 \Rightarrow t = 1$

times are different so boats do not collide

3 – (MAT/11_HL_Summer_2013_Q2) – *Vectors - Lines - Planes*

(a) $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$

Note: Award the above marks if the components are seen in the line below.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 0 & -2 & 2 \\ 1 & -3 & 1 \end{vmatrix} = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$$

(b) area $= \frac{1}{2}\left|\left(\overrightarrow{AB} \times \overrightarrow{AC}\right)\right|$

$$= \frac{1}{2}\sqrt{4^2 + 2^2 + 2^2} = \frac{1}{2}\sqrt{24}\left(= \sqrt{6}\right)$$

4 – (MAT/12_HL_Summer_2013_Q11) – *Vectors - Lines - Planes*

(a) (i) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 5\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$ (or in column vector form)

Note: Award ***A1*** if any one of the vectors, or its negative, representing the sides of the triangle is seen.

$$|\overrightarrow{AB}| = |5\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}| = \sqrt{30}$$
$$|\overrightarrow{BC}| = |-\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}| = \sqrt{11}$$
$$|\overrightarrow{CA}| = |-4\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}| = \sqrt{33}$$

Note: Award ***A1*** for two correct and ***A0*** for one correct.

(ii) **METHOD 1**

$$\cos \mathrm{BAC} = \frac{20+4+2}{\sqrt{30}\sqrt{33}}$$

Note: Award ***M1*** for an attempt at the use of the scalar product for two vectors representing the sides AB and AC, or their negatives, ***A1*** for the correct computation using their vectors.

$$= \frac{26}{\sqrt{990}} \quad \left(= \frac{26}{3\sqrt{110}} \right)$$

Note: Candidates who use the modulus need to justify it – the angle is not stated in the question to be acute.

METHOD 2

using the cosine rule

$$\cos \mathrm{BAC} = \frac{30+33-11}{2\sqrt{30}\sqrt{33}}$$
$$= \frac{26}{\sqrt{990}} \quad \left(= \frac{26}{3\sqrt{110}} \right)$$

(b) (i) $\overrightarrow{BC} \times \overrightarrow{CA} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & -3 & 1 \\ -4 & 4 & 1 \end{vmatrix}$

$$= ((-3)\times 1 - 1\times 4)\boldsymbol{i} + (1\times(-4) - (-1)\times 1)\boldsymbol{j} + ((-1)\times 4 - (-3)\times(-4))\boldsymbol{k}$$
$$= -7\boldsymbol{i} - 3\boldsymbol{j} - 16\boldsymbol{k}$$

(ii) the area of $\Delta \mathrm{ABC} = \frac{1}{2}\left|\overrightarrow{BC} \times \overrightarrow{CA}\right|$

$$\frac{1}{2}\sqrt{(-7)^2 + (-3)^2 + (-16)^2}$$
$$= \frac{1}{2}\sqrt{314}$$

(c) attempt at the use of "$(\boldsymbol{r}-\boldsymbol{a})\bullet\boldsymbol{n}=0$"

using $\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$, $\boldsymbol{a}=\overrightarrow{\mathrm{OA}}$ and $\boldsymbol{n}=-7\boldsymbol{i}-3\boldsymbol{j}-16\boldsymbol{k}$

$7x+3y+16z=47$

Note: Candidates who adopt a 2-parameter approach should be awarded, ***A1*** for correct 2-parameter equations for x, y and z; ***M1*** for a serious attempt at elimination of the parameters; ***A1*** for the final Cartesian equation.

(d) $\boldsymbol{r}=\overrightarrow{\mathrm{OA}}+t\overrightarrow{\mathrm{AB}}$ (or equivalent)

$\boldsymbol{r}=(-\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k})+t(5\boldsymbol{i}-\boldsymbol{j}-2\boldsymbol{k})$

Note: Award ***M1A0*** if "$\boldsymbol{r}=$" is missing.

Note: Accept forms of the equation starting with B or with the direction reversed.

(e) (i) $\overrightarrow{\mathrm{OD}}=(-\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k})+t(5\boldsymbol{i}-\boldsymbol{j}-2\boldsymbol{k})$

statement that $\overrightarrow{\mathrm{OD}}\bullet\overrightarrow{\mathrm{BC}}=0$

$$\begin{pmatrix}-1+5t\\2-t\\3-2t\end{pmatrix}\bullet\begin{pmatrix}-1\\-3\\1\end{pmatrix}=0$$

$-2-4t=0$ or $t=-\dfrac{1}{2}$

coordinates of D are $\left(-\dfrac{7}{2},\dfrac{5}{2},4\right)$

Note: Different forms of $\overrightarrow{\mathrm{OD}}$ give different values of t, but the same final answer.

(ii) $t<0 \Rightarrow \mathrm{D}$ is not between A and B

5 – (MAT/10_HL_Winter_2013_Q11) – *Vectors - Lines - Planes*

(a) $\overrightarrow{CA} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

$\overrightarrow{CB} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

Note: If $\overrightarrow{AC}$ and $\overrightarrow{BC}$ found correctly award ***(A1) (A0)***.

$\overrightarrow{CA} \times \overrightarrow{CB} = \begin{vmatrix} i & j & k \\ 1 & -2 & -1 \\ 2 & 0 & 1 \end{vmatrix}$

$\begin{pmatrix} -2 \\ -3 \\ 4 \end{pmatrix}$

(b) **METHOD 1**

$\frac{1}{2}\left|\overrightarrow{CA} \times \overrightarrow{CB}\right|$

$= \frac{1}{2}\sqrt{(-2)^2 + (-3)^2 + 4^2}$

$= \frac{\sqrt{29}}{2}$

METHOD 2

attempt to apply $\frac{1}{2}|CA||CB|\sin C$

$CA.CB = \sqrt{5}.\sqrt{6}\cos C \Rightarrow \cos C = \frac{1}{\sqrt{30}} \Rightarrow \sin C = \frac{\sqrt{29}}{\sqrt{30}}$

$\text{area} = \frac{\sqrt{29}}{2}$

(c) **METHOD 1**

$$\mathbf{r}.\begin{pmatrix}-2\\-3\\4\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}.\begin{pmatrix}-2\\-3\\4\end{pmatrix}$$

$\Rightarrow -2x-3y+4z=-2$

$\Rightarrow 2x+3y-4z=2$

METHOD 2

$-2x-3y+4z=\mathrm{d}$

substituting a point in the plane

$\mathrm{d}=-2$

$\Rightarrow -2x-3y+4z=-2$

$\Rightarrow 2x+3y-4z=2$

(d) **METHOD 1**

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -4 \\ 4 & -1 & -1 \end{vmatrix} = \begin{pmatrix} -7 \\ -14 \\ -14 \end{pmatrix}$$

$$\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$z = 0 \Rightarrow y = 0,\ x = 1$

$$L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Note: Do not award the final *A1* if $\mathbf{r} =$ is not seen.

METHOD 2

eliminate 1 of the variables, *eg* x

$-7y + 7z = 0$

introduce a parameter

$\Rightarrow z = \lambda,$

$y = \lambda,\ x = 1 + \frac{\lambda}{2}$

$$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \text{ or equivalent}$$

Note: Do not award the final *A1* if $\mathbf{r} =$ is not seen.

METHOD 3

$z = t$

write x and y in terms of $t \Rightarrow 4x - y = 4 + t,\ 2x + 3y = 2 + 4t$ or equivalent

attempt to eliminate x or y

x, y, z expressed in parameters

$\Rightarrow z = t,$

$y = t,\ x = 1 + \frac{t}{2}$

$$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \text{ or equivalent}$$

(e) **METHOD 1**

direction of the line is perpendicular to the normal of the plane

$$\begin{pmatrix}16\\ \alpha\\ -3\end{pmatrix}.\begin{pmatrix}1\\ 2\\ 2\end{pmatrix}=0$$

$16+2\alpha-6=0 \Rightarrow \alpha=-5$

METHOD 2

solving line/plane simultaneously

$16(1+\lambda)+2\alpha\lambda-6\lambda=\beta$

$16+(10+2\alpha)\lambda=\beta$

$\Rightarrow \alpha=-5$

METHOD 3

$$\begin{vmatrix}2 & 3 & -4\\ 4 & -1 & -1\\ 16 & \alpha & -3\end{vmatrix}=0$$

$2(3+\alpha)-3(-12+16)-4(4\alpha+16)=0$

$\Rightarrow \alpha=-5$

METHOD 4

attempt to use row reduction on augmented matrix

to obtain $\left(\begin{array}{ccc|c}2 & 3 & -4 & 2\\ 0 & -1 & 1 & 0\\ 0 & 0 & \alpha+5 & \beta-16\end{array}\right)$

$\Rightarrow \alpha=-5$

(f) $\alpha=-5$

$\beta\neq 16$

6 – (MAT/12_HL_Summer_2014_Q6) – *Vectors - Lines - Planes*

(a) $\overrightarrow{PR} = \boldsymbol{a} + \boldsymbol{b}$

$\overrightarrow{QS} = \boldsymbol{b} - \boldsymbol{a}$

(b) $\overrightarrow{PR} \cdot \overrightarrow{QS} = (\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{b} - \boldsymbol{a})$

$= |\boldsymbol{b}|^2 - |\boldsymbol{a}|^2$

for a rhombus $|\boldsymbol{a}| = |\boldsymbol{b}|$

hence $|\boldsymbol{b}|^2 - |\boldsymbol{a}|^2 = 0$

Note: Do not award the final *A1* unless *R1* is awarded.

hence the diagonals intersect at right angles

7 – (MAT/11_HL_Summer_2014_Q12) – *Vectors - Lines - Planes*

(a) $|\overrightarrow{OA}| = |\overrightarrow{CB}| = |\overrightarrow{OC}| = |\overrightarrow{AB}| = 6$ (therefore a rhombus)

Note: Award ***A1*** for two correct lengths, ***A2*** for all four.

Note: Award ***A1A0*** for $\overrightarrow{OA} = \overrightarrow{CB} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$ or $\overrightarrow{OC} = \overrightarrow{AB} = \begin{pmatrix} 0 \\ -\sqrt{24} \\ \sqrt{12} \end{pmatrix}$ if no magnitudes are shown.

$$\overrightarrow{OA} \text{ g} \overrightarrow{OC} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} \text{g} \begin{pmatrix} 0 \\ -\sqrt{24} \\ \sqrt{12} \end{pmatrix} = 0 \text{ (therefore a square)}$$

Note: Other arguments are possible with a minimum of three conditions.

(b) $M\left(3, -\frac{\sqrt{24}}{2}, \frac{\sqrt{12}}{2}\right)\left(= \left(3, -\sqrt{6}, \sqrt{3}\right)\right)$

(c) **METHOD 1**

$$\overrightarrow{OA} \times \overrightarrow{OC} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -\sqrt{24} \\ \sqrt{12} \end{pmatrix} = \begin{pmatrix} 0 \\ -6\sqrt{12} \\ -6\sqrt{24} \end{pmatrix} \left(= \begin{pmatrix} 0 \\ -12\sqrt{3} \\ -12\sqrt{6} \end{pmatrix} \right)$$

Note: Candidates may use other pairs of vectors.

equation of plane is $-6\sqrt{12}y - 6\sqrt{24}z = d$
any valid method showing that $d = 0$
$\Pi : y + \sqrt{2}z = 0$

METHOD 2

equation of plane is $ax + by + cz = d$
substituting O to find $d = 0$
substituting two points (A, B, C or M)
eg
$6a = 0$, $-\sqrt{24}b + \sqrt{12}c = 0$
$\Pi : y + \sqrt{2}z = 0$

(d) $\boldsymbol{r} = \begin{pmatrix} 3 \\ -\sqrt{6} \\ \sqrt{3} \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix}$

Note: Award *A1* for $\boldsymbol{r} =$, *A1A1* for two correct vectors.

(e) Using $y = 0$ to find λ
Substitute their λ into their equation from part (d)
D has coordinates $\left(3, 0, 3\sqrt{3}\right)$

(f) λ for point E is the negative of the λ for point D

Note: Other possible methods may be seen.

E has coordinates $\left(3, -2\sqrt{6}, -\sqrt{3}\right)$

Note: Award *A1* for each of the *y* and *z* coordinates.

(g) (i) $\overrightarrow{DA} \text{ g } \overrightarrow{DO} = \begin{pmatrix} 3 \\ 0 \\ -3\sqrt{3} \end{pmatrix} \text{g} \begin{pmatrix} -3 \\ 0 \\ -3\sqrt{3} \end{pmatrix} = 18$

$\cos O\hat{D}A = \dfrac{18}{\sqrt{36}\sqrt{36}} = \dfrac{1}{2}$

hence $O\hat{D}A = 60^{\circ}$

Note: Accept method showing OAD is equilateral.

(ii) OABCDE is a regular octahedron (accept equivalent description)

8 – (MAT/12_HL_Summer_2014_Q12) – *Vectors - Lines - Planes*

(a) direction vector $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$ or $\overrightarrow{BA} = \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix}$

$\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$ or $\boldsymbol{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$ or equivalent

Note: Do not award final ***A1*** unless '$r = \mathrm{K}$' (or equivalent) seen.
Allow FT on direction vector for final ***A1***.

(b) both lines expressed in parametric form:

L_1:

$x = 1 + t$

$y = 3t$

$z = 4 - 5t$

L_2:

$x = 1 + 3s$

$y = -2 + s$

$z = -2s + 1$

Notes: Award ***M1*** for an attempt to convert L_2 from Cartesian to parametric form.
Award ***A1*** for correct parametric equations for L_1 and L_2.
Allow ***M1A1*** at this stage if same parameter is used in both lines.

attempt to solve simultaneously for x and y:

$1 + t = 1 + 3s$

$3t = -2 + s$

$t = -\frac{3}{4}$, $s = -\frac{1}{4}$

substituting both values back into z values respectively gives $z = \frac{31}{4}$

and $z = \frac{3}{2}$ so a contradiction

therefore L_1 and L_2 are skew lines

(c) finding the cross product:

$$\begin{pmatrix}1\\3\\-5\end{pmatrix}\times\begin{pmatrix}3\\1\\-2\end{pmatrix}$$ ***(M1)***

$$=-\boldsymbol{i}-13\boldsymbol{j}-8\boldsymbol{k}$$ ***A1***

Note: Accept $\boldsymbol{i}+13\boldsymbol{j}+8\boldsymbol{k}$

$-1(0)-13(1)-8(-2)=3$ ***(M1)***

$\Rightarrow -x-13y-8z=3$ or equivalent ***A1***

(d) (i) $(\cos\theta=)\dfrac{\begin{pmatrix}k\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}}{\sqrt{k^2+1+1}\times\sqrt{1+1}}$ ***M1***

Note: Award ***M1*** for an attempt to use angle between two vectors formula.

$$\frac{\sqrt{3}}{2}=\frac{k+1}{\sqrt{2(k^2+2)}}$$ ***A1***

obtaining the quadratic equation

$4(k+1)^2=6(k^2+2)$ ***M1***

$k^2-4k+4=0$

$(k-2)^2=0$

$k=2$ ***A1***

(ii) $\boldsymbol{r}=\begin{pmatrix}3\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-1\end{pmatrix}$

substituting into the equation of the plane Π_2:

$3+2\lambda+\lambda=12$

$\lambda=3$

point P has the coordinates:

$(9, 3, -2)$

9 – (MAT/10_HL_Winter_2014_Q3) – *Vectors - Lines - Planes*

METHOD 1

$$\left|\overrightarrow{OP}\right| = \sqrt{(1+s)^2+(3+2s)^2+(1-s)^2} \quad (=\sqrt{6s^2+12s+11})$$

Note: Award *A1* if the square of the distance is found.

EITHER

attempt to differentiate: $\frac{d}{ds}\left|\overrightarrow{OP}\right|^2 \left(= 12s + 12\right)$

attempting to solve $\frac{d}{ds}\left|\overrightarrow{OP}\right|^2 = 0$ for s

$s = -1$

OR

attempt to differentiate: $\frac{d}{ds}\left|\overrightarrow{OP}\right| \quad \left(= \frac{6s+6}{\sqrt{6s^2+12s+11}}\right)$

attempting to solve $\frac{d}{ds}\left|\overrightarrow{OP}\right| = 0$ for s

$s = -1$

OR

attempt at completing the square: $\left(\left|\overrightarrow{OP}\right|^2 = 6(s+1)^2 + 5\right)$

minimum value
occurs at $s = -1$

THEN

the minimum length of $\overrightarrow{OP}$ is $\sqrt{5}$

METHOD 2

the length of $\overrightarrow{OP}$ is a minimum when $\overrightarrow{OP}$ is perpendicular to $\begin{pmatrix}1\\2\\-1\end{pmatrix}$

$$\begin{pmatrix}1+s\\3+2s\\1-s\end{pmatrix}\bullet\begin{pmatrix}1\\2\\-1\end{pmatrix} = 0$$

attempting to solve $1+s+6+4s-1+s=0\ (6s+6=0)$ for s

$s = -1$

$\left|\overrightarrow{OP}\right| = \sqrt{5}$

10 – (MAT/10_HL_Winter_2014_Q12) – *Vectors - Lines - Planes*

(a) (i) $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AC}$

$= \frac{1}{2}(\boldsymbol{c} - \boldsymbol{a})$

(ii) $\overrightarrow{BM} = \overrightarrow{BA} + \overrightarrow{AM}$

$= \boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{a})$

$\overrightarrow{BM} = \frac{1}{2}\boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}\boldsymbol{c}$

(b) (i) $\overrightarrow{RA} = \frac{1}{3}\overrightarrow{BA}$

$= \frac{1}{3}(\boldsymbol{a} - \boldsymbol{b})$

(ii) $\overrightarrow{RT} = \frac{2}{3}\overrightarrow{RS}$

$= \frac{2}{3}\left(\overrightarrow{RA} + \overrightarrow{AS}\right)$

$= \frac{2}{3}\left(\frac{1}{3}(\boldsymbol{a} - \boldsymbol{b}) + \frac{2}{3}(\boldsymbol{c} - \boldsymbol{a})\right)$ or equivalent.

$= \frac{2}{9}(\boldsymbol{a} - \boldsymbol{b}) + \frac{4}{9}(\boldsymbol{c} - \boldsymbol{a})$

$\overrightarrow{RT} = -\frac{2}{9}\boldsymbol{a} - \frac{2}{9}\boldsymbol{b} + \frac{4}{9}\boldsymbol{c}$

(c) $\overrightarrow{BT} = \overrightarrow{BR} + \overrightarrow{RT}$

$= \frac{2}{3}\overrightarrow{BA} + \overrightarrow{RT}$

$= \frac{2}{3}\boldsymbol{a} - \frac{2}{3}\boldsymbol{b} - \frac{2}{9}\boldsymbol{a} - \frac{2}{9}\boldsymbol{b} + \frac{4}{9}\boldsymbol{c}$

$\overrightarrow{BT} = \frac{8}{9}\left(\frac{1}{2}\boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}\boldsymbol{c}\right)$

point B is common to $\overrightarrow{BT}$ and $\overrightarrow{BM}$ and $\overrightarrow{BT} = \frac{8}{9}\overrightarrow{BM}$

so T lies on [BM]

11 – (MAT/11_HL_Summer_2015_Q13) – *Vectors - Lines - Planes*

(a) $\overrightarrow{OP} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k} + \lambda(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$

$\overrightarrow{OQ} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} + \mu(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$\overrightarrow{PQ} = \boldsymbol{i} - \boldsymbol{j} - 4\boldsymbol{k} - \lambda(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) + \mu(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$
$= (1 - \lambda + \mu)\boldsymbol{i} + (-1 - \lambda - \mu)\boldsymbol{j} + (-4 - \lambda + 2\mu)\boldsymbol{k}$

(b) **METHOD 1**

use of scalar product
perpendicular to $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ gives
$(1 - \lambda + \mu) + (-1 - \lambda - \mu) + (-4 - \lambda + 2\mu) = 0$
$\Rightarrow -3\lambda + 2\mu = 4$
perpendicular to $\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ gives
$(1 - \lambda + \mu) - (-1 - \lambda - \mu) + 2(-4 - \lambda + 2\mu) = 0$
$\Rightarrow -2\lambda + 6\mu = 6$
solving simultaneous equations gives $\lambda = -\frac{6}{7},\ \mu = \frac{5}{7}$

METHOD 2

$\boldsymbol{v} \times \boldsymbol{w} = 3\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$

$\overrightarrow{PQ} = a\left(3\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}\right)$

$1 - \lambda + \mu = 3a$
$-1 - \lambda - \mu = -a$
$-4 - \lambda + 2\mu = -2a$

solving simultaneous equations gives $\lambda = -\frac{6}{7},\ \mu = \frac{5}{7}$

(c) $\overrightarrow{PQ} = \frac{18}{7}\boldsymbol{i} - \frac{6}{7}\boldsymbol{j} - \frac{12}{7}\boldsymbol{k}$

shortest distance $= \left|\overrightarrow{PQ}\right| = \frac{6}{7}\sqrt{3^2 + (-1)^2 + (-2)^2} = \frac{6}{7}\sqrt{14}$

(d) **METHOD 1**

vector perpendicular to Π is given by vector product of $\boldsymbol{v}$ and $\boldsymbol{w}$
$\boldsymbol{v} \times \boldsymbol{w} = 3\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$
so equation of Π is $3x - y - 2z + d = 0$
through $(1, 2, 3) \Rightarrow d = 5$
so equation is $3x - y - 2z + 5 = 0$

METHOD 2

from part (b) $\overrightarrow{PQ} = \frac{18}{7}\boldsymbol{i} - \frac{6}{7}\boldsymbol{j} - \frac{12}{7}\boldsymbol{k}$ is a vector perpendicular to Π

so equation of Π is $\frac{18}{7}x - \frac{6}{7}y - \frac{12}{7}z + c = 0$

through $(1, 2, 3) \Rightarrow c = \frac{30}{7}$

so equation is $\frac{18}{7}x - \frac{6}{7}y - \frac{12}{7}z + \frac{30}{7} = 0 \quad (3x - y - 2z + 5 = 0)$

Note: Allow other methods *ie* via vector parametric equation.

(e) $\overrightarrow{OT} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} + \eta(3\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})$

$T = (2 + 3\eta, 1 - \eta, -1 - 2\eta)$ lies on Π implies

$3(2 + 3\eta) - (1 - \eta) - 2(-1 - 2\eta) + 5 = 0$

$\Rightarrow 12 + 14\eta = 0 \Rightarrow \eta = -\frac{6}{7}$

Note: If no marks awarded in (d) but correct vector product calculated in (e) award ***M1A1*** in (d).

(f) $\left|\overrightarrow{BT}\right| = \frac{6}{7}\sqrt{3^2 + (-1)^2 + (-2)^2} = \frac{6}{7}\sqrt{14}$

(g) they agree

Note: FT is inappropriate here.

$\overrightarrow{BT}$ is perpendicular to both Π and l_2

so its length is the shortest distance between Π and l_2 which is the shortest distance between l_1 and l_2

12 – (MAT/10_HL_Winter_2015_Q13) – *Vectors - Lines - Planes*

(a) $\overrightarrow{BR} = \overrightarrow{BA} + \overrightarrow{AR} \ \left(= \overrightarrow{BA} + \frac{1}{2}\overrightarrow{AC}\right)$ ***(M1)***

$= (\boldsymbol{a} - \boldsymbol{b}) + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{a})$

$= \frac{1}{2}\boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}\boldsymbol{c}$ ***A1***

(b) (i) $\mathrm{r}_{BR} = \boldsymbol{b} + \lambda\left(\frac{1}{2}\boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}\boldsymbol{c}\right)\left(= \frac{\lambda}{2}\boldsymbol{a} + (1-\lambda)\,\boldsymbol{b} + \frac{\lambda}{2}\boldsymbol{c}\right)$ ***A1A1***

Note: Award ***A1A0*** if the r = is omitted in an otherwise correct expression/equation.

(ii) $\overrightarrow{AQ} = -\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} + \frac{1}{2}\boldsymbol{c}$ ***(A1)***

$\mathrm{r}_{AQ} = \boldsymbol{a} + \mu\left(-\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} + \frac{1}{2}\boldsymbol{c}\right)\left(= (1-\mu)\,\boldsymbol{a} + \frac{\mu}{2}\boldsymbol{b} + \frac{\mu}{2}\boldsymbol{c}\right)$ ***A1***

(iii) when $\overrightarrow{AQ}$ and $\overrightarrow{BP}$ intersect we will have $\mathrm{r}_{BR} = \mathrm{r}_{AQ}$ ***(M1)***

$\frac{\lambda}{2}\boldsymbol{a} + (1-\lambda)\,\boldsymbol{b} + \frac{\lambda}{2}\boldsymbol{c} = (1-\mu)\,\boldsymbol{a} + \frac{\mu}{2}\boldsymbol{b} + \frac{\mu}{2}\boldsymbol{c}$

attempt to equate the coefficients of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ ***M1***

$$\left.\begin{aligned} \frac{\lambda}{2} &= 1-\mu \\ 1-\lambda &= \frac{\mu}{2} \\ \frac{\lambda}{2} &= \frac{\mu}{2} \end{aligned}\right\}$$ ***(A1)***

$\lambda = \frac{2}{3}$ or $\mu = \frac{2}{3}$ ***A1***

substituting parameters back into one of the equations ***M1***

$\overrightarrow{OG} = \frac{1}{2}\cdot\frac{2}{3}\boldsymbol{a} + \left(1 - \frac{2}{3}\right)\boldsymbol{b} + \frac{1}{2}\cdot\frac{2}{3}\boldsymbol{c} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$ ***AG***

(c) $\overrightarrow{\mathrm{CP}} = \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} - \boldsymbol{c}$

so we have that $\mathrm{r}_{\mathrm{CP}} = \boldsymbol{c} + \beta\left(\frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} - \boldsymbol{c}\right)$ and when $\beta = \frac{2}{3}$ the line passes through

the point G (*ie*, with position vector $\frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$)

hence $[\mathrm{AQ}]$, $[\mathrm{BR}]$ and $[\mathrm{CP}]$ all intersect in G

(d) $\overrightarrow{OG} = \frac{1}{3}\left(\begin{pmatrix}1\\3\\1\end{pmatrix} + \begin{pmatrix}3\\7\\-5\end{pmatrix} + \begin{pmatrix}2\\2\\1\end{pmatrix}\right) = \begin{pmatrix}2\\4\\-1\end{pmatrix}$ ***A1***

Note: This independent mark for the vector may be awarded wherever the vector is calculated.

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix}2\\4\\-6\end{pmatrix} \times \begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}-6\\-6\\-6\end{pmatrix}$ ***M1A1***

$\overrightarrow{GX} = \alpha\begin{pmatrix}1\\1\\1\end{pmatrix}$ ***(M1)***

volume of Tetrahedron given by $\frac{1}{3} \times$ Area $ABC \times GX$

$= \frac{1}{3}\left(\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|\right) \times GX = 12$ ***(M1)(A1)***

Note: Accept alternative methods, for example the use of a scalar triple product.

$= \frac{1}{6}\sqrt{(-6)^2 + (-6)^2 + (-6)^2} \times \sqrt{\alpha^2 + \alpha^2 + \alpha^2} = 12$ ***(A1)***

$= \frac{1}{6} 6\sqrt{3}\left|\alpha\right|\sqrt{3} = 12$

$\Rightarrow \left|\alpha\right| = 4$ ***A1***

Note: Condone absence of absolute value.

this gives us the position of X as $\begin{pmatrix}2\\4\\-1\end{pmatrix} \pm \begin{pmatrix}4\\4\\4\end{pmatrix}$

$X(6, 8, 3)$ or $(-2, 0, -5)$ ***A1***

13 – (MAT/11_HL_Summer_2016_Q8) – *Vectors - Lines - Planes*

$\boldsymbol{c} \bullet (\boldsymbol{b} - \boldsymbol{a}) = 0$

Note: Allow $c \bullet \overrightarrow{\mathrm{AB}} = 0$ or similar for ***M1***.

$\boldsymbol{c} \bullet \boldsymbol{b} = \boldsymbol{c} \bullet \boldsymbol{a}$

$\boldsymbol{b} \bullet (\boldsymbol{c} - \boldsymbol{a}) = 0$

$\boldsymbol{b} \bullet \boldsymbol{c} = \boldsymbol{b} \bullet \boldsymbol{a}$

$\boldsymbol{c} \bullet \boldsymbol{a} = \boldsymbol{b} \bullet \boldsymbol{a}$

$(\boldsymbol{c} - \boldsymbol{b}) \bullet \boldsymbol{a} = 0$

hence $\boldsymbol{a}$ is perpendicular to $\overrightarrow{\mathrm{BC}}$

14 – (MAT/12_HL_Summer_2016_Q10) – *Vectors - Lines - Planes*

(a) **EITHER**

$\boldsymbol{n} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ and $\boldsymbol{d} = \begin{pmatrix} p \\ 2 \\ 1 \end{pmatrix}$

and $\boldsymbol{n} \neq k\boldsymbol{d}$

OR

$\boldsymbol{n} \times \boldsymbol{d} = \begin{pmatrix} -5 \\ 3p - 1 \\ 2 - p \end{pmatrix}$

the vector product is non-zero for $p \in \mathbb{R}$

THEN

L is not perpendicular to Π

(b) **METHOD 1**

$(2 + p\lambda) + (q + 2\lambda) + 3(1 + \lambda) = 9$

$(q + 5) + (p + 5)\lambda = 9$

$p = -5$ and $q = 4$

METHOD 2

direction vector of line is perpendicular to plane, so

$\begin{pmatrix} p \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = 0$

$p = -5$

$(2, q, 1)$ is common to both L and Π

either $\begin{pmatrix} 2 \\ q \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = 9$ or by substituting into $x + y + 3z = 9$

$q = 4$

(c) (i) **METHOD 1**

α is the acute angle between $\boldsymbol{n}$ and L

if $\sin\theta = \frac{1}{\sqrt{11}}$ then $\cos\alpha = \frac{1}{\sqrt{11}}$

attempting to use $\cos\alpha = \frac{\boldsymbol{n}\cdot\boldsymbol{d}}{|\boldsymbol{n}||\boldsymbol{d}|}$ or $\sin\theta = \frac{\boldsymbol{n}\cdot\boldsymbol{d}}{|\boldsymbol{n}||\boldsymbol{d}|}$

$$\frac{p+5}{\sqrt{11}\times\sqrt{p^2+5}} = \frac{1}{\sqrt{11}}$$

$(p+5)^2 = p^2 + 5$

$10p = -20$ (or equivalent)

$p = -2$

METHOD 2

α is the angle between $\boldsymbol{n}$ and L

if $\sin\theta = \frac{1}{\sqrt{11}}$ then $\sin\alpha = \frac{\sqrt{10}}{\sqrt{11}}$

attempting to use $\sin\alpha = \frac{|\boldsymbol{n}\times\boldsymbol{d}|}{|\boldsymbol{n}||\boldsymbol{d}|}$

$$\frac{\sqrt{(-5)^2+(3p-1)^2+(2-p)^2}}{\sqrt{11}\times\sqrt{p^2+5}} = \frac{\sqrt{10}}{\sqrt{11}}$$

$p^2 - p + 3 = p^2 + 5$

$-p + 3 = 5$ (or equivalent)

$p = -2$

(ii) $p = -2$ and $z = -1 \Rightarrow \frac{x-2}{-2} = \frac{y-q}{2} = -2$

$x = 6$ and $y = q - 4$

this satisfies Π so $6 + q - 4 - 3 = 9$

$q = 10$

15 – (MAT/11_HL_Summer_2016_Q11) – *Vectors - Lines - Planes*

Note: Throughout the question condone vectors written horizontally.

(a) angle between planes is equal to the angles between the normal to the planes ***(M1)***

$$\begin{pmatrix}4\\1\\1\end{pmatrix}\bullet\begin{pmatrix}4\\3\\-1\end{pmatrix}=18$$ ***(A1)***

let θ be the angle between the normal to the planes

$$\cos\theta=\frac{18}{\sqrt{18}\sqrt{26}}=\sqrt{\frac{18}{26}}\left(\text{or equivalent, for example }\sqrt{\frac{324}{468}}\text{ or }\sqrt{\frac{9}{13}}\right)$$ ***M1A1***

(b) (i) **METHOD 1**

$$\begin{pmatrix}4\\1\\1\end{pmatrix}\times\begin{pmatrix}4\\3\\-1\end{pmatrix}=\begin{pmatrix}-4\\8\\8\end{pmatrix}$$ ***M1A1***

which is a multiple of $\begin{pmatrix}-1\\2\\2\end{pmatrix}$ ***R1AG***

Note: Allow any equivalent wording or $\begin{pmatrix}-4\\8\\8\end{pmatrix}=4\begin{pmatrix}-1\\2\\2\end{pmatrix}$, do not allow $\begin{pmatrix}-4\\8\\8\end{pmatrix}=\begin{pmatrix}-1\\2\\2\end{pmatrix}$.

METHOD 2

let $z=t$ (or equivalent)

solve simultaneously to get ***M1***

$y=t-4,\ x=3-0.5t$ ***A1***

hence direction vector is $\begin{pmatrix}-0.5\\1\\1\end{pmatrix}$

which is a multiple of $\begin{pmatrix}-1\\2\\2\end{pmatrix}$ ***R1AG***

METHOD 3

$$\begin{pmatrix}4\\1\\1\end{pmatrix}\bullet\begin{pmatrix}-1\\2\\2\end{pmatrix}=-4+2+2=0$$

$$\begin{pmatrix}4\\3\\-1\end{pmatrix}\bullet\begin{pmatrix}-1\\2\\2\end{pmatrix}=-4+6-2=0$$

Note: If only one scalar product is found award ***M0A0A0***.

(ii) $\Pi_1: 4+0+4=8$ and $\Pi_2: 4+0-4=0$

(iii) $\boldsymbol{r}=\begin{pmatrix}1\\0\\4\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\2\end{pmatrix}$

Note: ***A1*** for "$\boldsymbol{r}=$" and a correct point on the line, ***A1*** for a parameter and a correct direction vector.

(c) $\overrightarrow{AB}=\begin{pmatrix}a\\b\\1\end{pmatrix}-\begin{pmatrix}1\\0\\4\end{pmatrix}=\begin{pmatrix}a-1\\b\\-3\end{pmatrix}$

$$\begin{pmatrix}a-1\\b\\-3\end{pmatrix}\bullet\begin{pmatrix}-1\\2\\2\end{pmatrix}=0$$

Note: Award ***M0*** for $\begin{pmatrix}a\\b\\1\end{pmatrix}\bullet\begin{pmatrix}-1\\2\\2\end{pmatrix}=0$.

$-a+1+2b-6=0 \Rightarrow a-2b=-5$

lies on Π_1 so $4a+b+1=8 \Rightarrow 4a+b=7$

$a=1,\ b=3$

(d) $AB=\sqrt{0^2+3^2+(-3)^2}=3\sqrt{2}$

(e) **METHOD 1**

$$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{AP}\right| = 3\sqrt{2}$$

$$\overrightarrow{AP} = t\begin{pmatrix}-1\\2\\2\end{pmatrix}$$

$$|3t| = 3\sqrt{2} \Rightarrow t = \pm\sqrt{2}$$

$P\left(1-\sqrt{2},\ 2\sqrt{2},\ 4+2\sqrt{2}\right)$ and $\left(1+\sqrt{2},\ -2\sqrt{2},\ 4-2\sqrt{2}\right)$

METHOD 2

let P have coordinates $(1-\lambda,\ 2\lambda,\ 4+2\lambda)$

$$\overrightarrow{BA} = \begin{pmatrix}0\\-3\\3\end{pmatrix},\quad \overrightarrow{BP} = \begin{pmatrix}-\lambda\\2\lambda-3\\3+2\lambda\end{pmatrix}$$

$$\cos 45^0 = \frac{\overrightarrow{BA}\bullet\overrightarrow{BP}}{|BA||BP|}$$

Note: Award ***M1*** even if AB rather than BA is used in the scalar product.

$$\overrightarrow{BA}\bullet\overrightarrow{BP} = 18$$

$$\frac{1}{\sqrt{2}} = \frac{18}{\sqrt{18}\sqrt{9\lambda^2+18}}$$

$$\lambda = \pm\sqrt{2}$$

$P\left(1-\sqrt{2},\ 2\sqrt{2},\ 4+2\sqrt{2}\right)$ and $\left(1+\sqrt{2},\ -2\sqrt{2},\ 4-2\sqrt{2}\right)$

16 – (MAT/10_HL_Winter_2016_Q1) – *Vectors - Lines - Planes*

METHOD 1

for eliminating one variable from two equations ***(M1)***

eg, $\begin{cases} (x+y+z=3) \\ 2x+2z=8 \\ 2x+3z=11 \end{cases}$ ***A1A1***

for finding correctly one coordinate

eg, $\Rightarrow \begin{cases} (x+y+z=3) \\ (2x+2z=8) \\ z=3 \end{cases}$ ***A1***

for finding correctly the other two coordinates ***A1***

$\Rightarrow \begin{cases} x=1 \\ y=-1 \\ z=3 \end{cases}$

the intersection point has coordinates $(1, -1, 3)$

METHOD 2

for eliminating two variables from two equations or using row reduction ***(M1)***

eg, $\begin{cases} (x+y+z=3) \\ -2y=2 \\ z=3 \end{cases}$ **or** $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$ ***A1A1***

for finding correctly the other coordinates ***A1A1***

$\Rightarrow \begin{cases} x=1 \\ y=-1 \\ (z=3) \end{cases}$ **or** $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right)$

the intersection point has coordinates $(1, -1, 3)$

METHOD 3

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -2$$ *(A1)*

attempt to use Cramer's rule *M1*

$$x = \frac{\begin{vmatrix} 3 & 1 & 1 \\ 5 & -1 & 1 \\ 6 & 1 & 2 \end{vmatrix}}{-2} = \frac{-2}{-2} = 1$$ *A1*

$$y = \frac{\begin{vmatrix} 1 & 3 & 1 \\ 1 & 5 & 1 \\ 1 & 6 & 2 \end{vmatrix}}{-2} = \frac{2}{-2} = -1$$ *A1*

$$z = \frac{\begin{vmatrix} 1 & 1 & 3 \\ 1 & -1 & 5 \\ 1 & 1 & 6 \end{vmatrix}}{-2} = \frac{-6}{-2} = 3$$ *A1*

Note: Award ***M1*** only if candidate attempts to determine at least one of the variables using this method.

[5 marks]

17 – (MAT/10_HL_Winter_2016_Q4) – *Vectors - Lines - Planes*

(a) $\boldsymbol{a} \times \boldsymbol{b} = -12\boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k}$ *(M1)A1*

[2 marks]

(b) **METHOD 1**

$-12x - 2y - 3z = d$ *M1*

$-12 \times 1 - 2 \times 0 - 3(-1) = d$ *(M1)*

$\Rightarrow d = -9$ *A1*

$-12x - 2y - 3z = -9$ (or $12x + 2y + 3z = 9$)

METHOD 2

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -12 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -12 \\ -2 \\ -3 \end{pmatrix}$$ *M1A1*

$-12x - 2y - 3z = -9$ (or $12x + 2y + 3z = 9$) *A1*

[3 marks]

18 – (MAT/10_HL_Winter_2016_Q8) – *Vectors - Lines - Planes*

(a) **METHOD 1**

$$l_1 : \boldsymbol{r} = \begin{pmatrix} -3 \\ -2 \\ a \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \Rightarrow \begin{cases} x = -3 + \beta \\ y = -2 + 4\beta \\ z = a + 2\beta \end{cases}$$ ***M1***

$$\frac{6-(-3+\beta)}{3} = \frac{(-2+4\beta)-2}{4} \Rightarrow 4 = \frac{4\beta}{3} \Rightarrow \beta = 3$$ ***M1A1***

$$\frac{6-(-3+\beta)}{3} = 1-(a+2\beta) \Rightarrow 2 = -5-a \Rightarrow a = -7$$ ***A1***

METHOD 2

$$\begin{cases} -3 + \beta = 6 - 3\lambda \\ -2 + 4\beta = 4\lambda + 2 \\ a + 2\beta = 1 - \lambda \end{cases}$$ ***M1***

attempt to solve ***M1***

$\lambda = 2,\ \beta = 3$ ***A1***

$a = 1 - \lambda - 2\beta = -7$ ***A1***

[4 marks]

(b) $$\overrightarrow{OP} = \begin{pmatrix} -3 \\ -2 \\ -7 \end{pmatrix} + 3 \cdot \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$$ ***(M1)***

$$= \begin{pmatrix} 0 \\ 10 \\ -1 \end{pmatrix}$$ ***A1***

$\therefore \mathrm{P}(0, 10, -1)$

[2 marks]

19 – (MAT/11_HL_Summer_2017_Q5) – *Vectors - Lines - Planes*

(a) $\overrightarrow{AB} \times \overrightarrow{AD} = -\boldsymbol{i} + 10\boldsymbol{j} - 7\boldsymbol{k}$ ***M1A1***

area $= \left|\overrightarrow{AB} \times \overrightarrow{AD}\right| = \sqrt{1^2 + 10^2 + 7^2}$

$= 5\sqrt{6}\left(\sqrt{150}\right)$ ***A1***

[3 marks]

(b) **METHOD 1**

$\overrightarrow{AB} \cdot \overrightarrow{AD} = -4 - 2 - 6$ ***M1A1***

$= -12$

considering the sign of the answer

$\overrightarrow{AB} \cdot \overrightarrow{AD} < 0$, therefore angle $D\hat{A}B$ is obtuse ***M1***

(as it is a parallelogram), $A\hat{B}C$ is acute ***A1***

[4 marks]

METHOD 2

$\overrightarrow{BA} \cdot \overrightarrow{BC} = +4 + 2 + 6$ ***M1A1***

$= 12$

considering the sign of the answer ***M1***

$\overrightarrow{BA} \cdot \overrightarrow{BC} > 0 \Rightarrow A\hat{B}C$ is acute ***A1***

[4 marks]

20 – (MAT/10_HL_Winter_2017_Q2) – *Vectors - Lines - Planes*

(a) $\overrightarrow{AB} = \begin{pmatrix} 6 \\ -8 \\ 17 \end{pmatrix}$ ***(A1)***

$\boldsymbol{r} = \begin{pmatrix} 0 \\ 3 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -8 \\ 17 \end{pmatrix}$ or $\boldsymbol{r} = \begin{pmatrix} 6 \\ -5 \\ 11 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -8 \\ 17 \end{pmatrix}$ ***M1A1***

Note: Award ***M1A0*** if $\boldsymbol{r} =$ is not seen (or equivalent).

[3 marks]

(b) substitute line L in $\Pi : 4(6\lambda) - 3(3 - 8\lambda) + 2(-6 + 17\lambda) = 20$ ***M1***

$82\lambda = 41$

$\lambda = \frac{1}{2}$ ***(A1)***

$\boldsymbol{r} = \begin{pmatrix} 0 \\ 3 \\ -6 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 6 \\ -8 \\ 17 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ \frac{5}{2} \end{pmatrix}$

so coordinate is $\left(3, -1, \frac{5}{2}\right)$ ***A1***

Note: Accept coordinate expressed as position vector $\begin{pmatrix} 3 \\ -1 \\ \frac{5}{2} \end{pmatrix}$.

[3 marks]

21 – (MAT/10_HL_Winter_2017_Q9) – *Vectors - Lines - Planes*

(a) (i) $\overrightarrow{OF} = \frac{1}{7}\boldsymbol{b}$ ***A1***

(ii) $\overrightarrow{AF} = \overrightarrow{OF} - \overrightarrow{OA}$ ***(M1)***

$= \frac{1}{7}\boldsymbol{b} - \boldsymbol{a}$ ***A1***

[3 marks]

(b) (i) $\overrightarrow{OD} = \boldsymbol{a} + \lambda\left(\frac{1}{7}\boldsymbol{b} - \boldsymbol{a}\right)\left(= (1-\lambda)\boldsymbol{a} + \frac{\lambda}{7}\boldsymbol{b}\right)$ ***M1A1***

(ii) $\overrightarrow{OD} = \frac{1}{2}\boldsymbol{a} + \mu\left(-\frac{1}{2}\boldsymbol{a} + \boldsymbol{b}\right)\left(= \left(\frac{1}{2} - \frac{\mu}{2}\right)\boldsymbol{a} + \mu\boldsymbol{b}\right)$ ***M1A1***

[4 marks]

(c) equating coefficients: ***M1***

$\frac{\lambda}{7} = \mu,\ 1 - \lambda = \frac{1-\mu}{2}$ ***A1***

solving simultaneously: ***M1***

$\lambda = \frac{7}{13},\ \mu = \frac{1}{13}$ ***A1AG***

[4 marks]

(d) $\overrightarrow{CD} = \frac{1}{13}\overrightarrow{CB}$

$= \frac{1}{13}\left(\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}\right)\left(= -\frac{1}{26}\boldsymbol{a} + \frac{1}{13}\boldsymbol{b}\right)$ ***M1A1***

[2 marks]

(e) **METHOD 1**

area $\Delta ACD = \frac{1}{2} CD \times AC \times \sin A\hat{C}B$ ***(M1)***

area $\Delta ACB = \frac{1}{2} CB \times AC \times \sin A\hat{C}B$ ***(M1)***

ratio $\frac{\text{area } \Delta ACD}{\text{area } \Delta ACB} = \frac{CD}{CB} = \frac{1}{13}$ ***A1***

$k = \frac{\text{area } \Delta OAB}{\text{area } \Delta CAD} = \frac{13}{\text{area} \Delta CAB} \times \text{area} \Delta OAB$ ***(M1)***

$= 13 \times 2 = 26$ ***A1***

METHOD 2

area $\Delta OAB = \frac{1}{2} |\boldsymbol{a} \times \boldsymbol{b}|$ ***A1***

area $\Delta CAD = \frac{1}{2} \left| \overrightarrow{CA} \times \overrightarrow{CD} \right|$ or $\frac{1}{2} \left| \overrightarrow{CA} \times \overrightarrow{AD} \right|$ ***M1***

$= \frac{1}{2} \left| \frac{1}{2} \boldsymbol{a} \times \left(-\frac{1}{26} \boldsymbol{a} + \frac{1}{13} \boldsymbol{b} \right) \right|$

$= \frac{1}{2} \left| \frac{1}{2} \boldsymbol{a} \times \left(-\frac{1}{26} \boldsymbol{a} \right) + \frac{1}{2} \boldsymbol{a} \times \frac{1}{13} \boldsymbol{b} \right|$ ***(M1)***

$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{13} |\boldsymbol{a} \times \boldsymbol{b}| \left(= \frac{1}{52} |\boldsymbol{a} \times \boldsymbol{b}| \right)$ ***A1***

area $\Delta OAB = k(\text{area } \Delta CAD)$

$\frac{1}{2} |\boldsymbol{a} \times \boldsymbol{b}| = k \frac{1}{52} |\boldsymbol{a} \times \boldsymbol{b}|$

$k = 26$ ***A1***

[5 marks]

22 – (MAT/12_HL_Summer_2018_Q1) – *Vectors - Lines - Planes*

$\cos\theta = \frac{(3\boldsymbol{i} - 4\boldsymbol{j} - 5\boldsymbol{k}) \cdot (5\boldsymbol{i} - 4\boldsymbol{j} + 3\boldsymbol{k})}{|3\boldsymbol{i} - 4\boldsymbol{j} - 5\boldsymbol{k}||5\boldsymbol{i} - 4\boldsymbol{j} + 3\boldsymbol{k}|}$ ***(M1)***

$= \frac{16}{\sqrt{50}\sqrt{50}}$ ***A1A1***

Note: ***A1*** for correct numerator and ***A1*** for correct denominator.

$= \frac{8}{25} \left(= \frac{16}{50} = 0.32 \right)$ ***A1***

[4 marks]

23 – (MAT/12_HL_Summer_2018_Q9) – *Vectors - Lines - Planes*

(a) (i) a pair of opposite sides have equal length and are parallel ***R1***
hence ABCD is a parallelogram ***AG***

(ii) attempt to rewrite the given information in vector form ***M1***
$\boldsymbol{b}-\boldsymbol{a}=\boldsymbol{c}-\boldsymbol{d}$ ***A1***
rearranging $\boldsymbol{d}-\boldsymbol{a}=\boldsymbol{c}-\boldsymbol{b}$ ***M1***
hence $\overrightarrow{AD}=\overrightarrow{BC}$ ***AG***

Note: Candidates may correctly answer part i) by answering part ii) correctly and then deducing there are two pairs of parallel sides.

[4 marks]

(b) **EITHER**

use of $\overrightarrow{AB}=\overrightarrow{DC}$ ***(M1)***

$$\begin{pmatrix}2\\-3\\p+3\end{pmatrix}=\begin{pmatrix}q+1\\1-r\\4\end{pmatrix}$$ ***A1A1***

OR

use of $\overrightarrow{AD}=\overrightarrow{BC}$ ***(M1)***

$$\begin{pmatrix}-2\\r-2\\1\end{pmatrix}=\begin{pmatrix}q-3\\2\\2-p\end{pmatrix}$$ ***A1A1***

THEN

attempt to compare coefficients of $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ in their equation or statement to that effect ***M1***
clear demonstration that the given values satisfy their equation ***A1***
$p=1,\ q=1,\ r=4$ ***AG***

[5 marks]

(c) attempt at computing $\overrightarrow{AB}\times\overrightarrow{AD}$ (or equivalent) ***M1***

$$\begin{pmatrix}-11\\-10\\-2\end{pmatrix}$$ ***A1***

area $=\left|\overrightarrow{AB}\times\overrightarrow{AD}\right|\left(=\sqrt{225}\right)$ ***(M1)***

$=15$ ***A1***

[4 marks]

(d) valid attempt to find $\overrightarrow{OM}\left(=\frac{1}{2}(\boldsymbol{a}+\boldsymbol{c})\right)$ ***(M1)***

$\begin{pmatrix}1\\ \frac{3}{2}\\ -\frac{1}{2}\end{pmatrix}$ ***A1***

the equation is

$\boldsymbol{r}=\begin{pmatrix}1\\ \frac{3}{2}\\ -\frac{1}{2}\end{pmatrix}+t\begin{pmatrix}11\\ 10\\ 2\end{pmatrix}$ or equivalent ***M1A1***

Note: Award maximum ***M1A0*** if '$\mathbf{r}=\ldots$' (or equivalent) is not seen.

[4 marks]

(e) attempt to obtain the equation of the plane in the form $ax+by+cz=d$ ***M1***

$11x+10y+2z=25$ ***A1A1***

Note: ***A1*** for right hand side, ***A1*** for left hand side.

[3 marks]

(f) (i) putting two coordinates equal to zero ***(M1)***

$X\left(\frac{25}{11},0,0\right),\ Y\left(0,\frac{5}{2},0\right),\ Z\left(0,0,\frac{25}{2}\right)$ ***A1***

(ii) $YZ=\sqrt{\left(\frac{5}{2}\right)^2+\left(\frac{25}{2}\right)^2}$ ***M1***

$=\sqrt{\frac{325}{2}}\left(=\frac{5\sqrt{104}}{4}=\frac{5\sqrt{26}}{2}\right)$ ***A1***

[4 marks]

24 – (MAT/11_HL_Summer_2018_Q10) – *Vectors - Lines - Planes*

(a) recognising normal to plane or attempting to find cross product of two vectors lying in the plane ***(M1)***

for example, $\overrightarrow{AB} \times \overrightarrow{AD} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ ***(A1)***

$\Pi_1 : x + z = 1$ ***A1***

[3 marks]

(b) **EITHER**

$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 1 = \sqrt{2}\sqrt{2}\cos\theta$ ***M1A1***

OR

$\left| \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right| = \sqrt{3} = \sqrt{2}\sqrt{2}\sin\theta$ ***M1A1***

Note: ***M1*** is for an attempt to find the scalar or vector product of the two normal vectors.

$\Rightarrow \theta = 60^\circ \left(= \frac{\pi}{3} \right)$ ***A1***

angle between faces is $120^\circ \left(= \frac{2\pi}{3} \right)$ ***A1***

[4 marks]

(c) $\overrightarrow{DB} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ or $\overrightarrow{BD} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ ***(A1)***

$\Pi_3 : x + y - z = k$ ***(M1)***

$\Pi_3 : x + y - z = 0$ ***A1***

[3 marks]

(d) **METHOD 1**

line $\mathrm{AD}:(\mathbf{r}=)\begin{pmatrix}0\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\-1\end{pmatrix}$ ***M1A1***

intersects Π_3 when $\lambda-(1-\lambda)=0$ ***M1***

so $\lambda=\frac{1}{2}$ ***A1***

hence P is the midpoint of AD ***AG***

METHOD 2

midpoint of AD is $(0.5,\ 0,\ 0.5)$ ***(M1)A1***

substitute into $x+y-z=0$ ***M1***

$0.5+0-0.5=0$ ***A1***

hence P is the midpoint of AD ***AG***

[4 marks]

(e) **METHOD 1**

$\mathrm{OP}=\frac{1}{\sqrt{2}}$, $\mathrm{O\hat{P}Q}=90^\circ$, $\mathrm{O\hat{Q}P}=60^\circ$ ***A1A1A1***

$\mathrm{PQ}=\frac{1}{\sqrt{6}}$ ***A1***

area $=\frac{1}{2\sqrt{12}}=\frac{1}{4\sqrt{3}}=\frac{\sqrt{3}}{12}$ ***A1***

METHOD 2

line $\mathrm{BD}:(\mathbf{r}=)\begin{pmatrix}1\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\-1\\1\end{pmatrix}$

$\Rightarrow \lambda=\frac{2}{3}$ *(A1)*

$\overrightarrow{\mathrm{OQ}}=\begin{pmatrix}\frac{1}{3}\\\frac{1}{3}\\\frac{2}{3}\end{pmatrix}$ *A1*

area $=\frac{1}{2}\left|\overrightarrow{\mathrm{OP}}\times\overrightarrow{\mathrm{OQ}}\right|$ *M1*

$\overrightarrow{\mathrm{OP}}=\begin{pmatrix}\frac{1}{2}\\0\\\frac{1}{2}\end{pmatrix}$ *A1*

Note: This ***A1*** is dependent on ***M1***.

area $=\frac{\sqrt{3}}{12}$ *A1*

[5 marks]

25 – (MAT/10_HL_Winter_2018_Q5) – *Vectors - Lines - Planes*

(a) $\boldsymbol{a}\bullet\boldsymbol{b}=(1\times 0)+(1\times -t)+(t\times 4t)$ *(M1)*

$=-t+4t^2$ *A1*

[2 marks]

(b) recognition that $\boldsymbol{a}\bullet\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ *(M1)*

$\boldsymbol{a}\bullet\boldsymbol{b}<0$ or $-t+4t^2<0$ or $\cos\theta<0$ *R1*

Note: Allow $\leq$ for ***R1***.

attempt to solve using sketch or sign diagram *(M1)*

$0<t<\frac{1}{4}$ *A1*

[4 marks]

26 – (MAT/10_HL_Winter_2018_Q9) – *Vectors - Lines - Planes*

(a) **METHOD 1**

$$\boldsymbol{n} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2b \\ 0 \\ b-1 \end{pmatrix}$$ ***(M1)***

$$= \begin{pmatrix} b-1 \\ 4b \\ -2b \end{pmatrix}$$ ***(M1)A1***

$(0, 0, 0)$ on Π so $(b-1)x + 4by - 2bz = 0$ ***(M1)A1***

METHOD 2

using equation of the form $px + qy + rz = 0$ ***(M1)***

$(0, 1, 2)$ on $\Pi \Rightarrow q + 2r = 0$

$(2b, 0, b-1)$ on $\Pi \Rightarrow 2bp + r(b-1) = 0$ ***(M1)A1***

Note: Award ***(M1)A1*** for both equations seen.

solve for p, q, and r ***(M1)***

$(b-1)x + 4by - 2bz = 0$ ***A1***

[5 marks]

(b) M has coordinates $\left(b, 0, \frac{b-1}{2}\right)$ ***(A1)***

$$\boldsymbol{r} = \begin{pmatrix} b \\ 0 \\ \frac{b-1}{2} \end{pmatrix} + \lambda \begin{pmatrix} b-1 \\ 4b \\ -2b \end{pmatrix}$$ ***M1A1***

Note: Award ***M1A0*** if $\boldsymbol{r} =$ (or equivalent) is not seen.

Note: Allow equivalent forms such as $\frac{x-b}{b-1} = \frac{y}{4b} = \frac{2z-b+1}{-4b}$.

[3 marks]

(c) **METHOD 1**

$x = z = 0$ *(M1)*

Note: Award ***M1*** for either $x = 0$ or $z = 0$ or both.

$b + \lambda(b-1) = 0$ and $\frac{b-1}{2} - 2\lambda b = 0$ ***A1***

attempt to eliminate λ ***M1***

$\Rightarrow -\frac{b}{b-1} = \frac{b-1}{4b}$ *(A1)*

$-4b^2 = (b-1)^2$ ***A1***

EITHER

consideration of the signs of LHS and RHS *(M1)*
the LHS is negative and the RHS must be positive (or equivalent statement) ***R1***

OR

$-4b^2 = b^2 - 2b + 1$

$\Rightarrow 5b^2 - 2b + 1 = 0$

$\Delta = (-2)^2 - 4 \times 5 \times 1 = -16\ (< 0)$ ***M1***

$\therefore$ no real solutions ***R1***

THEN

so no point of intersection ***AG***

METHOD 2

$x = z = 0$ *(M1)*

Note: Award ***M1*** for either $x = 0$ or $z = 0$ or both.

$b + \lambda(b-1) = 0$ and $\frac{b-1}{2} - 2\lambda b = 0$ ***A1***

attempt to eliminate b ***M1***

$\Rightarrow \frac{\lambda}{1+\lambda} = \frac{1}{1-4\lambda}$ *(A1)*

$-4\lambda^2 = 1\left(\Rightarrow \lambda^2 = -\frac{1}{4}\right)$ ***A1***

consideration of the signs of LHS and RHS *(M1)*
there are no real solutions (or equivalent statement) ***R1***
so no point of intersection ***AG***

[7 marks]

27 – (MAT/11_HL_Summer_2019_Q1) – *Vectors - Lines - Planes*

$$\boldsymbol{a} \bullet \boldsymbol{b} = \begin{pmatrix} 2 \\ k \\ -1 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ k+2 \\ k \end{pmatrix}$$

$= -6 + k(k+2) - k$ ***A1***

$\boldsymbol{a} \bullet \boldsymbol{b} = 0$ ***(M1)***

$k^2 + k - 6 = 0$

attempt at solving their quadratic equation ***(M1)***

$(k+3)(k-2) = 0$

$k = -3, 2$ ***A1***

Note: Attempt at solving using $|\boldsymbol{a}||\boldsymbol{b}| = |\boldsymbol{a} \times \boldsymbol{b}|$ will be ***M1A0A0A0*** if neither answer found ***M1(A1)A1A0*** for one correct answer and ***M1(A1)A1A1*** for two correct answers.

28 – (MAT/12_HL_Summer_2019_Q2) – *Vectors - Lines - Planes*

(a) (i) $\overrightarrow{AB} = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$ ***A1***

(ii) $\overrightarrow{AC} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ ***A1***

Note: Accept row vectors or equivalent.

[2 marks]

(b) **METHOD 1**

attempt at vector product using $\overrightarrow{AB}$ and $\overrightarrow{AC}$. ***(M1)***

$\pm(2\boldsymbol{i} + 6\boldsymbol{j} + 6\boldsymbol{k})$ ***A1***

attempt to use area $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$ ***M1***

$= \frac{\sqrt{76}}{2}\left(= \sqrt{19}\right)$ ***A1***

[4 marks]

29 – (MAT/11_HL_Summer_2019_Q11) – *Vectors - Lines - Planes*

(a) (i) appreciation that two points distinct from P need to be chosen from each line ***M1***

$^4C_2 \times {}^3C_2$

$= 18$ ***A1***

(ii) **EITHER**

consider cases for triangles including P **or** triangles not including P ***M1***

$3\times 4 + 4\times {}^3C_2 + 3\times {}^4C_2$ ***(A1)(A1)***

Note: Award ***A1*** for 1st term, ***A1*** for 2nd & 3rd term.

OR

consider total number of ways to select 3 points and subtract those with 3 points on the same line ***M1***

$^8C_3 - {}^5C_3 - {}^4C_3$ ***(A1)(A1)***

Note: Award ***A1*** for 1st term, ***A1*** for 2nd & 3rd term.

$56-10-4$

THEN

$= 42$ ***A1***

[6 marks]

(b) **METHOD 1**

substitution of $(4, 6, 4)$ into both equations ***(M1)***

$\lambda = 3$ and $\mu = 1$ ***A1A1***

$(4, 6, 4)$ ***AG***

METHOD 2

attempting to solve two of the three parametric equations ***M1***

$\lambda = 3$ or $\mu = 1$ ***A1***

check both of the above give $(4, 6, 4)$ ***M1AG***

Note: If they have shown the curve intersects for all three coordinates they only need to check (4,6,4) with one of "λ" or "μ".

[3 marks]

(c) $\lambda = 2$ ***A1***

[1 mark]

(d) $\overrightarrow{PA} = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$, $\overrightarrow{PB} = \begin{pmatrix} -5 \\ -6 \\ -2 \end{pmatrix}$ ***A1A1***

Note: Award ***A1A0*** if both are given as coordinates.

[2 marks]

(e) **METHOD 1**

area triangle $\mathrm{ABP} = \frac{1}{2}\left|\overrightarrow{\mathrm{PB}} \times \overrightarrow{\mathrm{PA}}\right|$ ***M1***

$\left(= \frac{1}{2}\left|\begin{pmatrix} -5 \\ -6 \\ -2 \end{pmatrix} \times \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}\right|\right) = \frac{1}{2}\left|\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}\right|$ ***A1***

$= \frac{\sqrt{29}}{2}$ ***A1***

EITHER

$\overrightarrow{\mathrm{PC}} = 3\overrightarrow{\mathrm{PA}},\ \overrightarrow{\mathrm{PD}} = 3\overrightarrow{\mathrm{PB}}$ ***(M1)***

area triangle $\mathrm{PCD} = 9 \times$ area triangle ABP ***(M1)A1***

$= \frac{9\sqrt{29}}{2}$ ***A1***

OR

D has coordinates $(-11, -12, -2)$ ***A1***

area triangle $\mathrm{PCD} = \frac{1}{2}\left|\overrightarrow{\mathrm{PD}} \times \overrightarrow{\mathrm{PC}}\right| = \frac{1}{2}\left|\begin{pmatrix} -15 \\ -18 \\ -6 \end{pmatrix} \times \begin{pmatrix} -3 \\ -6 \\ -3 \end{pmatrix}\right|$ ***M1A1***

Note: A1 is for the correct vectors in the correct formula.

$= \frac{9\sqrt{29}}{2}$ ***A1***

THEN

area of $\mathrm{CDBA} = \frac{9\sqrt{29}}{2} - \frac{\sqrt{29}}{2}$

$= 4\sqrt{29}$ ***A1***

[8 marks]

30 – (MAT/10_HL_Winter_2019_Q3) – *Vectors - Lines - Planes*

attempt to eliminate a variable (or attempt to find det A) ***M1***

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 1 & 3 & -1 & 4 \\ 3 & -5 & a & b \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 0 & 7 & -3 & 3 \\ 0 & -14 & a+3 & b-12 \end{array}\right) \text{ (or } \det A = 14(a-3)\text{)}$$

(or two correct equations in two variables) ***A1***

$$\to \left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 0 & 7 & -3 & 3 \\ 0 & 0 & a-3 & b-6 \end{array}\right) \text{ (or solving } \det A = 0\text{)}$$

(or attempting to reduce to one variable, e.g. $(a-3)z = b-6$) ***M1***

$a = 3,\ b \neq 6$ ***A1A1***

[5 marks]

31 – (MAT/10_HL_Winter_2019_Q8) – *Vectors - Lines - Planes*

a vector normal to Π_p is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ***(A1)***

Note: Allow any scalar multiple of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, including $\begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix}$

attempt to find scalar product (or vector product) of direction vector of line with any scalar multiple of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ***M1***

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ \sin\theta \\ \cos\theta \end{pmatrix} = 5 \text{ (or } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 5 \\ \sin\theta \\ \cos\theta \end{pmatrix} = \begin{pmatrix} 0 \\ -\cos\theta \\ \sin\theta \end{pmatrix}\text{)}$$ ***A1***

(if α is the angle between the line and the normal to the plane)

$$\cos\alpha = \frac{5}{1 \times \sqrt{25 + \sin^2\theta + \cos^2\theta}} \text{ (or } \sin\alpha = \frac{1}{1 \times \sqrt{25 + \sin^2\theta + \cos^2\theta}}\text{)}$$ ***A1***

$$\Rightarrow \cos\alpha = \frac{5}{\sqrt{26}} \text{ or } \sin\alpha = \frac{1}{\sqrt{26}}$$ ***A1***

this is independent of p and θ, hence the angle between the line and the plane, $(90 - \alpha)$, is also independent of p and θ ***R1***

Note: The final ***R*** mark is independent, but is conditional on the candidate obtaining a value independent of p and θ.

[6 marks]

32 – (MAT/10_HL_Winter_2019_Q11) – *Vectors - Lines - Planes, Graphs*

(a) (i) $\overrightarrow{AV} = \begin{pmatrix} p \\ p \\ p-10 \end{pmatrix}$ ***A1***

$\overrightarrow{AB} \times \overrightarrow{AV} = \begin{pmatrix} 0 \\ 10 \\ -10 \end{pmatrix} \times \begin{pmatrix} p \\ p \\ p-10 \end{pmatrix} = \begin{pmatrix} 10(p-10)+10p \\ -10p \\ -10p \end{pmatrix}$ ***A1***

$= \begin{pmatrix} 20p-100 \\ -10p \\ -10p \end{pmatrix} = -10\begin{pmatrix} 10-2p \\ p \\ p \end{pmatrix}$ ***AG***

$\overrightarrow{AC} \times \overrightarrow{AV} = \begin{pmatrix} 10 \\ 0 \\ -10 \end{pmatrix} \times \begin{pmatrix} p \\ p \\ p-10 \end{pmatrix} = \begin{pmatrix} 10p \\ 100-20p \\ 10p \end{pmatrix} \left(= 10\begin{pmatrix} p \\ 10-2p \\ p \end{pmatrix} \right)$ ***A1***

(ii) attempt to find a scalar product ***M1***

$-10\begin{pmatrix} 10-2p \\ p \\ p \end{pmatrix} \bullet 10\begin{pmatrix} p \\ 10-2p \\ p \end{pmatrix} = 100\left(3p^2 - 20p\right)$

OR $-\begin{pmatrix} 10-2p \\ p \\ p \end{pmatrix} \bullet \begin{pmatrix} p \\ 10-2p \\ p \end{pmatrix} = 3p^2 - 20p$ ***A1***

attempt to find magnitude of either $\overrightarrow{AB} \times \overrightarrow{AV}$ or $\overrightarrow{AC} \times \overrightarrow{AV}$ ***M1***

$\left| -10\begin{pmatrix} 10-2p \\ p \\ p \end{pmatrix} \right| = \left| 10\begin{pmatrix} p \\ 10-2p \\ p \end{pmatrix} \right| = 10\sqrt{(10-2p)^2 + 2p^2}$ ***A1***

$100\left(3p^2 - 20p\right) = 100\left(\sqrt{(10-2p)^2 + 2p^2}\right)^2 \cos\theta$

$\cos\theta = \dfrac{3p^2 - 20p}{(10-2p)^2 + 2p^2}$ ***A1***

Note: Award ***A1*** for any intermediate step leading to the correct answer.

$= \dfrac{p(3p-20)}{6p^2 - 40p + 100}$ ***AG***

Note: Do not allow FT marks from part (a)(i).

[8 marks]

(b) (i) $p(3p-20)=0 \Rightarrow p=0$ or $p=\frac{20}{3}$ ***M1A1***

coordinates are $(0,0,0)$ and $\left(\frac{20}{3},\frac{20}{3},\frac{20}{3}\right)$ ***A1***

Note: Do not allow column vectors for the final ***A*** mark.

(ii) two points are mirror images in the plane
or opposite sides of the plane
or equidistant from the plane
or the line connecting the two Vs is perpendicular to the plane ***R1***

[4 marks]

(c) (i) geometrical consideration or attempt to solve $-1=\frac{p(3p-20)}{6p^2-40p+100}$ ***(M1)***

$p=\frac{10}{3}$, $\theta=\pi$ or $\theta=180$ ***A1A1***

(ii) $p\to\infty \Rightarrow \cos\theta \to \frac{1}{2}$ ***M1***

hence the asymptote has equation $\theta=\frac{\pi}{3}$ ***A1***

[5 marks]

33 - (MathAA/12_HL_Summer_2021_Q5) - *Vectors - Lines - Planes*

METHOD 1

use of $|\boldsymbol{a}\times\boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta$ on the LHS ***(M1)***

$|\boldsymbol{a}\times\boldsymbol{b}|^2 = |\boldsymbol{a}|^2|\boldsymbol{b}|^2\sin^2\theta$ ***A1***

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2(1-\cos^2\theta)$ ***M1***

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - |\boldsymbol{a}|^2|\boldsymbol{b}|^2\cos^2\theta$ OR $= |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (|\boldsymbol{a}||\boldsymbol{b}|\cos\theta)^2$ ***A1***

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a}\cdot\boldsymbol{b})^2$ ***AG***

METHOD 2

use of $\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ on the RHS ***(M1)***

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - |\boldsymbol{a}|^2|\boldsymbol{b}|^2\cos^2\theta$ ***A1***

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2(1-\cos^2\theta)$ ***M1***

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2\sin^2\theta$ OR $= (|\boldsymbol{a}||\boldsymbol{b}|\sin\theta)^2$ ***A1***

$= |\boldsymbol{a}\times\boldsymbol{b}|^2$ ***AG***

Note: If candidates attempt this question using cartesian vectors, e.g

$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ **and** $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$,

award full marks if fully developed solutions are seen.
Otherwise award no marks.

[4 marks]

34 – (MathAA/12_HL_Summer_2021_Q8) – *Vectors - Lines - Planes*

(a) **METHOD 1**

setting at least two components of l_1 and l_2 equal ***M1***

$$3+2\lambda=2+\mu \quad (1)$$
$$2-2\lambda=-\mu \quad (2)$$
$$-1+2\lambda=4+\mu \quad (3)$$

attempt to solve two of the equations eg. adding (1) and (2) ***M1***

gives a contradiction (no solution), eg $5=2$ ***R1***

so l_1 and l_2 do not intersect ***AG***

Note: For an error within the equations award ***M0M1R0.***
Note: The contradiction must be correct to award the ***R1.***

[3 marks]

METHOD 2

l_1 and l_2 are parallel, so l_1 and l_2 are either identical or distinct. ***R1***

Attempt to subtract two position vectors from each line,

e.g. $\begin{pmatrix}3\\2\\-1\end{pmatrix}-\begin{pmatrix}2\\0\\4\end{pmatrix}\left(=\begin{pmatrix}1\\2\\-5\end{pmatrix}\right)$ ***M1***

$\begin{pmatrix}3\\2\\-1\end{pmatrix}\neq k\begin{pmatrix}1\\-1\\1\end{pmatrix}$ ***A1***

[3 marks]

(b) **METHOD 1**

l_1 and l_2 are parallel (as $\begin{pmatrix}2\\-2\\2\end{pmatrix}$ is a multiple of $\begin{pmatrix}1\\-1\\1\end{pmatrix}$)

let A be $(3,2,-1)$ on l_1 and let B be $(2,0,4)$ on l_2

Attempt to find vector $\overrightarrow{\mathrm{AB}}\left(=\begin{pmatrix}-1\\-2\\5\end{pmatrix}\right)$ ***(M1)***

Distance required is $\dfrac{\left|\boldsymbol{v}\times\overrightarrow{\mathrm{AB}}\right|}{|\boldsymbol{v}|}$ ***M1***

$$= \frac{1}{\sqrt{3}}\left|\begin{pmatrix}1\\-1\\1\end{pmatrix}\times\begin{pmatrix}-1\\-2\\5\end{pmatrix}\right|$$ ***(A1)***

$$= \frac{1}{\sqrt{3}}\left|\begin{pmatrix}3\\6\\3\end{pmatrix}\right|$$ ***A1***

minimum distance is $\sqrt{18}\left(=3\sqrt{2}\right)$ ***A1***

[5 marks]

METHOD 2

l_1 and l_2 are parallel (as $\begin{pmatrix}2\\-2\\2\end{pmatrix}$ is a multiple of $\begin{pmatrix}1\\-1\\1\end{pmatrix}$)

let A be a fixed point on l_1 eg $(3,2,-1)$ and let B be a general point on l_2 $(2+\mu,-\mu,4+\mu)$

attempt to find vector $\overrightarrow{\mathrm{AB}}$ ***(M1)***

$$\overrightarrow{\mathrm{AB}} = \begin{pmatrix}-1\\-2\\5\end{pmatrix}+\mu\begin{pmatrix}1\\-1\\1\end{pmatrix}\ (\mu\in\mathbb{R})$$ ***A1***

$$\left|\overrightarrow{\mathrm{AB}}\right| = \sqrt{(-1+\mu)^2+(-2-\mu)^2+(5+\mu)^2}\ \left(=\sqrt{3\mu^2+12\mu+30}\right)$$ ***M1***

EITHER

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\left(\left|\overrightarrow{\mathrm{AB}}\right|^2\right)=0\Rightarrow 6\mu+12=0\Rightarrow\mu=-2$$ ***A1***

OR

$\left|\overrightarrow{\mathrm{AB}}\right| = \sqrt{3(\mu+2)^2+18}$ to obtain $\mu=-2$ ***A1***

THEN

minimum distance is $\sqrt{18}\left(=3\sqrt{2}\right)$ ***A1***

[5 marks]

METHOD 3

let A be $(3,2,-1)$ on l_1 and let B be $(2+\mu,-\mu,4+\mu)$ on l_2 ***(M1)***

(or let A be $(2,0,4)$ on l_2 and let B be $(3+2\lambda,2-2\lambda,-1+2\lambda)$ on l_1)

$\overrightarrow{\mathrm{AB}}=\begin{pmatrix}-1\\-2\\5\end{pmatrix}+\mu\begin{pmatrix}1\\-1\\1\end{pmatrix}$ $(\mu\in\mathbb{R})$ (or $\overrightarrow{\mathrm{AB}}=\begin{pmatrix}2\lambda+1\\-2\lambda+2\\2\lambda-5\end{pmatrix}$) ***A1***

$\begin{pmatrix}\mu-1\\-\mu-2\\\mu+5\end{pmatrix}.\begin{pmatrix}1\\-1\\1\end{pmatrix}=0$ (or $\begin{pmatrix}2\lambda+1\\-2\lambda+2\\2\lambda-5\end{pmatrix}.\begin{pmatrix}1\\-1\\1\end{pmatrix}=0$) ***M1***

$\mu=-2$ or $\lambda=1$ ***A1***

minimum distance is $\sqrt{18}\left(=3\sqrt{2}\right)$ ***A1***

[5 marks]

35 – (MathAA/11_HL_Summer_2021_Q11) – *Vectors - Lines - Planes*

(a) (i) $\frac{-1+1}{2} = 0 = 3-3$ ***A1***

the point $(-1,0,3)$ lies on L_1. ***AG***

(ii) attempt to set equal to a parameter or rearrange cartesian form ***(M1)***

$\frac{x+1}{2} = y = 3-z = \lambda \Rightarrow x = 2\lambda - 1,\ y = \lambda,\ z = 3-\lambda$ OR $\frac{x+1}{2} = \frac{y-0}{1} = \frac{z-3}{-1}$

correct direction vector $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ or equivalent seen in vector form ***(A1)***

$\boldsymbol{r} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ (or equivalent) ***A1***

Note: Award ***A0*** if $\boldsymbol{r} =$ is omitted.

[4 marks]

(b) attempt to use the scalar product formula ***(M1)***

$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix} = (\pm)\sqrt{6}\sqrt{a^2+2}\cos 45^\circ$ ***(A1)(A1)***

Note: Award ***A1*** for LHS and ***A1*** for RHS

$2a+2 = \frac{(\pm)\sqrt{6}\sqrt{a^2+2}\sqrt{2}}{2} \left(\Rightarrow 2a+2 = (\pm)\sqrt{3}\sqrt{a^2+2}\right)$ ***A1A1***

Note: Award ***A1*** for LHS and ***A1*** for RHS

$4a^2 + 8a + 4 = 3a^2 + 6$

$a^2 + 8a - 2 = 0$ ***A1***

attempt to solve their quadratic ***M1***

$a = \frac{-8 \pm \sqrt{64+8}}{2} = \frac{-8 \pm \sqrt{72}}{2} \left(= -4 \pm 3\sqrt{2}\right)$ ***A1***

[8 marks]

(c) **METHOD 1**

attempt to equate the parametric forms of L_1 and L_2 ***(M1)***

$$\begin{cases} 2\lambda - 1 = ta \\ \lambda = 1 + t \\ 3 - \lambda = 2 - t \end{cases}$$ ***A1***

attempt to solve equations by eliminating λ or t ***(M1)***

$2 + 2t - 1 = ta \Rightarrow 1 = t(a-2)$ or $2\lambda - 1 = (\lambda - 1)a \Rightarrow a - 1 = \lambda(a-2)$

Solutions exist unless $a - 2 = 0$

$k = 2$ ***A1***

Note: This ***A1*** is independent of the following marks.

$t = \frac{1}{a-2}$ or $\lambda = \frac{a-1}{a-2}$ ***A1***

A has coordinates $\left(\frac{a}{a-2}, 1 + \frac{1}{a-2}, 2 - \frac{1}{a-2}\right)\left(= \left(\frac{a}{a-2}, \frac{a-1}{a-2}, \frac{2a-5}{a-2}\right)\right)$ ***A2***

Note: Award ***A1*** for any two correct coordinates seen or final answer in vector form.

METHOD 2

no unique point of intersection implies direction vectors of L_1 and L_2 parallel

$k = 2$ ***A1***

Note: This ***A1*** is independent of the following marks.

attempt to equate the parametric forms of L_1 and L_2 ***(M1)***

$$\begin{cases} 2\lambda - 1 = ta \\ \lambda = 1 + t \\ 3 - \lambda = 2 - t \end{cases}$$ ***A1***

attempt to solve equations by eliminating λ or t ***(M1)***

$2 + 2t - 1 = ta \Rightarrow 1 = t(a-2)$ or $2\lambda - 1 = (\lambda - 1)a \Rightarrow a - 1 = \lambda(a-2)$

$t = \frac{1}{a-2}$ or $\lambda = \frac{a-1}{a-2}$ ***A1***

A has coordinates $\left(\frac{a}{a-2}, 1 + \frac{1}{a-2}, 2 - \frac{1}{a-2}\right)\left(= \left(\frac{a}{a-2}, \frac{a-1}{a-2}, \frac{2a-5}{a-2}\right)\right)$ ***A2***

Note: Award ***A1*** for any two correct coordinates seen or final answer in vector form.

[7 marks]

36 – (MathAA/11_HL_Summer_2022_Q11) – *Vectors - Lines - Planes*

(a) **METHOD 1**

attempt to eliminate a variable ***M1***

obtain a pair of equations in two variables

EITHER

$-3x+z=-3$ and ***A1***

$-3x+z=44$ ***A1***

OR

$-5x+y=-7$ and ***A1***

$-5x+y=40$ ***A1***

OR

$3x-z=3$ and ***A1***

$3x-z=-\frac{79}{5}$ ***A1***

THEN

the two lines are parallel ($-3\neq 44$ or $-7\neq 40$ or $3\neq -\frac{79}{5}$) ***R1***

Note: There are other possible pairs of equations in two variables.
To obtain the final ***R1***, at least the initial ***M1*** must have been awarded.

hence the three planes do not intersect ***AG***

[4 marks]

METHOD 2

vector product of the two normals $= \begin{pmatrix} -1 \\ -5 \\ -3 \end{pmatrix}$ (or equivalent) ***A1***

$\boldsymbol{r} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$ (or equivalent) ***A1***

Note: Award ***A0*** if "$r =$" is missing. Subsequent marks may still be awarded.

Attempt to substitute $(1+\lambda, -2+5\lambda, 3\lambda)$ in Π_3 ***M1***

$-9(1+\lambda)+3(-2+5\lambda)-2(3\lambda)=32$

$-15=32$, a contradiction ***R1***

hence the three planes do not intersect ***AG***

[4 marks]

METHOD 3

attempt to eliminate a variable ***M1***

$-3y+5z=6$ ***A1***

$-3y+5z=100$ ***A1***

$0=94$, a contradiction ***R1***

Note: Accept other equivalent alternatives. Accept other valid methods.
To obtain the final ***R1***, at least the initial ***M1*** must have been awarded.

hence the three planes do not intersect ***AG***

[4 marks]

(b) (i) $\Pi_1 : 2+2+0=4$ and $\Pi_2 : 1+4+0=5$ ***A1***

(ii) **METHOD 1**

attempt to find the vector product of the two normals ***M1***

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ -5 \\ -3 \end{pmatrix}$$ ***A1***

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$$ ***A1A1***

Note: Award ***A1A0*** if "$r =$" is missing.
Accept any multiple of the direction vector.

Working for (b)(ii) may be seen in part (a) Method 2. In this case penalize lack of "$r =$" only once.

METHOD 2

attempt to eliminate a variable from Π_1 and Π_2 ***M1***

$3x - z = 3$ OR $3y - 5z = -6$ OR $5x - y = 7$

Let $x = t$

substituting $x = t$ in $3x - z = 3$ to obtain

$z = -3 + 3t$ and $y = 5t - 7$ (for all three variables in parametric form) ***A1***

$$\boldsymbol{r} = \begin{pmatrix} 0 \\ -7 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$$ ***A1A1***

Note: Award ***A1A0*** if "$r =$" is missing.
Accept any multiple of the direction vector. Accept other position vectors which satisfy both the planes Π_1 and Π_2.

[5 marks]

(c) **METHOD 1**

the line connecting L and Π_3 is given by L_1

attempt to substitute position and direction vector to form L_1 ***(M1)***

$$\boldsymbol{s} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} -9 \\ 3 \\ -2 \end{pmatrix}$$ ***A1***

substitute $(1-9t, -2+3t, -2t)$ in Π_3 ***M1***

$$-9(1-9t)+3(-2+3t)-2(-2t)=32$$

$$94t = 47 \Rightarrow t = \frac{1}{2}$$ ***A1***

attempt to find distance between $(1,-2,0)$ and their point $\left(-\frac{7}{2}, -\frac{1}{2}, -1\right)$ ***(M1)***

$$= \left| \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -9 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \right| = \frac{1}{2}\sqrt{(-9)^2 + 3^2 + (-2)^2}$$

$$= \frac{\sqrt{94}}{2}$$ ***A1***

[6 marks]

METHOD 2

unit normal vector equation of Π_3 is given by $\dfrac{\begin{pmatrix}-9\\3\\2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}}{\sqrt{81+9+4}}$ ***(M1)***

$=\dfrac{32}{\sqrt{94}}$ ***A1***

let Π_4 be the plane parallel to Π_3 and passing through P,

then the normal vector equation of Π_4 is given by

$\begin{pmatrix}-9\\3\\2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-9\\3\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\0\end{pmatrix}=-15$ ***M1***

unit normal vector equation of Π_4 is given by

$\dfrac{\begin{pmatrix}-9\\3\\2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}}{\sqrt{81+9+4}}=\dfrac{-15}{\sqrt{94}}$ ***A1***

distance between the planes is $\dfrac{32}{\sqrt{94}}-\dfrac{-15}{\sqrt{94}}$ ***(M1)***

$=\dfrac{47}{\sqrt{94}}\left(=\dfrac{\sqrt{94}}{2}\right)$ ***A1***

[6 marks]

1 - (MAT/11_HL_Summer_2014_Q2) - *Statistics*

using the sum divided by 4 is 13
two of the numbers are 15
(as median is 14) we need a 13
fourth number is 9
numbers are 9, 13, 15, 15

2 - (MAT/11_HL_Summer_2015_Q2) - *Statistics*

(a) $\bar{x} = \dfrac{1\times 0 + 19\times 10}{20} = 9.5$

(b) median is 10

(c) (i) 19

(ii) 1

3 - (MAT/12_HL_Summer_2017_Q7) - *Statistics, Probability*

(a) $\mathrm{P}(X > 0) = 1 - \mathrm{P}(X = 0)$ ***(M1)***

$\Rightarrow 1 - \mathrm{e}^{-m} = \dfrac{3}{4}$ or equivalent ***A1***

$\Rightarrow m = \ln 4$ ***A1***

[3 marks]

(b) $\mathrm{P}(Y > 1) = 1 - \mathrm{P}(Y = 0) - \mathrm{P}(Y = 1)$ ***(M1)***

$= 1 - \mathrm{e}^{-2\ln 4} - \mathrm{e}^{-2\ln 4} \times 2\ln 4$ ***A1***

recognition that $2\ln 4 = \ln 16$ ***(A1)***

$\mathrm{P}(Y > 1) = \dfrac{15 - \ln 16}{16}$ ***A1***

[4 marks]

4 – (MAT/11_HL_Summer_2017_Q10) – *Probability, Statistics*

(a) attempt to equate integral to 1 (may appear later) ***M1***

$$k\int_0^6 \sin\left(\frac{\pi x}{6}\right)\mathrm{d}x = 1$$

correct integral ***A1***

$$k\left[-\frac{6}{\pi}\cos\left(\frac{\pi x}{6}\right)\right]_0^6 = 1$$

substituting limits ***M1***

$$-\frac{6}{\pi}(-1-1) = \frac{1}{k}$$

$$k = \frac{\pi}{12}$$ ***A1***

[4 marks]

(b) (i) mean $= 3$ ***A1***

(ii) median $= 3$ ***A1***

(iii) mode $= 3$ ***A1***

Note: Award ***A1A0A0*** for three equal answers in $(0, 6)$.

[3 marks]

(c) (i) $\frac{\pi}{12}\int_0^2 \sin\left(\frac{\pi x}{6}\right)\mathrm{d}x$ ***M1***

$$= \frac{\pi}{12}\left[-\frac{6}{\pi}\cos\left(\frac{\pi x}{6}\right)\right]_0^2$$ ***A1***

Note: Accept without the $\frac{\pi}{12}$ at this stage if it is added later.

$$\frac{\pi}{12}\left[-\frac{6}{\pi}\left(\cos\frac{\pi}{3} - 1\right)\right]$$ ***M1***

$$= \frac{1}{4}$$ ***AG***

(ii) from (c)(i) $Q_1 = 2$ ***(A1)***

as the graph is symmetrical about the middle value $x = 3 \Rightarrow Q_3 = 4$ ***(A1)***

so interquartile range is

$4 - 2$ ***A1***

$= 2$

[6 marks]

(d) $P(X \le 4 | X \ge 3) = \dfrac{P(3 \le X \le 4)}{P(X \ge 3)}$

$= \dfrac{\frac{1}{4}}{\frac{1}{2}}$ *(M1)*

$= \dfrac{1}{2}$ *A1*

[2 marks]

5 - (MAT/10_HL_Winter_2017_Q10) - *Statistics*

(a) **METHOD 1**

number of possible "deals" $= 4! = 24$ *A1*

consider ways of achieving "no matches" (Chloe winning):
Selena could deal B, C, D (*ie*, 3 possibilities)
as her first card *R1*

for each of these matches, there are only 3 possible combinations for the remaining 3 cards *R1*

so no. ways achieving no matches $= 3 \times 3 = 9$ *M1A1*

so probability Chloe wins $= \dfrac{9}{24} = \dfrac{3}{8}$ *A1AG*

METHOD 2

number of possible "deals" $= 4! = 24$ *A1*

consider ways of achieving a match (Selena winning)
Selena card A can match with Chloe card A, giving 6 possibilities for this happening *R1*

if Selena deals B as her first card, there are only 3 possible combinations for the remaining 3 cards. Similarly for dealing C and dealing D *R1*

so no. ways achieving one match is $= 6 + 3 + 3 + 3 = 15$ *M1A1*

so probability Chloe wins $= 1 - \dfrac{15}{24} = \dfrac{3}{8}$ *A1AG*

METHOD 3

systematic attempt to find number of outcomes where Chloe wins (no matches) (using tree diag. or otherwise) *M1*

9 found *A1*

each has probability $\dfrac{1}{4} \times \dfrac{1}{3} \times \dfrac{1}{2} \times 1$ *M1*

$= \dfrac{1}{24}$ *A1*

their 9 multiplied by their $\dfrac{1}{24}$ *M1A1*

$= \dfrac{3}{8}$ *AG*

[6 marks]

(b) (i) $X \sim B\left(50, \frac{3}{8}\right)$ ***(M1)***

$\mu = np = 50 \times \frac{3}{8} = \frac{150}{8}\left(= \frac{75}{4}\right)(= 18.75)$ ***(M1)A1***

(ii) $\sigma^2 = np(1-p) = 50 \times \frac{3}{8} \times \frac{5}{8} = \frac{750}{64}\left(= \frac{375}{32}\right)(= 11.7)$ ***(M1)A1***

[5 marks]

6 – (MathAA/11_HL_Summer_2021_Q3) – *Statistics*

(a) attempt to use definition of outlier

$1.5 \times 20 + Q_3$ ***(M1)***

$1.5 \times 20 + U \geq 75$ $(\Rightarrow U \geq 45$, accept $U > 45)$ OR $1.5 \times 20 + Q_3 = 75$ ***A1***

minimum value of $U = 45$ ***A1***

[3 marks]

(b) attempt to use interquartile range ***(M1)***

$U - L = 20$ (may be seen in part (a)) OR $L \geq 25$ (accept $L > 25$)

minimum value of $L = 25$ ***A1***

[2 marks]

7 – (MathAA/11_HL_Summer_2022_Q2) – *Statistics*

(a) $\text{IQR} = 10 - 6\ (= 4)$ ***(A1)***

attempt to find $Q_3 + 1.5 \times \text{IQR}$ ***(M1)***

$10 + 6$

16 ***A1***

[3 marks]

(b) (i) choosing $c = \frac{1}{2}a - 9$ ***(M1)***

$\frac{1}{2} \times 42 - 9$

= 12 (years old) ***A1***

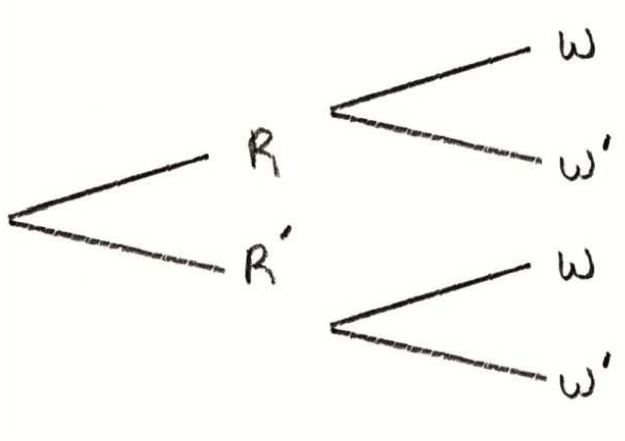
(ii) attempt to solve system by substitution or elimination ***(M1)***

34 (years old) ***A1***

[4 marks]

1 – (MAT/12_HL_Summer_2012_Q3) – *Probability*

(a) let R be "it rains" and W be "the 'Tigers' soccer team win"



(b) $$\mathrm{P}(W) = \frac{2}{5} \times \frac{2}{7} + \frac{3}{5} \times \frac{4}{7}$$
$$= \frac{16}{35}$$

(c) $$\mathrm{P}(R|W) = \frac{\frac{2}{5} \times \frac{2}{7}}{\frac{16}{35}}$$
$$= \frac{1}{4}$$

2 – (MAT/12_HL_Summer_2012_Q11) – *Functions - Roots, Probability*

(a) $f(x) \geq \frac{1}{25}$

$g(x) \in \mathbb{R}, g(x) \geq 0$

(b) $f \circ g(x) = \dfrac{2\left(\frac{3x-4}{10}\right)^2 + 3}{75}$

$= \dfrac{\frac{2(9x^2 - 24x + 16)}{100} + 3}{75}$

$= \dfrac{9x^2 - 24x + 166}{3750}$

(c) (i) **METHOD 1**

$y = \dfrac{2x^2 + 3}{75}$

$x^2 = \dfrac{75y - 3}{2}$

$x = \sqrt{\dfrac{75y - 3}{2}}$

$\Rightarrow f^{-1}(x) = \sqrt{\dfrac{75x - 3}{2}}$

Note:	Accept ± in line 3 for the ***(A1)*** but not in line 4 for the ***A1***. Award the ***A1*** only if written in the form $f^{-1}(x) =$.

METHOD 2

$y = \dfrac{2x^2 + 3}{75}$

$x = \dfrac{2y^2 + 3}{75}$

$y = \sqrt{\dfrac{75x - 3}{2}}$

$\Rightarrow f^{-1}(x) = \sqrt{\dfrac{75x - 3}{2}}$

Note:	Accept ± in line 3 for the ***(A1)*** but not in line 4 for the ***A1***. Award the ***A1*** only if written in the form $f^{-1}(x) =$.

(ii) domain: $x \geq \frac{1}{25}$; range: $f^{-1}(x) \geq 0$

(d) probabilities from $f(x)$:

X	0	1	2	3	4
$P(X=x)$	$\frac{3}{75}$	$\frac{5}{75}$	$\frac{11}{75}$	$\frac{21}{75}$	$\frac{35}{75}$

Note: Award ***A1*** for one error, ***A0*** otherwise.

probabilities from $g(x)$:

X	0	1	2	3	4
$P(X=x)$	$\frac{4}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{5}{10}$	$\frac{8}{10}$

Note: Award ***A1*** for one error, ***A0*** otherwise.

only in the case of $f(x)$ does $\sum P(X=x)=1$, hence only $f(x)$ can be used as a probability mass function

(e)

$$E(x)=\sum x\cdot \mathrm{P}(X=x)$$

$$=\frac{5}{75}+\frac{22}{75}+\frac{63}{75}+\frac{140}{75}=\frac{230}{75}\left(=\frac{46}{15}\right)$$

3 – (MAT/10_HL_Winter_2012_Q5) – *Probability*

(a) 0

(b) $\int_0^1 f(x)\,\mathrm{d}x = 1$

$\Rightarrow a = \dfrac{1}{\int_0^1 e^{-x}\,\mathrm{d}x}$

$\Rightarrow a = \dfrac{1}{\left[-e^{-x}\right]_0^1}$

$\Rightarrow a = \dfrac{e}{e-1}$ (or equivalent)

Note: Award first ***A1*** for correct integration of $\int e^{-x}\,\mathrm{d}x$. This ***A1*** is independent of previous ***M*** mark.

(c) $\mathrm{E}(X) = \int_0^1 xf(x)\,\mathrm{d}x \left(= a\int_0^1 xe^{-x}\,\mathrm{d}x\right)$

attempt to integrate by parts

$= a\left[-xe^{-x} - e^{-x}\right]_0^1$

$= a\left(\dfrac{e-2}{e}\right)$

$= \dfrac{e-2}{e-1}$ (or equivalent)

4 – (MAT/11_HL_Summer_2013_Q4) – *Probability*

$\int_0^{\frac{\pi}{2}} x\sin x\,\mathrm{d}x$

$= \left[-x\cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x\,\mathrm{d}x$

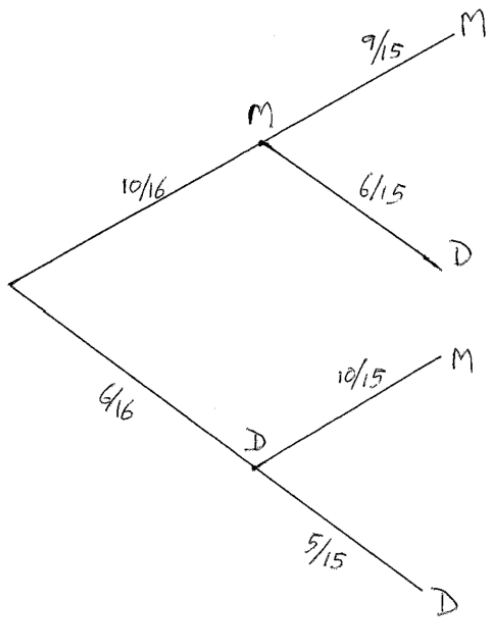
Note: Condone the absence of limits or wrong limits to this point.

$= \left[-x\cos x + \sin x\right]_0^{\frac{\pi}{2}}$

$= 1$

5 – (MAT/12_HL_Summer_2013_Q4) – *Probability*

(a)



Note:	Award ***A1*** for the initial level probabilities, ***A1*** for each of the second level branch probabilities.

(b) $\frac{10}{16}\times\frac{9}{15}+\frac{6}{16}\times\frac{5}{15}$

$=\frac{120}{240}\left(=\frac{1}{2}\right)$

6 – (MAT/11_HL_Summer_2013_Q9) – *Probability*

METHOD 1

$\mathrm{P}(A'|B') = 1 - \mathrm{P}(A|B') = 1 - 0.6 = 0.4$

$\mathrm{P}(A'|B') = \dfrac{\mathrm{P}(A' \cap B')}{\mathrm{P}(B')}$

$\mathrm{P}(A' \cap B') = \mathrm{P}\big((A \cup B)'\big) = 1 - 0.7 = 0.3$

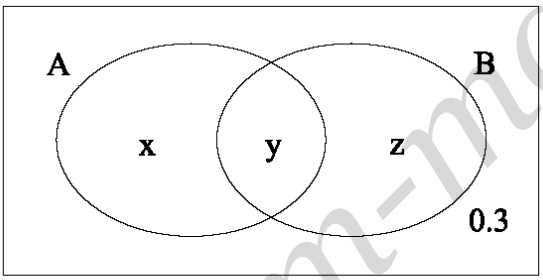
$0.4 = \dfrac{0.3}{\mathrm{P}(B')} \Rightarrow \mathrm{P}(B') = 0.75$

$\mathrm{P}(B) = 0.25$

(this method can be illustrated using a tree diagram)

METHOD 2

$\mathrm{P}\big((A \cup B)'\big) = 1 - 0.7 = 0.3$



$\mathrm{P}(A|B') = \dfrac{x}{x + 0.3} = 0.6$

$x = 0.6x + 0.18$

$0.4x = 0.18$

$x = 0.45$

$\mathrm{P}(A \cup B) = x + y + z$

$\mathrm{P}(B) = y + z = 0.7 - 0.45$

$= 0.25$

METHOD 3

$\dfrac{\mathrm{P}(A \cap B')}{\mathrm{P}(B')} = 0.6$ (or $\mathrm{P}(A \cap B') = 0.6\mathrm{P}(B')$)

$\mathrm{P}(A \cap B') = \mathrm{P}(A \cup B) - \mathrm{P}(B)$

$\mathrm{P}(B') = 1 - \mathrm{P}(B)$

$0.7 - \mathrm{P}(B) = 0.6 - 0.6\mathrm{P}(B)$

$0.1 = 0.4\mathrm{P}(B)$

$\mathrm{P}(B) = \dfrac{1}{4}$

7 – (MAT/11_HL_Summer_2013_Q13) – *Probability, Binomial Theorem*

(a) $B\left(6, \frac{2}{3}\right)$

$p(4) = \binom{6}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2$

$\binom{6}{4} = 15$

$= 15 \times \frac{2^4}{3^6} = \frac{80}{243}$

(b) (i) 2 outcomes for each of the 6 games or $2^6 = 64$

(ii) $(1+x)^6 = \binom{6}{0} + \binom{6}{1}x + \binom{6}{2}x^2 + \binom{6}{3}x^3 + \binom{6}{4}x^4 + \binom{6}{5}x^5 + \binom{6}{6}x^6$

Note: Accept ${}^nC_r$ notation or $1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$

setting $x = 1$ in both sides of the expression

Note: Do not award ***R1*** if the right hand side is not in the correct form.

$64 = \binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}$

(iii) the total number of outcomes = number of ways Alfred can win no games, plus the number of ways he can win one game *etc*.

(c) (i) Let $\mathrm{P}(x, y)$ be the probability that Alfred wins x games on the first day and y on the second.

$\mathrm{P}(4, 2) = \binom{6}{4} \times \left(\frac{2}{3}\right)^4 \times \left(\frac{1}{3}\right)^2 \times \binom{6}{2} \times \left(\frac{2}{3}\right)^2 \times \left(\frac{1}{3}\right)^4$

$\binom{6}{2}^2\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^6$ or $\binom{6}{4}^2\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^6$

$r = 2$ or 4, $s = t = 6$

(ii) $\mathrm{P}(\text{Total}=6)=$

$\mathrm{P}(0,6)+\mathrm{P}(1,5)+\mathrm{P}(2,4)+\mathrm{P}(3,3)+\mathrm{P}(4,2)+\mathrm{P}(5,1)+\mathrm{P}(6,0)$

$$=\binom{6}{0}^2\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^6+\binom{6}{1}^2\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^6+\ldots+\binom{6}{6}^2\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^6$$

$$=\frac{2^6}{3^{12}}\left(\binom{6}{0}^2+\binom{6}{1}^2+\binom{6}{2}^2+\binom{6}{3}^2+\binom{6}{4}^2+\binom{6}{5}^2+\binom{6}{6}^2\right)$$

Note: Accept any valid sum of 7 probabilities.

(iii) use of $\binom{6}{i}=\binom{6}{6-i}$

(can be used either here or in (c)(ii))

$$\mathrm{P}(\text{wins 6 out of 12})=\binom{12}{6}\times\left(\frac{2}{3}\right)^6\times\left(\frac{1}{3}\right)^6=\frac{2^6}{3^{12}}\binom{12}{6}$$

$$\frac{2^6}{3^{12}}\left(\binom{6}{0}^2+\binom{6}{1}^2+\binom{6}{2}^2+\binom{6}{3}^2+\binom{6}{4}^2+\binom{6}{5}^2+\binom{6}{6}^2\right)=\frac{2^6}{3^{12}}\binom{12}{6}$$

therefore $\binom{6}{0}^2+\binom{6}{1}^2+\binom{6}{2}^2+\binom{6}{3}^2+\binom{6}{4}^2+\binom{6}{5}^2+\binom{6}{6}^2=\binom{12}{6}$

(d) (i) $\mathrm{E}(A)=\sum_{r=0}^{n}r\binom{n}{r}\left(\frac{2}{3}\right)^r\left(\frac{1}{3}\right)^{n-r}=\sum_{r=0}^{n}r\binom{n}{r}\frac{2^r}{3^n}$

$(a=2,\ b=3)$

Note: ***M0A0*** for $a=2$, $b=3$ without any method.

(ii) $n(1+x)^{n-1}=\sum_{r=1}^{n}\binom{n}{r}rx^{r-1}$

(sigma notation not necessary)

(if sigma notation used also allow lower limit to be $r=0$)

let $x=2$

$$n3^{n-1}=\sum_{r=1}^{n}\binom{n}{r}r2^{r-1}$$

multiply by 2 and divide by 3^n

$$\frac{2n}{3}=\sum_{r=1}^{n}\binom{n}{r}r\frac{2^r}{3^n}\quad\left(=\sum_{r=0}^{n}\binom{n}{r}\frac{2^r}{3^n}\right)$$

8 – (MAT/10_HL_Winter_2013_Q2) – *Probability*

(a) $\frac{1}{6}+\frac{1}{2}+\frac{3}{10}+a=1 \Rightarrow a=\frac{1}{30}$

(b) $E(X)=\frac{1}{2}+2\times\frac{3}{10}+3\times\frac{1}{30}$

$=\frac{6}{5}$

Note: Do not award ***FT*** marks if a is outside $[0, 1]$.

(c) $\mathrm{E}(X^2)=\frac{1}{2}+2^2\times\frac{3}{10}+3^2\times\frac{1}{30}=2$

attempt to apply $\mathrm{Var}(X)=\mathrm{E}\left(X^2\right)-\left(\mathrm{E}(X)\right)^2$

$\left(=2-\frac{36}{25}\right)=\frac{14}{25}$

9 – (MAT/12_HL_Summer_2014_Q1) – *Probability*

(a) $\mathrm{P}(A \cap B) = \mathrm{P}(A \mid B) \times P(B)$

$\mathrm{P}(A \cap B) = \frac{2}{11} \times \frac{11}{20}$

$= \frac{1}{10}$

(b) $\mathrm{P}(A \cup B) = \mathrm{P}(A) + P(B) - \mathrm{P}(A \cap B)$

$\mathrm{P}(A \cup B) = \frac{2}{5} + \frac{11}{20} - \frac{1}{10}$

$= \frac{17}{20}$

(c) No – events A and B are not independent

EITHER

$\mathrm{P}(A \mid B) \neq \mathrm{P}(A)$

$\left(\frac{2}{11} \neq \frac{2}{5}\right)$

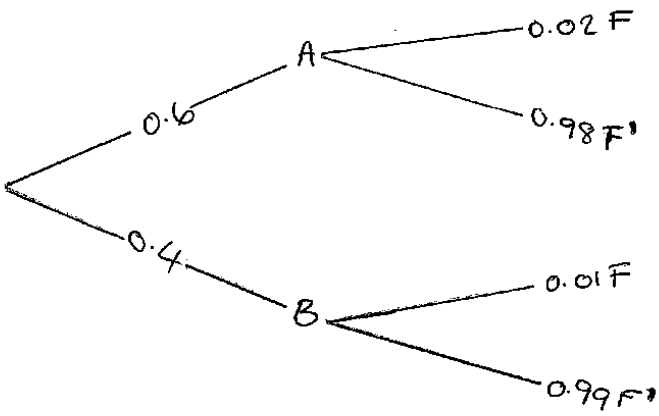
OR

$\mathrm{P}(A) \times P(B) \neq \mathrm{P}(A \cap B)$

$\frac{2}{5} \times \frac{11}{20} = \frac{11}{50} \neq \frac{1}{10}$

10 – (MAT/12_HL_Summer_2014_Q11) – *Probability*

(a) (i)



A1A1

Note: Award ***A1*** for a correctly labelled tree diagram and ***A1*** for correct probabilities.

(ii) $\mathrm{P}(F) = 0.6 \times 0.02 + 0.4 \times 0.01$ ***(M1)***

$= 0.016$ ***A1***

(iii) $\mathrm{P}(A \mid F) = \dfrac{\mathrm{P}(A \cap F)}{\mathrm{P}(F)}$

$= \dfrac{0.6 \times 0.02}{0.016} \left(= \dfrac{0.012}{0.016} \right)$ ***M1***

$= 0.75$ ***A1***

(b) (i) **METHOD 1**

$$\mathrm{P}(X=2)=\frac{{}^{3}C_{2}\times{}^{4}C_{1}}{{}^{7}C_{3}}$$

$$=\frac{12}{35}$$

METHOD 2

$$\frac{3}{7}\times\frac{2}{6}\times\frac{4}{5}\times 3$$

$$=\frac{12}{35}$$

(ii)

x	0	1	2	3
$\mathrm{P}(X=x)$	$\frac{4}{35}$	$\frac{18}{35}$	$\frac{12}{35}$	$\frac{1}{35}$

Note: Award *A1* if $\frac{4}{35}$, $\frac{18}{35}$ or $\frac{1}{35}$ is obtained.

(iii) $\mathrm{E}(X)=\sum x\,\mathrm{P}(X=x)$

$$\mathrm{E}(X)=0\times\frac{4}{35}+1\times\frac{18}{35}+2\times\frac{12}{35}+3\times\frac{1}{35}$$

$$=\frac{45}{35}=\left(\frac{9}{7}\right)$$

11 - (MAT/10_HL_Winter_2014_Q4) - *Probability*

(a) (i) use of $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$

$$\mathrm{P}(A \cup B) = 0.2 + 0.5$$
$$= 0.7$$

(ii) use of $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A)\mathrm{P}(B)$

$$\mathrm{P}(A \cup B) = 0.2 + 0.5 - 0.1$$
$$= 0.6$$

(b) $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

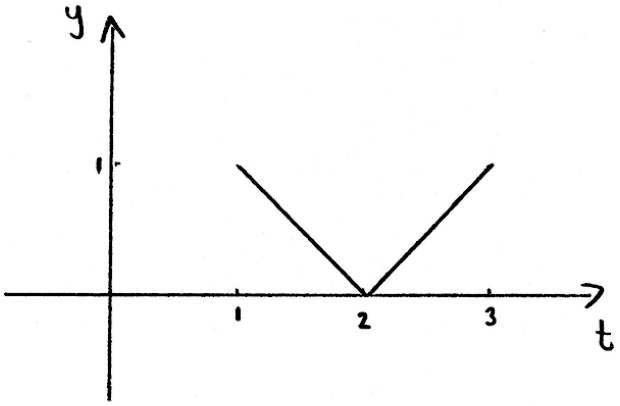
$\mathrm{P}(A \mid B)$ is a maximum when $\mathrm{P}(A \cap B) = \mathrm{P}(A)$

$\mathrm{P}(A \mid B)$ is a minimum when $\mathrm{P}(A \cap B) = 0$

$0 \le \mathrm{P}(A \mid B) \le 0.4$

12 – (MAT/10_HL_Winter_2014_Q9) – *Probability*

(a)



$|2-t|$ correct for $[1, 2]$

$|2-t|$ correct for $[2, 3]$

(b) **EITHER**

let q_1 be the lower quartile and let q_3 be the upper quartile

let $d = 2 - q_1 \left(= q_3 - 2\right)$ and so $\text{IQR} = 2d$ by symmetry

use of area formulae to obtain $\frac{1}{2}d^2 = \frac{1}{4}$

(or equivalent)

$d = \frac{1}{\sqrt{2}}$ or the value of at least one q.

OR

let q_1 be the lower quartile

consider $\int_1^{q_1} (2-t)\,dt = \frac{1}{4}$

obtain $q_1 = 2 - \frac{1}{\sqrt{2}}$

THEN

$\text{IQR} = \sqrt{2}$

13 - (MAT/12_HL_Summer_2015_Q1) - *Probability*

(a) $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$\mathrm{P}(A \cap B) = 0.25 + 0.6 - 0.7$

$= 0.15$

(b) **EITHER**

$\mathrm{P}(A)\mathrm{P}(B)\left(= 0.25 \times 0.6\right) = 0.15$

$= \mathrm{P}(A \cap B)$ so independent

OR

$P\left(A \middle| B\right) = \dfrac{P\left(A \cap B\right)}{P\left(B\right)} = \dfrac{0.15}{0.6} = 0.25$

$= P\left(A\right)$ so independent

14 – (MAT/11_HL_Summer_2015_Q10) – *Probability*

METHOD 1

to have 3 consecutive losses there must be exactly 5, 4 or 3 losses

the probability of exactly 5 losses (must be 3 consecutive) is $\left(\frac{1}{3}\right)^5$

the probability of exactly 4 losses (with 3 consecutive) is $4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)$

Note: First ***A1*** is for the factor 4 and second ***A1*** for the other 2 factors.

the probability of exactly 3 losses (with 3 consecutive) is $3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2$

Note: First ***A1*** is for the factor 3 and second ***A1*** for the other 2 factors.

(Since the events are mutually exclusive)

the total probability is $\frac{1+8+12}{3^5}=\frac{21}{243}\left(=\frac{7}{81}\right)$

METHOD 2

Roy loses his job if
A – first 3 games are all lost (so the last 2 games can be any result)
B – first 3 games are not all lost, but middle 3 games are all lost (so the first game is not a loss and the last game can be any result)
or C – first 3 games are not all lost, middle 3 games are not all lost but last 3 games are all lost, (so the first game can be any result but the second game is not a loss)

for A 4th & 5th games can be anything

$$\mathrm{P}(A)=\left(\frac{1}{3}\right)^3=\frac{1}{27}$$

for B 1st game not a loss & 5th game can be anything

$$\mathrm{P}(B)=\frac{2}{3}\times\left(\frac{1}{3}\right)^3=\frac{2}{81}$$

for C 1st game anything, 2nd game not a loss

$$\mathrm{P}(C)=1\times\frac{2}{3}\times\left(\frac{1}{3}\right)^3=\frac{2}{81}$$

(Since the events are mutually exclusive)

total probability is $\frac{1}{27}+\frac{2}{81}+\frac{2}{81}=\frac{7}{81}$

15 – (MAT/10_HL_Winter_2015_Q6) – *Probability*

(a) probability that Darren wins $\mathrm{P}(W)+\mathrm{P}(RRW)+\mathrm{P}(RRRRW)$ ***(M1)***

Note: Only award ***M1*** if three terms are seen or are implied by the following numerical equivalent.

Note: Accept equivalent tree diagram for method mark.

$$=\frac{2}{6}+\frac{4}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}+\frac{4}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{1}{3}\cdot\frac{2}{2}\quad\left(=\frac{1}{3}+\frac{1}{5}+\frac{1}{15}\right)$$ ***A2***

Note: ***A1*** for two correct.

$$=\frac{3}{5}$$ ***A1***

(b) **METHOD 1**

the probability that Darren wins is given by

$\mathrm{P}(W)+\mathrm{P}(RRW)+\mathrm{P}(RRRRW)+\ldots$ ***(M1)***

Note: Accept equivalent tree diagram with correctly indicated path for method mark.

$$\mathrm{P}(\text{Darren Win})=\frac{1}{3}+\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3}+\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3}+\ldots$$

or $=\frac{1}{3}\left(1+\frac{4}{9}+\left(\frac{4}{9}\right)^2+\ldots\right)$ ***A1***

$$=\frac{1}{3}\left(\frac{1}{1-\frac{4}{9}}\right)$$ ***A1***

$$=\frac{3}{5}$$ ***AG***

METHOD 2

$\mathrm{P}(\text{Darren wins})=\mathrm{P}$

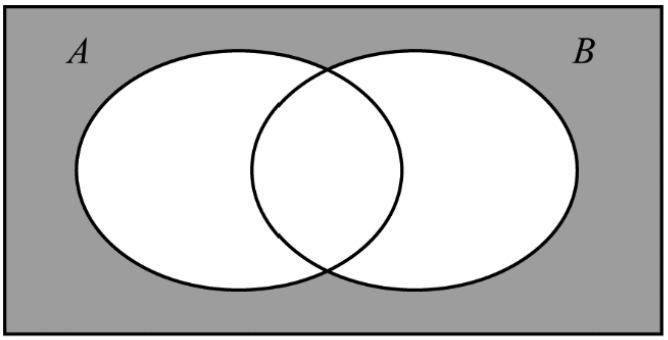
$$\mathrm{P}=\frac{1}{3}+\frac{4}{9}\mathrm{P}$$ ***M1A2***

$$\frac{5}{9}\mathrm{P}=\frac{1}{3}$$

$$\mathrm{P}=\frac{3}{5}$$ ***AG***

16 – (MAT/11_HL_Summer_2016_Q4) – *Probability*

(a)



(b) $P(A' \mid B') = \dfrac{P(A' \cap B')}{P(B')}$

$P(B') = 0.1 + 0.2 = 0.3$

$P(A' \cap B') = 0.1$

$P(A' \mid B') = \dfrac{0.1}{0.3} = \dfrac{1}{3}$

17 – (MAT/12_HL_Summer_2016_Q5) – *Probability*

(a) $X \sim \mathrm{B}(5,\ p)$

$\mathrm{P}(X = 4) = \binom{5}{4} p^4 (1 - p)$ (or equivalent)

(b) (i) $\dfrac{\mathrm{d}}{\mathrm{d}p}\left(5p^4 - 5p^5\right) = 20p^3 - 25p^4$

$5p^3(4 - 5p) = 0 \Rightarrow p = \dfrac{4}{5}$

Note: Do not award the final ***A1*** if $p = 0$ is included in the answer.

(ii) $\mathrm{E}(X) = np = 5\left(\dfrac{4}{5}\right)$

$= 4$

18 – (MAT/12_HL_Summer_2016_Q7) – *Probability*

(a) $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A)\mathrm{P}(B)$

$= p + p - p^2$

$= 2p - p^2$

(b) $\mathrm{P}(A \mid A \cup B) = \dfrac{\mathrm{P}\left(A \cap (A \cup B)\right)}{\mathrm{P}(A \cup B)}$

Note: Allow $\mathrm{P}\left(A \cap A \cup B\right)$ if seen on the numerator.

$= \dfrac{\mathrm{P}(A)}{\mathrm{P}(A \cup B)}$

$= \dfrac{p}{2p - p^2}$

$= \dfrac{1}{2 - p}$

19 – (MAT/10_HL_Winter_2016_Q2) – *Probability*

(a)

x	1	2	4	6
$\mathrm{P}(X = x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

A1A1

Note: Award ***A1*** for each correct row.

[2 marks]

(b) $\mathrm{E}(X) = 1 \times \frac{1}{6} + 2 \times \frac{1}{3} + 4 \times \frac{1}{3} + 6 \times \frac{1}{6}$ ***(M1)***

$= \frac{19}{6} \left(= 3\frac{1}{6} \right)$ ***A1***

Note: If the probabilities in (a) are not values between 0 and 1 or lead to $\mathrm{E}(X) > 6$ award ***M1A0*** to correct method using the incorrect probabilities; otherwise allow ***FT*** marks.

[2 marks]

20 – (MAT/10_HL_Winter_2016_Q10) – *Probability*

(a) **METHOD 1**

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ ***M1***

$= \mathrm{P}(A) + \mathrm{P}(A \cap B) + \mathrm{P}(A' \cap B) - \mathrm{P}(A \cap B)$ ***M1A1***

$= \mathrm{P}(A) + \mathrm{P}(A' \cap B)$ ***AG***

METHOD 2

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ ***M1***

$= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A|B) \times \mathrm{P}(B)$ ***M1***

$= \mathrm{P}(A) + \left(1 - \mathrm{P}(A|B)\right) \times \mathrm{P}(B)$

$= \mathrm{P}(A) + \mathrm{P}(A'|B) \times \mathrm{P}(B)$ ***A1***

$= \mathrm{P}(A) + \mathrm{P}(A' \cap B)$ ***AG***

[3 marks]

(b) (i) use $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(A' \cap B)$ and $\mathrm{P}(A' \cap B) = \mathrm{P}(B \mid A')\mathrm{P}(A')$ ***(M1)***

$\frac{4}{9} = \mathrm{P}(A) + \frac{1}{6}\left(1 - \mathrm{P}(A)\right)$ ***A1***

$8 = 18\mathrm{P}(A) + 3\left(1 - \mathrm{P}(A)\right)$ ***M1***

$\mathrm{P}(A) = \frac{1}{3}$ ***AG***

(ii) **METHOD 1**

$\mathrm{P}(B) = \mathrm{P}(A \cap B) + \mathrm{P}(A' \cap B)$ ***M1***

$= \mathrm{P}(B \mid A)\mathrm{P}(A) + \mathrm{P}(B \mid A')\mathrm{P}(A')$ ***M1***

$= \frac{1}{3} \times \frac{1}{3} + \frac{1}{6} \times \frac{2}{3} = \frac{2}{9}$ ***A1***

METHOD 2

$\mathrm{P}(A \cap B) = \mathrm{P}(B \mid A)\mathrm{P}(A) \Rightarrow \mathrm{P}(A \cap B) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$ ***M1***

$\mathrm{P}(B) = \mathrm{P}(A \cup B) + \mathrm{P}(A \cap B) - \mathrm{P}(A)$ ***M1***

$\mathrm{P}(B) = \frac{4}{9} + \frac{1}{9} - \frac{1}{3} = \frac{2}{9}$ ***A1***

[6 marks]

21 – (MAT/11_HL_Summer_2017_Q4) – *Probability*

METHOD 1

total number of arrangements $7!$ ***(A1)***

number of ways for girls and boys to sit together $= 3! \times 4! \times 2$ ***(M1)(A1)***

Note: Award ***M1A0*** if the 2 is missing.

probability $= \dfrac{3! \times 4! \times 2}{7!}$ ***M1***

Note: Award ***M1*** for attempting to write as a probability.

$\dfrac{2 \times 3 \times 4! \times 2}{7 \times 6 \times 5 \times 4!}$

$= \dfrac{2}{35}$ ***A1***

Note: Award ***A0*** if not fully simplified.

METHOD 2

$\dfrac{3}{7} \times \dfrac{2}{6} \times \dfrac{1}{5} + \dfrac{4}{7} \times \dfrac{3}{6} \times \dfrac{2}{5} \times \dfrac{1}{4}$ ***(M1)A1A1***

Note: Accept $\dfrac{3}{7} \times \dfrac{2}{6} \times \dfrac{1}{5} \times 2$ or $\dfrac{4}{7} \times \dfrac{3}{6} \times \dfrac{2}{5} \times \dfrac{1}{4} \times 2$.

$= \dfrac{2}{35}$ ***(M1)A1***

Note: Award ***A0*** if not fully simplified.

[5 marks]

22 – (MAT/12_HL_Summer_2017_Q7) – *Statistics, Probability*

(a) $\mathrm{P}(X > 0) = 1 - \mathrm{P}(X = 0)$ ***(M1)***

$\Rightarrow 1 - \mathrm{e}^{-m} = \dfrac{3}{4}$ or equivalent ***A1***

$\Rightarrow m = \ln 4$ ***A1***

[3 marks]

(b) $\mathrm{P}(Y > 1) = 1 - \mathrm{P}(Y = 0) - \mathrm{P}(Y = 1)$ ***(M1)***

$= 1 - \mathrm{e}^{-2\ln 4} - \mathrm{e}^{-2\ln 4} \times 2\ln 4$ ***A1***

recognition that $2\ln 4 = \ln 16$ ***(A1)***

$\mathrm{P}(Y > 1) = \dfrac{15 - \ln 16}{16}$ ***A1***

[4 marks]

23 – (MAT/11_HL_Summer_2017_Q10) – *Probability, Statistics*

(a) attempt to equate integral to 1 (may appear later) ***M1***

$$k\int_0^6 \sin\left(\frac{\pi x}{6}\right)\mathrm{d}x = 1$$

correct integral ***A1***

$$k\left[-\frac{6}{\pi}\cos\left(\frac{\pi x}{6}\right)\right]_0^6 = 1$$

substituting limits ***M1***

$$-\frac{6}{\pi}(-1-1) = \frac{1}{k}$$

$$k = \frac{\pi}{12}$$ ***A1***

[4 marks]

(b) (i) mean $= 3$ ***A1***

(ii) median $= 3$ ***A1***

(iii) mode $= 3$ ***A1***

Note: Award ***A1A0A0*** for three equal answers in $(0, 6)$.

[3 marks]

(c) (i) $\frac{\pi}{12}\int_0^2 \sin\left(\frac{\pi x}{6}\right)\mathrm{d}x$ ***M1***

$$= \frac{\pi}{12}\left[-\frac{6}{\pi}\cos\left(\frac{\pi x}{6}\right)\right]_0^2$$ ***A1***

Note: Accept without the $\frac{\pi}{12}$ at this stage if it is added later.

$$\frac{\pi}{12}\left[-\frac{6}{\pi}\left(\cos\frac{\pi}{3} - 1\right)\right]$$ ***M1***

$$= \frac{1}{4}$$ ***AG***

(ii) from (c)(i) $Q_1 = 2$ ***(A1)***

as the graph is symmetrical about the middle value $x = 3 \Rightarrow Q_3 = 4$ ***(A1)***

so interquartile range is

$4 - 2$ ***A1***

$= 2$

[6 marks]

(d) $P(X \le 4 | X \ge 3) = \dfrac{P(3 \le X \le 4)}{P(X \ge 3)}$

$= \dfrac{\frac{1}{4}}{\frac{1}{2}}$ ***(M1)***

$= \dfrac{1}{2}$ ***A1***

[2 marks]

24 – (MAT/11_HL_Summer_2018_Q3) – *Probability*

(a) $a = \dfrac{3}{16}$ and $b = \dfrac{5}{16}$ ***(M1)A1A1***

[3 marks]

Note: Award ***M1*** for consideration of the possible outcomes when rolling the two dice.

(b) $\mathrm{E}(T) = \dfrac{1+6+15+28}{16} = \dfrac{25}{8}\left(= 3.125\right)$ ***(M1)A1***

Note: Allow follow through from part (a) even if probabilities do not add up to 1.

[2 marks]

25 – (MAT/12_HL_Summer_2018_Q3) – *Probability*

(a) equating sum of probabilities to 1 ($p + 0.5 - p + 0.25 + 0.125 + p^3 = 1$) ***M1***

$p^3 = 0.125 = \dfrac{1}{8}$

$p = 0.5$ ***A1***

[2 marks]

(b) (i) $\mu = 0 \times 0.5 + 1 \times 0 + 2 \times 0.25 + 3 \times 0.125 + 4 \times 0.125$ ***M1***

$= 1.375 \left(= \dfrac{11}{8}\right)$ ***A1***

(ii) $\mathrm{P}(X > \mu) = \mathrm{P}(X = 2) + \mathrm{P}(X = 3) + \mathrm{P}(X = 4)$ ***(M1)***

$= 0.5$ ***A1***

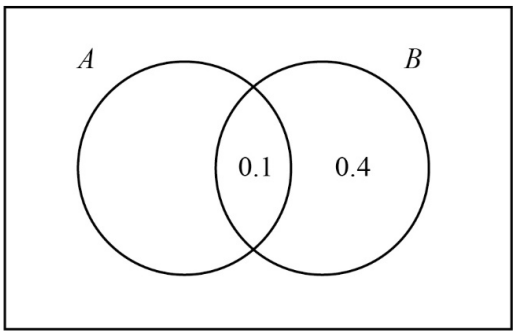
Note: Do not award follow through ***A*** marks in (b)(i) from an incorrect value of p.

Note: Award ***M*** marks in both (b)(i) and (b)(ii) provided no negative probabilities, and provided a numerical value for μ has been found.

[4 marks]

26 – (MAT/10_HL_Winter_2018_Q1) – *Probability*

(a)



(M1)

Note: Award ***M1*** for a Venn diagram with at least one probability in the correct region.

EITHER

$\mathrm{P}(A \cap B') = 0.3$ ***(A1)***

$\mathrm{P}(A \cup B) = 0.3 + 0.4 + 0.1 = 0.8$ ***A1***

OR

$\mathrm{P}(B) = 0.5$ ***(A1)***

$\mathrm{P}(A \cup B) = 0.5 + 0.4 - 0.1 = 0.8$ ***A1***

[3 marks]

(b) **METHOD 1**

$\mathrm{P}(A)\mathrm{P}(B) = 0.4 \times 0.5$ ***M1***

$= 0.2$ ***A1***

statement that their $\mathrm{P}(A)\mathrm{P}(B) \neq \mathrm{P}(A \cap B)$ ***R1***

Note: Award ***R1*** for correct reasoning from their value.

$\Rightarrow A, B$ not independent ***AG***

METHOD 2

$\mathrm{P}(A|B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \dfrac{0.1}{0.5}$ ***M1***

$= 0.2$ ***A1***

statement that their $\mathrm{P}(A|B) \neq \mathrm{P}(A)$ ***R1***

Note: Award ***R1*** for correct reasoning from their value.

$\Rightarrow A, B$ not independent ***AG***

Note: Accept equivalent argument using $\mathrm{P}(B|A) = 0.25$.

[3 marks]

27 – (MAT/11_HL_Summer_2019_Q6) – *Probability*

(a) use of symmetry *eg* diagram ***(M1)***

$P(X > \mu + 5) = 0.2$ ***A1***

[2 marks]

(b) **EITHER**

$P(X < \mu + 5 \mid X > \mu - 5) = \dfrac{P(X > \mu - 5 \cap X < \mu + 5)}{P(X > \mu - 5)}$ ***(M1)***

$= \dfrac{P(\mu - 5 < X < \mu + 5)}{P(X > \mu - 5)}$ ***(A1)***

$= \dfrac{0.6}{0.8}$ ***A1A1***

Note: ***A1*** for denominator is independent of the previous ***A*** marks.

OR

use of diagram ***(M1)***

Note: Only award ***(M1)*** if the region $\mu - 5 < X < \mu + 5$ is indicated and used.

$P(X > \mu - 5) = 0.8 \quad P(\mu - 5 < X < \mu + 5) = 0.6$ ***(A1)***

Note: Probabilities can be shown on the diagram.

$= \dfrac{0.6}{0.8}$ ***M1A1***

THEN

$= \dfrac{3}{4} = (0.75)$ ***A1***

[5 marks]

28 – (MAT/11_HL_Summer_2019_Q10) – *Remainder & Factor Theorem, Probability*

(a) (i) $p(2) = 8 - 12 + 16 - 24$ ***(M1)***

Note: Award ***M1*** for a valid attempt at remainder theorem or polynomial division.

$= -12$ ***A1***

remainder $= -12$

(ii) $p(3) = 27 - 27 + 24 - 24 = 0$ ***A1***

remainder $= 0$

[3 marks]

(b) $x = 3$ (is a zero) ***A1***

Note: Can be seen anywhere.

EITHER

factorise to get $(x-3)\left(x^2+8\right)$ ***(M1)A1***

$x^2 + 8 \neq 0$ (for $x \in \mathbb{R}$) (or equivalent statement) ***R1***

Note: Award ***R1*** if correct two complex roots are given.

OR

$p'(x) = 3x^2 - 6x + 8$ ***A1***

attempting to show $p'(x) \neq 0$ ***M1***

eg discriminant $= 36 - 96 < 0$, completing the square

no turning points ***R1***

THEN

only one real zero (as the curve is continuous) ***AG***

[4 marks]

(c) new graph is $y = p(2x)$ ***(M1)***

stretch parallel to the x-axis (with $x = 0$ invariant), scale factor 0.5 ***A1***

[2 marks]

Note: Accept "horizontal" instead of "parallel to the x-axis".

(d) $\frac{6\lambda^3 e^{-\lambda}}{6} = \frac{3\lambda^2 e^{-\lambda}}{2} - 2\lambda e^{-\lambda} + 3e^{-\lambda}$ ***M1A1***

Note: Allow factorials in the denominator for ***A1***.

$2\lambda^3 - 3\lambda^2 + 4\lambda - 6 = 0$ ***A1***

Note: Accept any correct cubic equation without factorials and $e^{-\lambda}$.

EITHER

$4\left(2\lambda^3 - 3\lambda^2 + 4\lambda - 6\right) = 8\lambda^3 - 12\lambda^2 + 16\lambda - 24 = 0$ ***(M1)***

$2\lambda = 3$ ***(A1)***

OR

$(2\lambda - 3)\left(\lambda^2 + 2\right) = 0$ ***(M1)(A1)***

THEN

$\lambda = 1.5$ ***A1***

[6 marks]

29 – (MAT/12_HL_Summer_2019_Q10) – *Integration, Probability*

(a) mode is 0 ***A1***

[1 mark]

(b) (i) attempt at integration by parts ***(M1)***

$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}, dv = dx$

$= x\arcsin x - \int \frac{x\,dx}{\sqrt{1-x^2}}$ ***A1***

$= x\arcsin x + \sqrt{1-x^2}\,(+c)$ ***A1***

(ii) $\int_0^1 (\pi - \arcsin x)\,dx = \left[\pi x - x\arcsin x - \sqrt{1-x^2}\right]_0^1$ ***A1***

$= \left(\pi - \frac{\pi}{2} - 0\right) - (0 - 0 - 1) = \frac{\pi}{2} + 1$

$= \frac{\pi + 2}{2}$ ***A1***

$\int_0^1 k(\pi - \arcsin x)\,dx = 1$ ***(M1)***

Note: This line can be seen (or implied) anywhere.

Note: Do not allow ***FT A*** marks from bi to bii.

$k\left(\frac{\pi+2}{2}\right) = 1$

$\Rightarrow k = \frac{2}{2+\pi}$ ***AG***

[6 marks]

(c) (i) attempt to use product rule to differentiate ***M1***

$\frac{dy}{dx} = x\arcsin x + \frac{x^2}{2\sqrt{1-x^2}} - \frac{1}{4\sqrt{1-x^2}} - \frac{x^2}{4\sqrt{1-x^2}} + \frac{\sqrt{1-x^2}}{4}$ ***A2***

Note: Award ***A2*** for all terms correct, ***A1*** for 4 correct terms.

$= x\arcsin x + \frac{2x^2}{4\sqrt{1-x^2}} - \frac{1}{4\sqrt{1-x^2}} - \frac{x^2}{4\sqrt{1-x^2}} + \frac{1-x^2}{4\sqrt{1-x^2}}$ ***A1***

Note: Award ***A1*** for equivalent combination of correct terms over a common denominator.

$= x\arcsin x$ ***AG***

(ii) $\mathrm{E}(X) = k\int_0^1 x(\pi - \arcsin x)\,\mathrm{d}x$ ***M1***

$= k\int_0^1 (\pi x - x\arcsin x)\,\mathrm{d}x$

$= k\left[\frac{\pi x^2}{2} - \frac{x^2}{2}\arcsin x + \frac{1}{4}\arcsin x - \frac{x}{4}\sqrt{1-x^2}\right]_0^1$ ***A1A1***

Note: Award ***A1*** for first term, ***A1*** for next 3 terms.

$= k\left[\left(\frac{\pi}{2} - \frac{\pi}{4} + \frac{\pi}{8}\right) - (0)\right]$ ***A1***

$= \left(\frac{2}{2+\pi}\right)\frac{3\pi}{8}$ ***A1***

$= \frac{3\pi}{4(\pi+2)}$ ***AG***

[9 marks]

30 – (MAT/10_HL_Winter_2019_Q1) – *Probability*

(a) $p = 1 - \frac{1}{2} - \frac{1}{5} - \frac{1}{5}$ ***(M1)***

$= \frac{1}{10}$ ***A1***

[2 marks]

(b) attempt to find $\mathrm{E}(X)$ ***(M1)***

$\frac{1}{2} + 1 + 2 + \frac{N}{10} = 10$ ***A1***

$\Rightarrow N = 65$ ***A1***

Note: Do not allow FT in part (b) if their p is outside the range $0 < p < 1$.

[3 marks]

31 – (MAT/10_HL_Winter_2020_Q1) – *Probability*

$\mathrm{E}(X) = (0 \times p) + \left(1 \times \frac{1}{4}\right) + \left(2 \times \frac{1}{6}\right) + 3q \left(= \frac{19}{12}\right)$ ***(M1)***

$\left(\Rightarrow \frac{1}{4} + \frac{1}{3} + 3q = \frac{19}{12}\right)$

$q = \frac{1}{3}$ ***A1***

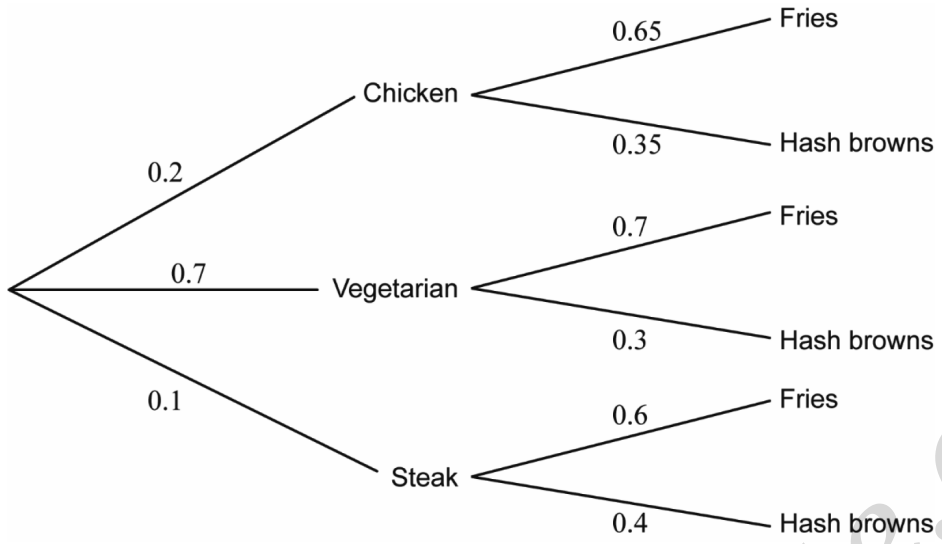
$p + \frac{1}{4} + \frac{1}{6} + q = 1$ ***(M1)***

$\left(\Rightarrow p + q = \frac{7}{12}\right)$

$p = \frac{1}{4}$ ***A1***

32 – (MAT/10_HL_Winter_2020_Q3) – *Probability*

(a)



A1A1

Note: Award ***A1*** for probabilities for type of omelette and ***A1*** for probabilities for fries / hash browns.

[2 marks]

(b) $(0.2\times0.65)+(0.7\times0.7)+(0.1\times0.6)$ ***(M1)***

$=0.68\left(=\frac{17}{25}\right)$ ***A1***

[2 marks]

(c) $\frac{\text{P(ordered fries and did not order chicken omelette)}}{\text{P(did not order chicken omelette)}}$ ***(M1)***

$\frac{0.7\times0.7+0.1\times0.6}{0.7+0.1}\left(=\frac{0.49+0.06}{0.8}=\frac{0.55}{0.8}\right)$ ***(A1)***

$=\frac{55}{80}\left(=\frac{11}{16}\right)$ ***A1***

[3 marks]

33 – (MathAA/11_HL_Summer_2021_Q10) – *Probability*

(a) recognising probabilities sum to 1 ***(M1)***

$p+p+p+\frac{1}{2}p=1$

$p=\frac{2}{7}$ ***A1***

[2 marks]

(b) valid attempt to find $\mathrm{E}(X)$ ***(M1)***

$1\times p+2\times p+3\times p+4\times\frac{1}{2}p\left(=8p\right)$

$\mathrm{E}(X)=\frac{16}{7}$ ***A1***

[2 marks]

(c) (i) $0\le r\le 1$ ***A1***

(ii) Attempt to find a value of q ***(M1)***

$0\le 1-3q\le 1$ OR $r=0\Rightarrow q=\frac{1}{3}$ OR $r=1\Rightarrow q=0$

$0\le q\le\frac{1}{3}$ ***A1***

[3 marks]

(d) $\mathrm{E}(Y)=1\times q+2\times q+3\times q+4\times r\left(=2+2r \text{ OR } 4-6q\right)$ ***(A1)***

one correct boundary value ***A1***

$1\times\frac{1}{3}+2\times\frac{1}{3}+3\times\frac{1}{3}+4\times 0\left(=2\right)$ OR

$1\times 0+2\times 0+3\times 0+4\times 1\left(=4\right)$ OR

$2+2(0)\left(=2\right)$ OR

$2+2(1)\left(=4\right)$ OR

$4-6(0)\left(=4\right)$ OR $4-6\left(\frac{1}{3}\right)\left(=2\right)$

$2\le\mathrm{E}(Y)\le 4$ ***A1***

[3 marks]

(e) **METHOD 1**

evidence of choosing at least four correct outcomes from
1&2, 1&3, 1&4, 2&3, 2&4, 3&4 ***(M1)***

$\frac{6}{7}q+\frac{6}{7}r$ OR $3pq+3pr$ OR $pq+pq+p(1-3q)+pq+p(1-3q)+p(1-3q)$ ***(A1)***

solving for either q or r ***M1***

$\frac{6}{7}(q+1-3q)=\frac{1}{2}$ OR $\frac{6}{7}\left(\frac{1-r}{3}+r\right)=\frac{1}{2}$ OR $3pq+3p(1-3q)=\frac{1}{2}$

OR $3p\left(\frac{1-r}{3}\right)+3pr=\frac{1}{2}$

EITHER two correct values

$q=\frac{5}{24}$ and $r=\frac{3}{8}$ ***A1A1***

OR one correct value

$q=\frac{5}{24}$ OR $r=\frac{3}{8}$ ***A1***

substituting their value for q or r ***A1***

$4-6\left(\frac{5}{24}\right)$ OR $2+2\left(\frac{3}{8}\right)$

THEN

$\mathrm{E}(Y)=\frac{11}{4}$ ***A1***

[6 marks]

METHOD 2 (solving for $\mathrm{E}(Y)$)
evidence of choosing at least four correct outcomes from
1&2, 1&3, 1&4, 2&3, 2&4, 3&4 ***(M1)***

$\frac{6}{7}q+\frac{6}{7}r$ OR $3pq+3pr$ OR $pq+pq+p(1-3q)+pq+p(1-3q)+p(1-3q)$ ***(A1)***

rearranging to make q the subject ***M1***

$q=\frac{4-\mathrm{E}(Y)}{6}$

$3pq+3p(1-3q)=\frac{1}{2}$ ***M1***

$\frac{6}{7}\times\left(\frac{4-\mathrm{E}(Y)}{6}\right)+\frac{6}{7}\left(1-3\left(\frac{4-\mathrm{E}(Y)}{6}\right)\right)=\frac{1}{2}$ ***A1***

$\frac{2(\mathrm{E}(Y)-1)}{7}=\frac{1}{2}$

$\mathrm{E}(Y)=\frac{11}{4}$ ***A1***

[6 marks]

34 – (MathAA/10_HL_Winter_2021_Q4) – *Probability*

(a) valid approach to find $\mathrm{P}(R)$ ***(M1)***

tree diagram (must include probabilty of picking box) with correct required probabilities

OR $\mathrm{P}(R\cap B_1)+\mathrm{P}(R\cap B_2)$ OR $\mathrm{P}(R\mid B_1)\mathrm{P}(B_1)+\mathrm{P}(R\mid B_2)\mathrm{P}(B_2)$

$\frac{5}{7}\cdot\frac{1}{2}+\frac{4}{7}\cdot\frac{1}{2}$ ***(A1)***

$\mathrm{P}(R)=\frac{9}{14}$ ***A1***

[3 marks]

(b) events A and R are not independent, since $\frac{9}{14}\cdot\frac{1}{2}\neq\frac{5}{14}$ OR $\frac{5}{7}\neq\frac{9}{14}$ OR $\frac{5}{9}\neq\frac{1}{2}$

OR an explanation e.g. different number of red balls in each box ***A2***

Note: Both conclusion and reasoning are required. Do not split the ***A2***.

[2 marks]

Total [5 marks]

35 – (MathAA/11_HL_Summer_2022_Q7) – *Probability*

(a) attempt to integrate $\frac{k}{\sqrt{4-3x^2}}$ ***(M1)***

$= k\left[\frac{1}{\sqrt{3}}\arcsin\left(\frac{\sqrt{3}}{2}x\right)\right]$ ***A1***

Note: Award ***(M1)A0*** for $\arcsin\left(\frac{\sqrt{3}}{2}x\right)$.

Condone absence of k up to this stage.

equating their integrand to 1 ***M1***

$k\left[\frac{1}{\sqrt{3}}\arcsin\left(\frac{\sqrt{3}}{2}x\right)\right]_0^1 = 1$

$k = \frac{3\sqrt{3}}{\pi}$ ***A1***

[4 marks]

(b) $\mathrm{E}(X)=\frac{3\sqrt{3}}{\pi}\int_0^1 \frac{x}{\sqrt{4-3x^2}}\,\mathrm{d}x$ ***A1***

Note: Condone absence of limits if seen at a later stage.

EITHER

attempt to integrate by inspection ***(M1)***

$=\frac{3\sqrt{3}}{\pi}\times -\frac{1}{6}\int -6x\left(4-3x^2\right)^{-\frac{1}{2}}\,\mathrm{d}x$

$=\frac{3\sqrt{3}}{\pi}\left[-\frac{1}{3}\sqrt{4-3x^2}\right]_0^1$ ***A1***

Note: Condone the use of k up to this stage.

OR

for example, $u=4-3x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}=-6x$

Note: Other substitutions may be used. For example, $u=-3x^2$.

$=-\frac{\sqrt{3}}{2\pi}\int_4^1 u^{-\frac{1}{2}}\,\mathrm{d}u$ ***M1***

Note: Condone absence of limits up to this stage.

$=-\frac{\sqrt{3}}{2\pi}\left[2\sqrt{u}\right]_4^1$ ***A1***

Note: Condone the use of k up to this stage.

THEN

$=\frac{\sqrt{3}}{\pi}$ ***A1***

Note: Award ***A0M1A1A0*** for their $k\left[-\frac{1}{3}\sqrt{4-3x^2}\right]$ or $k\left[-2\sqrt{u}\right]$ for working with incorrect or no limits.

[4 marks]

36 – (MathAA/12_HL_Summer_2022_Q8) – *Probability*

let m be the median

EITHER

attempts to find the area of the required triangle ***M1***

base is $(m-a)$ ***(A1)***

and height is $\frac{2}{(b-a)(c-a)}(m-a)$

area $=\frac{1}{2}(m-a)\times\frac{2}{(b-a)(c-a)}(m-a)\ \left(=\frac{(m-a)^2}{(b-a)(c-a)}\right)$ ***A1***

OR

attempts to integrate the correct function ***M1***

$\int_a^m \frac{2}{(b-a)(c-a)}(x-a)\,\mathrm{d}x$

$=\frac{2}{(b-a)(c-a)}\left[\frac{1}{2}(x-a)^2\right]_a^m$ OR $\frac{2}{(b-a)(c-a)}\left[\frac{x^2}{2}-ax\right]_a^m$ ***A1A1***

Note: Award ***A1*** for correct integration and ***A1*** for correct limits.

THEN

sets up (their) $\int_a^m \frac{2}{(b-a)(c-a)}(x-a)\,\mathrm{d}x$ or area $=\frac{1}{2}$ ***M1***

Note: Award ***M0A0A0M1A0A0*** if candidates conclude that $m>c$ and set up their area or sum of integrals $=\frac{1}{2}$.

$\frac{(m-a)^2}{(b-a)(c-a)}=\frac{1}{2}$

$m=a\pm\sqrt{\frac{(b-a)(c-a)}{2}}$ ***(A1)***

as $m>a$, rejects $m=a-\sqrt{\frac{(b-a)(c-a)}{2}}$

so $m=a+\sqrt{\frac{(b-a)(c-a)}{2}}$ ***A1***

[6 marks]

37 – (MathAA/12_HL_Summer_2022_Q10) – *Probability*

(a) uses $\sum \mathrm{P}(X=x)=1$ to form a linear equation in p and q ***(M1)***

correct equation in terms of p and q from summing to 1 ***A1***

$p+0.3+q+0.1=1$ OR $p+q=0.6$ (or equivalent)

uses $\mathrm{E}(X)=2$ to form a linear equation in p and q ***(M1)***

correct equation in terms of p and q from $\mathrm{E}(X)=2$ ***A1***

$p+0.6+3q+0.4=2$ OR $p+3q=1$ (or equivalent)

Note: The marks for using $\sum \mathrm{P}(X=x)=1$ and the marks for using $\mathrm{E}(X)=2$ may be awarded independently of each other.

evidence of correctly solving these equations simultaneously ***A1***

for example, $2q=0.4 \Rightarrow q=0.2$ or $p+3\times(0.6-p)=1 \Rightarrow p=0.4$

so $p=0.4$ and $q=0.2$ ***AG***

[5 marks]

(b) valid approach ***(M1)***

$\mathrm{P}(X>2)=\mathrm{P}(X=3)+\mathrm{P}(X=4)$ OR $\mathrm{P}(X>2)=1-\mathrm{P}(X=1)-\mathrm{P}(X=2)$

$=0.3$ ***A1***

[2 marks]

(c) recognises at least one of the valid scores (6, 7, or 8) required to win the game ***(M1)***

Note: Award ***M0*** if candidate also considers scores other than 6, 7, or 8 (such as 5).

let T represent the score on the last two rolls

a score of 6 is obtained by rolling $(2,4),(4,2)$ or $(3,3)$

$\mathrm{P}(T=6)=2(0.3)(0.1)+(0.2)^2 \ (=0.1)$ ***A1***

a score of 7 is obtained by rolling $(3,4)$ or $(4,3)$

$\mathrm{P}(T=7)=2(0.2)(0.1) \ (=0.04)$ ***A1***

a score of 8 is obtained by rolling $(4,4)$

$\mathrm{P}(T=8)=(0.1)^2 \ (=0.01)$ ***A1***

Note: The above 3 ***A1*** marks are independent of each other.

$\mathrm{P}(\text{Nicky wins})=0.1+0.04+0.01$

$=0.15$ ***A1***

[5 marks]

(d) $3+b=8$ ***(M1)***

$b=5$ ***A1***

[2 marks]

(e) **METHOD 1**

EITHER

$P(S=5)=\frac{4}{16}$

$P(S=a+2)=\frac{4}{16}$ ***A1***

$\Rightarrow a+2=5$

OR

$P(S=6)=\frac{3}{16}$

$P(S=a+3)=\frac{2}{16}$ and $P(S=5+1)=\frac{1}{16}$ ***A1***

$\Rightarrow a+3=6$

OR

$P(S=4)=\frac{3}{16}$

$P(S=a+1)=\frac{2}{16}$ and $P(S=1+3)=\frac{1}{16}$ ***A1***

$\Rightarrow a+1=4$

THEN

$\Rightarrow a=3$ ***A1***

Note: Award ***A0A0*** for $a=3$ obtained without working/reasoning/justification.

[2 marks]

METHOD 2

EITHER

correctly lists a relevant part of the sample space ***A1***

for example, $\{S=4\}=\{(3,1),(1,a),(1,a)\}$ or $\{S=5\}=\{(2,a),(2,a),(2,a),(2,a)\}$

or $\{S=6\}=\{(3,a),(3,a),(1,5)\}$

$a+3=6$

OR

eliminates possibilities (exhaustion) for $a<5$

convincingly shows that $a\neq 2,4$ ***A1***

$a\neq 4$, for example, $\mathrm{P}(S=7)=\frac{2}{16}$ from $(2,5),(2,5)$ and so $(3,a),(3,a)\Rightarrow a+3\neq 7$

THEN

$\Rightarrow a=3$ ***A1***

[2 marks]

38 – (MathAA/10_HL_Winter_2022_Q6) – *Probability*

(a) $\mathrm{P}(A \cap B) = 0.24$ ***A1***

[1 mark]

(b) $\mathrm{P}(A \cup B) = 1.1 - \mathrm{P}(A \cap B)$ ***(A1)***

$(0 \le)\mathrm{P}(A \cup B) \le 1$ ***(M1)***

Note: This may be conveyed in a clearly labelled diagram or written explanation where $\mathrm{P}(A \cup B) = 1$

the minimum value of $\mathrm{P}(A \cap B)$ is 0.1 ***A1***

[3 marks]

(c) A is a subset of B (so $\mathrm{P}(A \cap B) = \mathrm{P}(A)$). ***R1***

Note: This may be conveyed in a clearly labelled diagram where A is completely inside B, or in a written explanation indicating that $\mathrm{P}(A \cap B) = \mathrm{P}(A)$

so the maximum value of $\mathrm{P}(A \cap B)$ is 0.3 ***A1***

Note: Do not award ***R0A1***.

[2 marks]

1 - (MAT/21_HL_Summer_2012_Q1) - *Differentiation*

$$\frac{dy}{dx} = 3x^2 - 12x + k$$

For use of discriminant $b^2 - 4ac = 0$ or completing the square $3(x-2)^2 + k - 12$

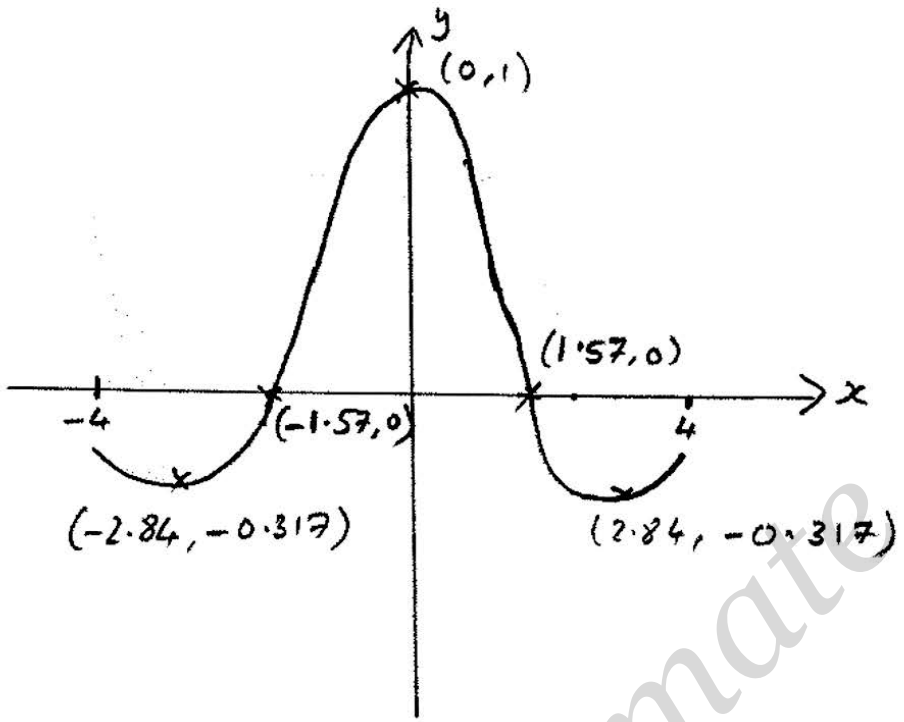
$144 - 12k = 0$

Note: Accept trial and error, sketches of parabolas with vertex (2,0) or use of second derivative.

$k = 12$

2 – (MAT/22_HL_Summer_2012_Q6) – *Differentiation*

(a)



Note:	Award *A1* for correct shape. Do not penalise if too large a domain is used, *A1* for correct x-intercepts, *A1* for correct coordinates of two minimum points, *A1* for correct coordinates of maximum point. Accept answers which correctly indicate the position of the intercepts, maximum point and minimum points.

(b) gradient at $x = 1$ is -0.786

(c) gradient of normal is $\frac{-1}{-0.786} (= 1.272\ldots)$

when $x = 1$, $y = 0.3820...$

Equation of normal is $y - 0.382 = 1.27(x - 1)$

$(\Rightarrow y = 1.27x - 0.890)$

3 - (MAT/21_HL_Summer_2012_Q10) - *Differentiation*

attempt to find intersections

intersections are $\left(\frac{10}{m+2}, \frac{10m}{m+2}\right)$ and $\left(\frac{10m}{2m-1}, -\frac{10}{2m-1}\right)$

area of triangle $= \frac{1}{2} \times \frac{\sqrt{100+100m^2}}{(m+2)} \times \frac{\sqrt{100+100m^2}}{(2m-1)}$

$= \frac{50(1+m^2)}{(m+2)(2m-1)}$

minimum when $m = 3$

4 – (MAT/21_HL_Summer_2012_Q11) – *Differentiation*

(a) $(3.79, -5)$

(b) $p = 1.57$ or $\frac{\pi}{2}$, $q = 6.00$

(c) $f'(x) = 3\cos x - 4\sin x$
$3\cos x - 4\sin x = 3 \Rightarrow x = 4.43\ldots$
$(y = -4)$

Coordinates are $(4.43, -4)$

(d) $m_{\text{normal}} = -\frac{1}{m_{\text{tangent}}}$

gradient at P is -4 so gradient of normal at P is $\frac{1}{4}$

gradient at Q is 4 so gradient of normal at Q is $-\frac{1}{4}$

equation of normal at P is $y - 3 = \frac{1}{4}(x - 1.570\ldots)$ $(\text{or } y = 0.25x + 2.60\ldots)$

equation of normal at Q is $y - 3 = -\frac{1}{4}\left(x - 5.999\ldots\right)\left(\text{or } y = -0.25x + \underbrace{4.499\ldots}\right)$

Note: Award the previous two ***M1*** even if the gradients are incorrect in $y - b = m(x - a)$ where (a, b) are coordinates of P and Q (or in $y = mx + c$ with c determined using coordinates of Pand Q.

intersect at $(3.79, 3.55)$

Note: Award ***N2*** for 3.79 without other working.

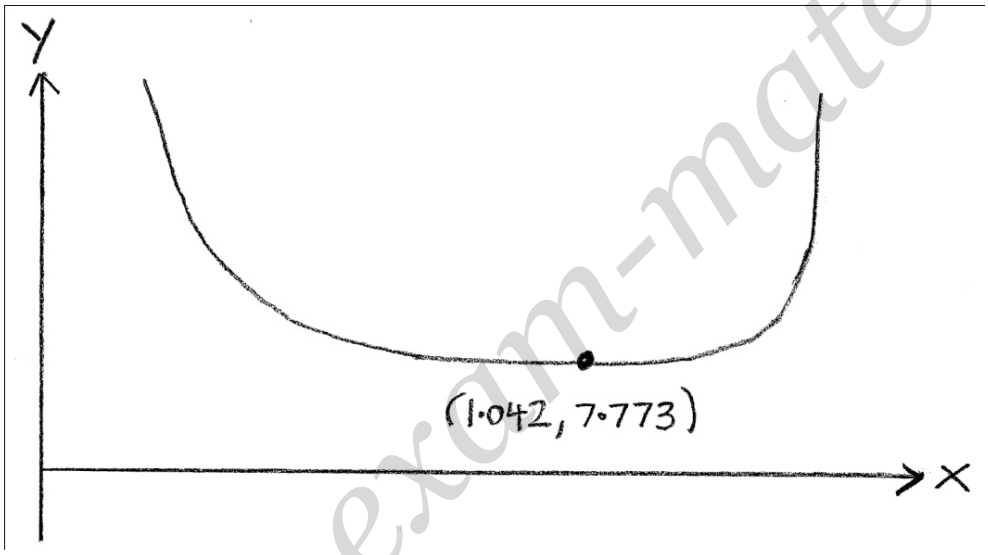
5 – (MAT/20_HL_Winter_2012_Q12) – *Differentiation*

(a) $L = \text{CA} + \text{AD}$

$$\sin\alpha = \frac{a}{\text{CA}} \Rightarrow \text{CA} = \frac{a}{\sin\alpha}$$
$$\cos\alpha = \frac{b}{\text{AD}} \Rightarrow \text{AD} = \frac{b}{\cos\alpha}$$
$$L = \frac{a}{\sin\alpha} + \frac{b}{\cos\alpha}$$

(b) $a = 5$ and $b = 1 \Rightarrow L = \dfrac{5}{\sin\alpha} + \dfrac{1}{\cos\alpha}$

METHOD 1



minimum from graph $\Rightarrow$ $L = 7.77$
minimum of L gives the max length of the painting

METHOD 2

$$\frac{\mathrm{d}L}{\mathrm{d}\alpha} = \frac{-5\cos\alpha}{\sin^2\alpha} + \frac{\sin\alpha}{\cos^2\alpha}$$
$$\frac{\mathrm{d}L}{\mathrm{d}\alpha} = 0 \Rightarrow \frac{\sin^3\alpha}{\cos^3\alpha} = 5 \Rightarrow \tan\alpha = \sqrt[3]{5} \quad (\alpha = 1.0416\ldots)$$
minimum of L gives the max length of the painting
maximum length $= 7.77$

(c) $\dfrac{\mathrm{d}L}{\mathrm{d}\alpha} = \dfrac{-3k\cos\alpha}{\sin^2\alpha} + \dfrac{k\sin\alpha}{\cos^2\alpha}$ (or equivalent)

(d) $\dfrac{\mathrm{d}L}{\mathrm{d}\alpha} = \dfrac{-3k\cos^3\alpha + k\sin^3\alpha}{\sin^2\alpha\cos^2\alpha}$

$\dfrac{\mathrm{d}L}{\mathrm{d}\alpha} = 0 \Rightarrow \dfrac{\sin^3\alpha}{\cos^3\alpha} = \dfrac{3k}{k} \Rightarrow \tan\alpha = \sqrt[3]{3} \quad (\alpha = 0.96454\ldots)$

$\tan\alpha = \sqrt[3]{3} \Rightarrow \dfrac{1}{\cos\alpha} = \sqrt{1+\sqrt[3]{9}} \qquad (1.755\ldots)$

and $\dfrac{1}{\sin\alpha} = \dfrac{\sqrt{1+\sqrt[3]{9}}}{\sqrt[3]{3}} \qquad (1.216\ldots)$

$L = 3k\left(\dfrac{\sqrt{1+\sqrt[3]{9}}}{\sqrt[3]{3}}\right) + k\sqrt{1+\sqrt[3]{9}} \qquad (L = 5.405598\ldots k)$

(e) $L \le 8 \Rightarrow k \ge 1.48$

the minimum value is 1.48

6 – (MAT/22_HL_Summer_2013_Q12) – *Differentiation, Integration*

(a) (i) **METHOD 1**

$$\frac{dy}{dx} = -\sin x + \cos x$$

$$y\frac{dy}{dx} = (\cos x + \sin x)(-\sin x + \cos x)$$

$$= \cos^2 x - \sin^2 x$$

$$= \cos 2x$$

METHOD 2

$$y^2 = (\sin x + \cos x)^2$$

$$2y\frac{dy}{dx} = 2(\cos x + \sin x)(\cos x - \sin x)$$

$$y\frac{dy}{dx} = \cos^2 x - \sin^2 x$$

$$= \cos 2x$$

(ii) attempting to separate variables $\int y\,dy = \int \cos 2x\,dx$

$$\frac{1}{2}y^2 = \frac{1}{2}\sin 2x + C$$

Note:	Award ***A1*** for a correct LHS and ***A1*** for a correct RHS.

$$y = \pm(\sin 2x + A)^{\frac{1}{2}}$$

(iii) $\sin 2x + A \equiv (\cos x + \sin x)^2$

$(\cos x + \sin x)^2 = \cos^2 x + 2\sin x\cos x + \sin^2 x$

use of $\sin 2x \equiv 2\sin x\cos x$.

$A = 1$

(b) (i) substituting $x=\frac{\pi}{4}$ and $y=2$ into $y=(\sin 2x+A)^{\frac{1}{2}}$

so $g(x)=(\sin 2x+3)^{\frac{1}{2}}$.

range g is $\left[\sqrt{2}, 2\right]$

Note: Accept $[1.41, 2]$. Award ***A1*** for each correct endpoint and ***A1*** for the correct closed interval.

(ii) $\int_0^{\frac{\pi}{2}}(\sin 2x+3)^{\frac{1}{2}}\,\mathrm{d}x$

$=2.99$

(iii) $\pi\int_0^{\frac{\pi}{2}}(\sin 2x+3)\,\mathrm{d}x-\pi(1)\left(\frac{\pi}{2}\right)$ (or equivalent)

Note: Award ***(M1)(A1)(A1)*** for $\pi\int_0^{\frac{\pi}{2}}(\sin 2x+2)\mathrm{d}x$

$=17.946-4.935\ \left(=\frac{\pi}{2}(3\pi+2)-\pi\left(\frac{\pi}{2}\right)\right)$

$=13.0$

Note: Award ***A1*** for $\pi(\pi+1)$.

7 – (MAT/21_HL_Summer_2013_Q13) – *Differentiation*

(a) $f(2) = 9$

$f^{-1}(x) = (x-1)^{\frac{1}{3}}$

$(f^{-1})'(x) = \frac{1}{3}(x-1)^{-\frac{2}{3}}$

$(f^{-1})'(9) = \frac{1}{12}$

$f'(x) = 3x^2$

$\frac{1}{f'(2)} = \frac{1}{3 \times 4} = \frac{1}{12}$

Note: The last ***M1*** and ***A1*** are independent of previous marks.

(b) $g'(x) = e^{x^2} + 2x^2 e^{x^2}$

$g'(x) > 0$ as each part is positive

(c) to find the x-coordinate on $y = g(x)$ solve

$2 = xe^{x^2}$

$x = 0.89605022078\ldots$

gradient $= (g^{-1})'(2) = \dfrac{1}{g'(0.896\ldots)}$

$= \dfrac{1}{e^{(0.896\ldots)^2}\left(1 + 2\times(0.896\ldots)^2\right)} = 0.172$ to 3sf

(using the $\dfrac{dy}{dx}$ function on gdc $g'(0.896\ldots) = 5.7716028\ldots$

$\dfrac{1}{g'(0.896\ldots)} = 0.173$)

(d) (i) $(x^3+1)e^{(x^3+1)^2} = 2$

$x = -0.470191\ldots$

(ii) **METHOD 1**

$(g\circ f)'(x) = 3x^2e^{(x^3+1)^2}\left(2(x^3+1)^2+1\right)$

$(g\circ f)'(-0.470191\ldots) = 3.85755\ldots$

$h'(2) = \dfrac{1}{3.85755\ldots} = 0.259\ (232\ldots)$

Note: The solution can be found without the student obtaining the explicit form of the composite function.

METHOD 2

$h(x) = (f^{-1}\circ g^{-1})(x)$

$h'(x) = (f^{-1})'\left(g^{-1}(x)\right)\times(g^{-1})'(x)$

$= \dfrac{1}{3}\left(g^{-1}(x)-1\right)^{-\frac{2}{3}}\times(g^{-1})'(x)$

$h'(2) = \dfrac{1}{3}\left(g^{-1}(2)-1\right)^{-\frac{2}{3}}\times(g^{-1})'(2)$

$= \dfrac{1}{3}(0.89605\ldots-1)^{-\frac{2}{3}}\times 0.171933\ldots$

$= 0.259\ (232\ldots)$

8 – (MAT/22_HL_Summer_2013_Q13) – *Differentiation*

(a) **EITHER**

$$\theta = \pi - \arctan\left(\frac{8}{x}\right) - \arctan\left(\frac{13}{20-x}\right) \text{ (or equivalent)}$$

Note: Accept $\theta = 180° - \arctan\left(\frac{8}{x}\right) - \arctan\left(\frac{13}{20-x}\right)$ (or equivalent).

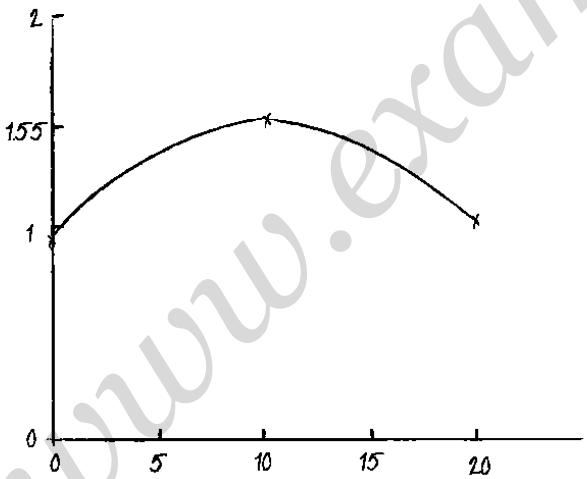
OR

$$\theta = \arctan\left(\frac{x}{8}\right) + \arctan\left(\frac{20-x}{13}\right) \text{ (or equivalent)}$$

(b) (i) $\theta = 0.994 \ \left(= \arctan\frac{20}{13}\right)$

(ii) $\theta = 1.19 \ \left(= \arctan\frac{5}{2}\right)$

(c) correct shape.
correct domain indicated.



(d) attempting to differentiate one $\arctan(f(x))$ term

EITHER

$$\theta = \pi - \arctan\left(\frac{8}{x}\right) - \arctan\left(\frac{13}{20-x}\right)$$

$$\frac{d\theta}{dx} = \frac{8}{x^2} \times \frac{1}{1+\left(\frac{8}{x}\right)^2} - \frac{13}{(20-x)^2} \times \frac{1}{1+\left(\frac{13}{20-x}\right)^2}$$

OR

$$\theta = \arctan\left(\frac{x}{8}\right) + \arctan\left(\frac{20-x}{13}\right)$$

$$\frac{d\theta}{dx} = \frac{\frac{1}{8}}{1+\left(\frac{x}{8}\right)^2} + \frac{-\frac{1}{13}}{1+\left(\frac{20-x}{13}\right)^2}$$

THEN

$$= \frac{8}{x^2+64} - \frac{13}{569-40x+x^2}$$

$$= \frac{8(569-40x+x^2)-13(x^2+64)}{(x^2+64)(x^2-40x+569)}$$

$$= \frac{5(744-64x-x^2)}{(x^2+64)(x^2-40x+569)}$$

(e) Maximum light intensity at P occurs when $\frac{d\theta}{dx} = 0$.

either attempting to solve $\frac{d\theta}{dx} = 0$ for x or using the graph of either θ or $\frac{d\theta}{dx}$

$x = 10.05$ (m)

(f) $\frac{dx}{dt} = 0.5$

At $x = 10$, $\frac{d\theta}{dx} = 0.000453$ $(= \frac{5}{11\,029})$.

use of $\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt}$

$\frac{d\theta}{dt} = 0.000227$ $(= \frac{5}{22058})$ (rad s^{-1})

9 – (MAT/22_HL_Summer_2014_Q9) – *Differentiation*

METHOD 1

volume of a cone is $V = \frac{1}{3}\pi r^2 h$

given $h = r$, $V = \frac{1}{3}\pi h^3$

$\frac{dV}{dh} = \pi h^2$

when $h = 4$, $\frac{dV}{dt} = \pi \times 4^2 \times 0.5$ (using $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$)

$\frac{dV}{dt} = 8\pi \ (= 25.1) \ (\text{cm}^3\,\text{min}^{-1})$

METHOD 2

volume of a cone is $V = \frac{1}{3}\pi r^2 h$

given $h = r$, $V = \frac{1}{3}\pi h^3$

$\frac{dV}{dt} = \frac{1}{3}\pi \times 3h^2 \times \frac{dh}{dt}$

when $h = 4$, $\frac{dV}{dt} = \pi \times 4^2 \times 0.5$

$\frac{dV}{dt} = 8\pi \ (= 25.1) \ (\text{cm}^3\,\text{min}^{-1})$

METHOD 3

$V = \frac{1}{3}\pi r^2 h$

$\frac{dV}{dt} = \frac{1}{3}\pi\left(2r\,h\frac{dr}{dt} + r^2\frac{dh}{dt}\right)$

Note: Award ***M1*** for attempted implicit differentiation and ***A1*** for each correct term on the RHS.

when $h = 4$, $r = 4$, $\frac{dV}{dt} = \frac{1}{3}\pi\left(2 \times 4 \times 4 \times 0.5 + 4^2 \times 0.5\right)$

$\frac{dV}{dt} = 8\pi \ (= 25.1) \ (\text{cm}^3\,\text{min}^{-1})$

10 – (MAT/21_HL_Summer_2014_Q10) – *Functions - Roots, Differentiation*

(a) $x \to -\infty \Rightarrow y \to -\frac{1}{2}$ so $y = -\frac{1}{2}$ is an asymptote

$e^x - 2 = 0 \Rightarrow x = \ln 2$ so $x = \ln 2 (= 0.693)$ is an asymptote

(b) (i) $f'(x) = \dfrac{2\left(e^x - 2\right)e^{2x} - \left(e^{2x} + 1\right)e^x}{\left(e^x - 2\right)^2}$

$= \dfrac{e^{3x} - 4e^{2x} - e^x}{\left(e^x - 2\right)^2}$

(ii) $f'(x) = 0$ when $e^{3x} - 4e^{2x} - e^x = 0$

$e^x\left(e^{2x} - 4e^x - 1\right) = 0$

$e^x = 0,\ e^x = -0.236,\ e^x = 4.24\ \left(\text{or } e^x = 2 \pm \sqrt{5}\right)$

Note: Award *A1* for zero, *A1* for other two solutions.
Accept any answers which show a zero, a negative and a positive.

as $e^x > 0$ exactly one solution

Note: Do not award marks for purely graphical solution.

(iii) $(1.44, 8.47)$

(c) $f'(0) = -4$

so gradient of normal is $\frac{1}{4}$

$f(0) = -2$

so equation of L_1 is $y = \frac{1}{4}x - 2$

(d) $f'(x) = \frac{1}{4}$

so $x = 1.46$

$f(1.46) = 8.47$

equation of L_2 is $y - 8.47 = \frac{1}{4}(x - 1.46)$

(or $y = \frac{1}{4}x + 8.11$)

11 – (MAT/22_HL_Summer_2014_Q10) – *Differentiation*

(a) **METHOD 1**

expanding the brackets first:

$x^4 + 2x^2y^2 + y^4 = 4xy^2$

$4x^3 + 4xy^2 + 4x^2y\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 4y^2 + 8xy\frac{dy}{dx}$

Note:	Award ***M1*** for an attempt at implicit differentiation. Award ***A1*** for each side correct.

$\frac{dy}{dx} = \frac{-x^3 - xy^2 + y^2}{yx^2 - 2xy + y^3}$ or equivalent

METHOD 2

$2\left(x^2 + y^2\right)\left(2x + 2y\frac{dy}{dx}\right) = 4y^2 + 8xy\frac{dy}{dx}$

Note:	Award ***M1*** for an attempt at implicit differentiation. Award ***A1*** for each side correct.

$\left(x^2 + y^2\right)\left(x + y\frac{dy}{dx}\right) = y^2 + 2xy\frac{dy}{dx}$

$x^3 + x^2y\frac{dy}{dx} + y^2x + y^3\frac{dy}{dx} = y^2 + 2xy\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{-x^3 - xy^2 + y^2}{yx^2 - 2xy + y^3}$ or equivalent

(b) **METHOD 1**

at $(1, 1)$, $\frac{dy}{dx}$ is undefined

$y = 1$

METHOD 2

gradient of normal $= -\frac{1}{\frac{dy}{dx}} = -\frac{\left(yx^2 - 2xy + y^3\right)}{\left(-x^3 - xy^2 + y^2\right)}$

at $(1, 1)$ gradient $= 0$

$y = 1$

12 – (MAT/22_HL_Summer_2014_Q12) – *Differentiation*

(a) $C = \text{AX} \times 5k + \text{XB} \times k$ ***(M1)***

Note: Award ***(M1)*** for attempting to express the cost in terms of AX, XB and k.

$= 5k\sqrt{450^2 + x^2} + (1000 - x)k$ ***A1***

$= 5k\sqrt{202500 + x^2} + (1000 - x)k$ ***AG***

(b) (i) $\dfrac{\mathrm{d}C}{\mathrm{d}x} = k\left[\dfrac{5 \times 2x}{2\sqrt{202500 + x^2}} - 1\right] = k\left(\dfrac{5x}{\sqrt{202500 + x^2}} - 1\right)$ ***M1A1***

(ii) attempting to solve $\frac{dC}{dx}=0$ *M1*

$\frac{5x}{\sqrt{202500+x^2}}=1$ *(A1)*

$x=91.9\text{ (m)}\left(=\frac{75\sqrt{6}}{2}\text{(m)}\right)$ *A1*

METHOD 1
for example,

at $x=91$ $\frac{dC}{dx}=-0.00895k<0$ *M1*

at $x=92$ $\frac{dC}{dx}=0.001506k>0$ *A1*

Note: Award *M1* for attempting to find the gradient either side of $x=91.9$ and *A1* for two correct values.

thus $x=91.9$ gives a minimum *AG*

METHOD 2

$$\frac{d^2C}{dx^2}=\frac{1012500k}{\left(x^2+202500\right)^{\frac{3}{2}}}$$

at $x=91.9$ $\frac{d^2C}{dx^2}=0.010451k>0$ *(M1)A1*

Note: Award *M1* for attempting to find the second derivative and *A1* for the correct value.

Note: If $\frac{d^2C}{dx^2}$ is obtained and its value at $x=91.9$ is not calculated, award *(M1)A1* for correct reasoning *eg*, both numerator and denominator are positive at $x=91.9$.

thus $x=91.9$ gives a minimum *AG*

METHOD 3

Sketching the graph of either C versus x or $\frac{dC}{dx}$ versus x. *M1*

Clearly indicating that $x=91.9$ gives the minimum on their graph. *A1*

(c) $C_{\min} = 3205k$

Note:	Accept $3200k$. Accept $3204\ k$.

(d) $\arctan\left(\dfrac{450}{91.855865\text{K}}\right) = 78.463\text{K}^{\circ}$

$180 - 78.463\text{K} = 101.537\text{K}$

$= 102^{\circ}$

(e) (i) when $\theta = 120^{\circ},\ x = 260\,(\text{m})\ \left(\dfrac{450}{\sqrt{3}}(\text{m})\right)$

(ii) $\dfrac{133.728\text{K}}{3204.5407685\text{K}} \times 100\%$

$= 4.17\,(\%)$

13 – (MAT/20_HL_Winter_2014_Q10) – *Differentiation*

(a) use of $A = \frac{1}{2}qr\sin\theta$ to obtain $A = \frac{1}{2}(x+2)(5-x)^2\sin 30^\circ$

$= \frac{1}{4}(x+2)(25-10x+x^2)$

$A = \frac{1}{4}(x^3 - 8x^2 + 5x + 50)$

(b) (i) $\frac{dA}{dx} = \frac{1}{4}\left(3x^2 - 16x + 5\right) = \frac{1}{4}(3x-1)(x-5)$

(ii) **METHOD 1**

EITHER

$$\frac{dA}{dx} = \frac{1}{4}\left(3\left(\frac{1}{3}\right)^2 - 16\left(\frac{1}{3}\right) + 5\right) = 0$$

OR

$$\frac{dA}{dx} = \frac{1}{4}\left(3\left(\frac{1}{3}\right) - 1\right)\left(\left(\frac{1}{3}\right) - 5\right) = 0$$

THEN

so $\frac{dA}{dx} = 0$ when $x = \frac{1}{3}$

METHOD 2

solving $\frac{dA}{dx} = 0$ for x

$-2 < x < 5 \Rightarrow x = \frac{1}{3}$

so $\frac{dA}{dx} = 0$ when $x = \frac{1}{3}$

METHOD 3

a correct graph of $\frac{dA}{dx}$ versus x

the graph clearly showing that $\frac{dA}{dx} = 0$ when $x = \frac{1}{3}$

so $\frac{dA}{dx} = 0$ when $x = \frac{1}{3}$

(c) (i) $\frac{d^2A}{dx^2} = \frac{1}{2}(3x-8)$

for $x = \frac{1}{3}$, $\frac{d^2A}{dx^2} = -3.5 (< 0)$

so $x = \frac{1}{3}$ gives the maximum area of triangle PQR

(ii) $A_{max} = \frac{343}{27} (= 12.7) (\text{cm}^2)$

(iii) $\text{PQ} = \frac{7}{3} (\text{cm})$ and $\text{PR} = \left(\frac{14}{3}\right)^2 (\text{cm})$

$\text{QR}^2 = \left(\frac{7}{3}\right)^2 + \left(\frac{14}{3}\right)^4 - 2\left(\frac{7}{3}\right)\left(\frac{14}{3}\right)^2 \cos 30^\circ$

$= 391.702\ldots$

$\text{QR} = 19.8 (\text{cm})$

14 – (MAT/20_HL_Winter_2014_Q13) – *Differentiation, Integration*

(a) attempting to use $V = \pi\int_a^b x^2 \mathrm{d}y$

attempting to express x^2 in terms of y *ie* $x^2 = 4(y+16)$

for $y = h$, $V = 4\pi\int_0^h y + 16\,\mathrm{d}y$

$V = 4\pi\left(\frac{h^2}{2} + 16h\right)$

(b) (i) **METHOD 1**

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t}$

$\frac{\mathrm{d}V}{\mathrm{d}h} = 4\pi(h+16)$

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{4\pi(h+16)} \times \frac{-250\sqrt{h}}{\pi(h+16)}$

Note: Award ***M1*** for substitution into $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t}$.

$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{250\sqrt{h}}{4\pi^2(h+16)^2}$

METHOD 2

$\frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi(h+16)\frac{\mathrm{d}h}{\mathrm{d}t}$ (implicit differentiation)

$\frac{-250\sqrt{h}}{\pi(h+16)} = 4\pi(h+16)\frac{\mathrm{d}h}{\mathrm{d}t}$ (or equivalent)

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{4\pi(h+16)} \times \frac{-250\sqrt{h}}{\pi(h+16)}$

$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{250\sqrt{h}}{4\pi^2(h+16)^2}$

(ii) $\frac{\mathrm{d}t}{\mathrm{d}h} = -\frac{4\pi^2(h+16)^2}{250\sqrt{h}}$

$t = \int -\frac{4\pi^2(h+16)^2}{250\sqrt{h}}\,\mathrm{d}h$

$t = \int -\frac{4\pi^2(h^2+32h+256)}{250\sqrt{h}}\,\mathrm{d}h$

$t = \frac{-4\pi^2}{250}\int\left(h^{\frac{3}{2}} + 32h^{\frac{1}{2}} + 256h^{-\frac{1}{2}}\right)\mathrm{d}h$

(iii) **METHOD 1**

$$t=\frac{-4\pi^2}{250}\int_{48}^{0}\left(h^{\frac{3}{2}}+32h^{\frac{1}{2}}+256h^{-\frac{1}{2}}\right)\mathrm{d}h$$ ***(M1)***

$t=2688.756\ldots$ (s) ***(A1)***

45 minutes (correct to the nearest minute) ***A1***

METHOD 2

$$t=\frac{-4\pi^2}{250}\left(\frac{2}{5}h^{\frac{5}{2}}+\frac{64}{3}h^{\frac{3}{2}}+512h^{\frac{1}{2}}\right)+c$$

when $t=0, h=48 \Rightarrow c=2688.756\ldots\left(c=\frac{4\pi^2}{250}\left(\frac{2}{5}\times 48^{\frac{5}{2}}+\frac{64}{3}\times 48^{\frac{3}{2}}+512\times 48^{\frac{1}{2}}\right)\right)$

when $h=0$, $t=2688.756\ldots\left(t=\frac{4\pi^2}{250}\left(\frac{2}{5}\times 48^{\frac{5}{2}}+\frac{64}{3}\times 48^{\frac{3}{2}}+512\times 48^{\frac{1}{2}}\right)\right)$(s)

45 minutes (correct to the nearest minute)

(c) **EITHER**

the depth stabilises when $\frac{\mathrm{d}V}{\mathrm{d}t}=0$ *ie* $8.5-\frac{250\sqrt{h}}{\pi(h+16)}=0$ ***R1***

attempting to solve $8.5-\frac{250\sqrt{h}}{\pi(h+16)}=0$ for h ***(M1)***

OR

the depth stabilises when $\frac{\mathrm{d}h}{\mathrm{d}t}=0$ *ie* $\frac{1}{4\pi(h+16)}\left(8.5-\frac{250\sqrt{h}}{\pi(h+16)}\right)=0$ ***R1***

attempting to solve $\frac{1}{4\pi(h+16)}\left(8.5-\frac{250\sqrt{h}}{\pi(h+16)}\right)=0$ for h ***(M1)***

THEN

$h=5.06$ (cm) ***A1***

15 – (MAT/21_HL_Summer_2015_Q5) – *Differentiation*

$V = 200\pi r^2$

Note: Allow $V = \pi h r^2$ if value of h is substituted later in the question.

EITHER

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 200\pi 2r\frac{\mathrm{d}r}{\mathrm{d}t}$$

Note: Award ***M1*** for an attempt at implicit differentiation.

at $r = 2$ we have $30 = 200\pi 4\frac{\mathrm{d}r}{\mathrm{d}t}$

OR

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\frac{\mathrm{d}V}{\mathrm{d}t}}{\frac{\mathrm{d}V}{\mathrm{d}r}}$$

$$\frac{\mathrm{d}V}{\mathrm{d}r} = 400\pi r$$

$r = 2$ we have $\frac{\mathrm{d}V}{\mathrm{d}r} = 800\pi$

THEN

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{30}{800\pi}\left(= \frac{3}{80\pi} \quad = 0.0119\right) \quad (\mathrm{cm\,s^{-1}})$$

16 – (MAT/21_HL_Summer_2015_Q6) – *Differentiation*

$$f'(x) = 3x^2 + e^x$$

Note: Accept labelled diagram showing the graph $y = f'(x)$ above the x-axis; do not accept unlabelled graphs nor graph of $y = f(x)$.

EITHER

this is always > 0
so the function is (strictly) increasing
and thus $1-1$

OR

this is always > 0 (accept $\neq 0$)
so there are no turning points
and thus $1-1$

17 – (MAT/21_HL_Summer_2015_Q9) – *Differentiation*

$x = 0 \Rightarrow y = 1$

$y'(0) = 1.367879...$

Note: The exact answer is $y'(0) = \frac{e+1}{e} = 1 + \frac{1}{e}$.

so gradient of normal is $\frac{-1}{1.367879...} \left(= -0.731058...\right)$

equation of normal is $y = -0.731058...x + c$

gives $y = -0.731x + 1$

18 – (MAT/22_HL_Summer_2015_Q11) – *Differentiation*

(a) attempt at implicit differentiation

$$2x-5x\frac{\mathrm{d}y}{\mathrm{d}x}-5y+2y\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

Note: ***A1*** for differentiation of x^2-5xy, ***A1*** for differentiation of y^2 and 7.

$$2x-5y+\frac{\mathrm{d}y}{\mathrm{d}x}(2y-5x)=0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{5y-2x}{2y-5x}$$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{5\times1-2\times6}{2\times1-5\times6}=\frac{1}{4}$

gradient of normal $=-4$

equation of normal $y=-4x+c$

substitution of $(6,1)$

$y=-4x+25$

Note: Accept $y-1=-4(x-6)$

(c) setting $\frac{5y-2x}{2y-5x}=1$

$y=-x$

substituting into original equation

$x^2+5x^2+x^2=7$

$7x^2=7$

$x=\pm1$

points $(1,-1)$ and $(-1,1)$

distance $=\sqrt{8}\left(=2\sqrt{2}\right)$

19 – (MAT/20_HL_Winter_2015_Q13) – *Differentiation*

(a) $f'(x) = 30e^{-\frac{x^2}{400}} \cdot -\frac{2x}{400}\left(= -\frac{3x}{20}e^{-\frac{x^2}{400}}\right)$

Note: Award ***M1*** for attempting to use the chain rule.

$$f''(x) = -\frac{3}{20}e^{-\frac{x^2}{400}} + \frac{3x^2}{4000}e^{-\frac{x^2}{400}}\left(= \frac{3}{20}e^{-\frac{x^2}{400}}\left(\frac{x^2}{200} - 1\right)\right)$$

Note: Award ***M1*** for attempting to use the product rule.

(b) the roof function has maximum gradient when $f''(x) = 0$

Note: Award ***(M1)*** for attempting to find $f''\left(-\sqrt{200}\right)$.

EITHER
$= 0$

OR
$f''(x) = 0 \Rightarrow x = \pm\sqrt{200}$

THEN
valid argument for maximum such as reference to an appropriate graph or change in the sign of $f''(x)$ *eg* $f''(-15) = 0.010...(> 0)$ and $f''(-14) = -0.001...(< 0)$

$\Rightarrow x = -\sqrt{200}$

(c) $A = 2a \cdot 30\mathrm{e}^{-\frac{a^2}{400}} \left(= 60a\mathrm{e}^{-\frac{a^2}{400}} = -400 f'(a) \right)$ ***(M1)(A1)***

EITHER

$\frac{\mathrm{d}A}{\mathrm{d}a} = 60a\mathrm{e}^{-\frac{a^2}{400}} \cdot -\frac{a}{200} + 60\mathrm{e}^{-\frac{a^2}{400}} = 0 \Rightarrow a = \sqrt{200}$ $\left(-400 f''(a) = 0 \Rightarrow a = \sqrt{200}\right)$

M1A1

OR

by symmetry *eg* $a = -\sqrt{200}$ found in (b) or $A_{\max}$ coincides with $f''(a) = 0$ ***R1***

$\Rightarrow a = \sqrt{200}$ ***A1***

THEN

$A_{\max} = 60 \cdot \sqrt{200}\mathrm{e}^{-\frac{200}{400}}$ ***M1***

$= 600\sqrt{2}\mathrm{e}^{-\frac{1}{2}}$ ***AG***

(d) (i) perimeter $= 4a + 60\mathrm{e}^{-\frac{a^2}{400}}$ ***A1A1***

Note: Condone use of x.

(ii) $I(a) = \dfrac{4a + 60\mathrm{e}^{-\frac{a^2}{400}}}{60a\mathrm{e}^{-\frac{a^2}{400}}}$ ***(A1)***

graphing $I(a)$ or other valid method to find the minimum ***(M1)***

$a = 12.6$ ***A1***

(iii) area under roof $= \int_{-20}^{20} 30\mathrm{e}^{-\frac{x^2}{400}}\,\mathrm{d}x$ ***M1***

$= 896.18\ldots$ ***(A1)***

area of living space $= 60 \cdot (12.6\ldots) \cdot \mathrm{e}^{-\frac{(12.6\ldots)^2}{400}} = 508.56\ldots$ ***(A1)***

percentage of empty space $= 43.3\%$ ***A1***

20 – (MAT/22_HL_Summer_2016_Q7) – *Differentiation*

(a) $3x^2 + 3y^2\frac{dy}{dx} = 4\left(y + x\frac{dy}{dx}\right)$

$\left(3y^2 - 4x\right)\frac{dy}{dx} = 4y - 3x^2$

$\frac{dy}{dx} = \frac{4y - 3x^2}{3y^2 - 4x}$

(b) $\frac{dy}{dx} = 0 \Rightarrow 4y - 3x^2 = 0$

substituting $x = k$ and $y = \frac{3}{4}k^2$ into $x^3 + y^3 = 4xy$

$k^3 + \frac{27}{64}k^6 = 3k^3$

attempting to solve $k^3 + \frac{27}{64}k^6 = 3k^3$ for k

$k = 1.68\left(= \frac{4}{3}\sqrt[3]{2}\right)$

21 – (MAT/22_HL_Summer_2016_Q11) – *Differentiation*

(a) **EITHER**

$\alpha = \arctan\frac{7}{10} - \arctan\frac{5}{10}$ $(= 34.992\ldots^\circ - 26.5651\ldots^\circ)$ ***(M1)(A1)(A1)***

Note: Award ***(M1)*** for $\alpha = \mathrm{A\hat{P}T} - \mathrm{B\hat{P}T}$, ***(A1)*** for a correct $\mathrm{A\hat{P}T}$ and ***(A1)*** for a correct $\mathrm{B\hat{P}T}$.

OR

$\alpha = \arctan 2 - \arctan\frac{10}{7}$ $(= 63.434\ldots^\circ - 55.008\ldots^\circ)$ ***(M1)(A1)(A1)***

Note: Award ***(M1)*** for $\alpha = \mathrm{P\hat{B}T} - \mathrm{P\hat{A}T}$, ***(A1)*** for a correct $\mathrm{P\hat{B}T}$ and ***(A1)*** for a correct $\mathrm{P\hat{A}T}$.

OR

$\alpha = \arccos\left(\frac{125 + 149 - 4}{2 \times \sqrt{125} \times \sqrt{149}}\right)$ ***(M1)(A1)(A1)***

Note: Award ***(M1)*** for use of cosine rule, ***(A1)*** for a correct numerator and ***(A1)*** for a correct denominator.

THEN

$= 8.43^\circ$ ***A1***

(b) **EITHER**

$$\tan\alpha = \frac{\frac{7}{x} - \frac{5}{x}}{1 + \left(\frac{7}{x}\right)\left(\frac{5}{x}\right)}$$ ***M1A1A1***

Note: Award ***M1*** for use of $\tan(A - B)$, ***A1*** for a correct numerator and ***A1*** for a correct denominator.

$$= \frac{\frac{2}{x}}{1 + \frac{35}{x^2}}$$ ***M1***

OR

$$\tan\alpha = \frac{\frac{x}{5} - \frac{x}{7}}{1 + \left(\frac{x}{5}\right)\left(\frac{x}{7}\right)}$$ ***M1A1A1***

Note: Award ***M1*** for use of $\tan(A - B)$, ***A1*** for a correct numerator and ***A1*** for a correct denominator.

$$= \frac{\frac{2x}{35}}{1 + \frac{x^2}{35}}$$ ***M1***

OR

$$\cos\alpha = \frac{x^2 + 35}{\sqrt{(x^2 + 25)(x^2 + 49)}}$$ ***M1A1***

Note: Award ***M1*** for either use of the cosine rule or use of $\cos(A - B)$.

$$\sin\alpha = \frac{2x}{\sqrt{(x^2 + 25)(x^2 + 49)}}$$ ***A1***

$$\tan\alpha = \frac{\frac{2x}{\sqrt{(x^2 + 25)(x^2 + 49)}}}{\frac{x^2 + 35}{\sqrt{(x^2 + 25)(x^2 + 49)}}}$$ ***M1***

THEN

$$\tan\alpha = \frac{2x}{x^2 + 35}$$ ***AG***

(c) (i) $\frac{\mathrm{d}}{\mathrm{d}x}(\tan\alpha) = \frac{2\left(x^2+35\right)-(2x)(2x)}{\left(x^2+35\right)^2} \left(= \frac{70-2x^2}{\left(x^2+35\right)^2}\right)$

Note: Award ***M1*** for attempting product or quotient rule differentiation, ***A1*** for a correct numerator and ***A1*** for a correct denominator.

(ii) **METHOD 1**

EITHER

$\frac{\mathrm{d}}{\mathrm{d}x}(\tan\alpha) = 0 \Rightarrow 70 - 2x^2 = 0$

$x = \sqrt{35}(\mathrm{m})\left(= 5.9161\ldots(\mathrm{m})\right)$

$\tan\alpha = \frac{1}{\sqrt{35}}(= 0.16903\ldots)$

OR

attempting to locate the stationary point on the graph of

$\tan\alpha = \frac{2x}{x^2+35}$

$x = 5.9161\ldots\ (\mathrm{m})\left(= \sqrt{35}\,(\mathrm{m})\right)$

$\tan\alpha = 0.16903\ldots\left(= \frac{1}{\sqrt{35}}\right)$

THEN

$\alpha = 9.59^\circ$

METHOD 2

EITHER

$\alpha = \arctan\left(\frac{2x}{x^2+35}\right) \Rightarrow \frac{d\alpha}{dx} = \frac{70-2x^2}{\left(x^2+35\right)^2+4x^2}$ ***M1***

$\frac{d\alpha}{dx} = 0 \Rightarrow x = \sqrt{35}(\text{m})\left(= 5.9161\ldots(\text{m})\right)$ ***A1***

OR

attempting to locate the stationary point on the graph of

$\alpha = \arctan\left(\frac{2x}{x^2+35}\right)$ ***(M1)***

$x = 5.9161\ldots(\text{m})\ \left(= \sqrt{35}\,(\text{m})\right)$ ***A1***

THEN

$\alpha = 0.1674\ldots\left(= \arctan\frac{1}{\sqrt{35}}\right)$ ***(A1)***

$= 9.59^\circ$ ***A1***

(iii) $\frac{d^2}{dx^2}(\tan\alpha) = \frac{\left(x^2+35\right)^2(-4x)-(2)(2x)\left(x^2+35\right)\left(70-2x^2\right)}{\left(x^2+35\right)^4}\ \left(=\frac{4x\left(x^2-105\right)}{\left(x^2+35\right)^3}\right)$

M1A1

substituting $x = \sqrt{35}\ (= 5.9161\ldots)$ into $\frac{d^2}{dx^2}(\tan\alpha)$ ***M1***

$\frac{d^2}{dx^2}(\tan\alpha) < 0\ (= -0.004829\ldots)$ and so $\alpha = 9.59^\circ$ is the maximum value of α ***R1***

α never exceeds 10° ***AG***

(d) attempting to solve $\frac{2x}{x^2+35} \geq \tan 7^\circ$

Note: Award ***(M1)*** for attempting to solve $\frac{2x}{x^2+35} = \tan 7^\circ$.

$x = 2.55$ and $x = 13.7$

$2.55 \leq x \leq 13.7$ (m)

22 – (MAT/21_HL_Summer_2016_Q12) – *Differentiation*

(a) $a^2 = 5 - 1$

$a = 2$

(b) $2y\dfrac{dy}{dx} - \left(2x\dfrac{dy}{dx} + 2y\right) = -e^x$

Note: Award ***M1*** for an attempt at implicit differentiation, ***A1*** for each part.

$$\frac{dy}{dx} = \frac{2y - e^x}{2(y - x)}$$

(c) at $x = 0$, $\dfrac{dy}{dx} = \dfrac{3}{4}$

finding the negative reciprocal of a number

gradient of normal is $-\dfrac{4}{3}$

$y = -\dfrac{4}{3}x + 2$

(d) substituting linear expression

$\left(-\dfrac{4}{}x + 2\right)^2 - 2x\left(-\dfrac{4}{3}x + 2\right) + e^x - 5 = 0$ or equivalent

$x = 1.56$

$y = -0.0779$

$(1.56\ ,\ -0.0779)$

(e) $\dfrac{dv}{dx} = 3y^2\dfrac{dy}{dx}$

$\dfrac{dv}{dx} = 3 \times 4 \times \dfrac{3}{4} = 9$

23 – (MAT/22_HL_Summer_2016_Q12) – *Differentiation, Integration*

(a) (i)

$$\frac{1}{4\left(\frac{e^x+e^{-x}}{2}\right)-2\left(\frac{e^x-e^{-x}}{2}\right)}$$

$$=\frac{1}{2\left(e^x+e^{-x}\right)-\left(e^x-e^{-x}\right)}$$

$$=\frac{1}{e^x+3e^{-x}}$$

$$=\frac{e^x}{e^{2x}+3}$$

(ii) $u=e^x \Rightarrow du=e^x dx$

$$\int\frac{e^x}{e^{2x}+3}dx=\int\frac{1}{u^2+3}du$$

(when $x=0$, $u=1$ and when $x=\ln 3$, $u=3$)

$$\int_1^3\frac{1}{u^2+3}du=\left[\frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)\right]_1^3$$

$$\left(=\left[\frac{1}{\sqrt{3}}\arctan\left(\frac{e^x}{\sqrt{3}}\right)\right]_0^{\ln 3}\right)$$

$$=\frac{\pi\sqrt{3}}{9}-\frac{\pi\sqrt{3}}{18}$$

$$=\frac{\pi\sqrt{3}}{18}$$

(b) (i) $(n+1)e^{2x} - 2ke^{x} + (n-1) = 0$

$$e^{x} = \frac{2k \pm \sqrt{4k^2 - 4\left(n^2 - 1\right)}}{2(n+1)}$$

$$x = \ln\left(\frac{k \pm \sqrt{k^2 - n^2 + 1}}{n+1}\right)$$

(ii) for two real solutions, we require $k > \sqrt{k^2 - n^2 + 1}$

and we also require $k^2 - n^2 + 1 > 0$

$k^2 > n^2 - 1$

$\Rightarrow k > \sqrt{n^2 - 1}$ $(k \in \mathbb{R}^+)$

(c) (i) **METHOD 1**

$$t(x) = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

$$t'(x) = \frac{\left(e^{x} + e^{-x}\right)^2 - \left(e^{x} - e^{-x}\right)^2}{\left(e^{x} + e^{-x}\right)^2}$$

$$t'(x) = \frac{\left(\frac{e^{x} + e^{-x}}{2}\right)^2 - \left(\frac{e^{x} - e^{-x}}{2}\right)^2}{\left(\frac{e^{x} + e^{-x}}{2}\right)^2}$$

$$= \frac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$$

METHOD 2

$$t'(x) = \frac{f(x)g'(x) - g(x)f'(x)}{[f(x)]^2}$$

$g'(x) = f(x)$ and $f'(x) = g(x)$

$$= \frac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$$

METHOD 3

$t(x) = \left(e^x - e^{-x}\right)\left(e^x + e^{-x}\right)^{-1}$

$t'(x) = 1 - \dfrac{\left(e^x - e^{-x}\right)^2}{\left(e^x + e^{-x}\right)^2}$

$= 1 - \dfrac{[g(x)]^2}{[f(x)]^2}$

$= \dfrac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$

METHOD 4

$t'(x) = \dfrac{g'(x)}{f(x)} - \dfrac{g(x)f'(x)}{[f(x)]^2}$

$g'(x) = f(x)$ and $f'(x) = g(x)$ gives $t'(x) = 1 - \dfrac{[g(x)]^2}{[f(x)]^2}$

$= \dfrac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$

(ii) **METHOD 1**

$[f(x)]^2 > [g(x)]^2$ (or equivalent)

$[f(x)]^2 > 0$

hence $t'(x) > 0$, $x \in \mathbb{R}$

METHOD 2

$[f(x)]^2 - [g(x)]^2 = 1$ (or equivalent) ***M1A1***

$[f(x)]^2 > 0$ ***R1***

hence $t'(x) > 0$, $x \in \mathbb{R}$ ***AG***

Note: Award as above for use of either $f(x) = \dfrac{e^x + e^{-x}}{2}$ and $g(x) = \dfrac{e^x - e^{-x}}{2}$ or $e^x + e^{-x}$ and $e^x - e^{-x}$.

METHOD 3

$$t'(x) = \frac{4}{\left(e^x + e^{-x}\right)^2}$$

$\left(e^x + e^{-x}\right)^2 > 0$ ***M1A1***

$\dfrac{4}{\left(e^x + e^{-x}\right)^2} > 0$ ***R1***

hence $t'(x) > 0$, $x \in \mathbb{R}$ ***AG***

24 – (MAT/20_HL_Winter_2016_Q6) – *Differentiation*

METHOD 1

substituting for x and attempting to solve for y (or vice versa) ***(M1)***

$y = (\pm)\ 0.11821\ldots$ ***(A1)***

EITHER

$145x + 143y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad \left(\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{145x}{143y}\right)$ ***M1A1***

OR

$145x\frac{\mathrm{d}x}{\mathrm{d}t} + 143y\frac{\mathrm{d}y}{\mathrm{d}t} = 0$ ***M1A1***

THEN

attempting to find $\frac{\mathrm{d}y}{\mathrm{d}t} \quad \left(\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{145\left(3.2\times10^{-3}\right)}{143\left((\pm)\ 0.11821\ldots\right)}\times\left(7.75\times10^{-5}\right)\right)$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}t} = \pm 2.13\times10^{-6}$ ***A1***

Note: Award all marks except the final ***A1*** to candidates who do not consider $\pm$.

METHOD 2

$y = (\pm)\sqrt{\frac{1-72.5x^2}{71.5}}$ ***M1A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = (\pm)0.0274\ldots$ ***(M1)(A1)***

$\frac{\mathrm{d}y}{\mathrm{d}t} = (\pm)0.0274\ldots\times7.75\times10^{-5}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}t} = \pm 2.13\times10^{-6}$ ***A1***

Note: Award all marks except the final ***A1*** to candidates who do not consider $\pm$.

[6 marks]

25 - (MAT/20_HL_Winter_2016_Q10) - *Functions - Roots, Differentiation, Integration*

(a) attempting to solve either $2e^x - 1 = 0$ or $2e^x - 1 \neq 0$ for x ***(M1)***

$D = \mathbb{R} \setminus \{-\ln 2\}$ (or equivalent *eg* $x \neq -\ln 2$) ***A1***

Note: Accept $D = \mathbb{R} \setminus \{-0.693\}$ or equivalent *eg* $x \neq -0.693$.

[2 marks]

(b) considering $\lim\limits_{x \to -\ln 2} f(x)$ ***(M1)***

$x = -\ln 2 \ (x = -0.693)$ ***A1***

considering one of $\lim\limits_{x \to -\infty} f(x)$ or $\lim\limits_{x \to +\infty} f(x)$ ***M1***

$\lim\limits_{x \to -\infty} f(x) = -2 \Rightarrow y = -2$ ***A1***

$\lim\limits_{x \to +\infty} f(x) = -\frac{1}{2} \Rightarrow y = -\frac{1}{2}$ ***A1***

Note: Award ***A0A0*** for $y = -2$ and $y = -\frac{1}{2}$ stated without any justification.

[5 marks]

(c) $f'(x) = \dfrac{-e^x\left(2e^x - 1\right) - 2e^x\left(2 - e^x\right)}{\left(2e^x - 1\right)^2}$ ***M1A1A1***

$= -\dfrac{3e^x}{\left(2e^x - 1\right)^2}$ ***AG***

[3 marks]

(d) $f'(x) < 0$ (for all $x \in D$) $\Rightarrow f$ is (strictly) decreasing ***R1***

Note: Award ***R1*** for a statement such as $f'(x) \neq 0$ and so the graph of f has no turning points.

one branch is above the upper horizontal asymptote and the other branch is below the lower horizontal asymptote ***R1***

f has an inverse ***AG***

$-\infty < x < -2 \cup -\frac{1}{2} < x < \infty$ ***A2***

Note: Award ***A2*** if the domain of the inverse is seen in either part (d) or in part (e).

[4 marks]

(e) $x = \dfrac{2 - e^y}{2e^y - 1}$ ***M1***

Note: Award ***M1*** for interchanging x and y (can be done at a later stage).

$2xe^y - x = 2 - e^y$ ***M1***

$e^y(2x+1) = x + 2$ ***A1***

$f^{-1}(x) = \ln\left(\dfrac{x+2}{2x+1}\right)$ $\left(f^{-1}(x) = \ln(x+2) - \ln(2x+1)\right)$ ***A1***

[4 marks]

(f) use of $V = \pi\int_a^b x^2\,dy$ ***(M1)***

$= \pi\int_0^1 \left(\ln\left(\dfrac{y+2}{2y+1}\right)\right)^2 dy$ ***(A1)(A1)***

Note: Award ***(A1)*** for the correct integrand and ***(A1)*** for the limits.

$= 0.331$ ***A1***

[4 marks]

26 – (MAT/21_HL_Summer_2017_Q2) – *Differentiation*

(a) $y + x\dfrac{dy}{dx} - \dfrac{1}{y}\dfrac{dy}{dx} = 0$ ***M1A1A1***

Note: Award ***A1*** for the first two terms, ***A1*** for the third term and the 0.

$\dfrac{dy}{dx} = \dfrac{y^2}{1 - xy}$ ***A1***

Note: Accept $\dfrac{-y^2}{\ln y}$.

Note: Accept $\dfrac{-y}{x - \dfrac{1}{y}}$.

[4 marks]

(b) $m_T = \dfrac{e^2}{1 - e \times \dfrac{2}{e}}$ ***(M1)***

$m_T = -e^2$ ***(A1)***

$y - e = -e^2x + 2e$

$-e^2x - y + 3e = 0$ or equivalent ***A1***

Note: Accept $y = -7.39x + 8.15$.

[3 marks]

27 – (MAT/22_HL_Summer_2017_Q2) – *Differentiation, Integration*

(a) **METHOD 1**

$4x^2 + y^2 = 7$

$8x + 2y\frac{dy}{dx} = 0$ ***(M1)(A1)***

$\frac{dy}{dx} = -\frac{4x}{y}$

Note: Award ***M1A1*** for finding $\frac{dy}{dx} = -2.309\ldots$ using any alternative method.

hence gradient of normal $= \frac{y}{4x}$ ***(M1)***

hence gradient of normal at $\left(1, \sqrt{3}\right)$ is $\frac{\sqrt{3}}{4} (= 0.433)$ ***(A1)***

hence equation of normal is $y - \sqrt{3} = \frac{\sqrt{3}}{4}(x - 1)$ ***(M1)A1***

$\left(y = \frac{\sqrt{3}}{4}x + \frac{3\sqrt{3}}{4}\right)$ $\left(y = 0.433x + 1.30\right)$

METHOD 2

$4x^2 + y^2 = 7$

$y = \sqrt{7 - 4x^2}$ ***(M1)***

$\frac{dy}{dx} = -\frac{4x}{\sqrt{7 - 4x^2}}$ ***(A1)***

Note: Award ***M1A1*** for finding $\frac{dy}{dx} = -2.309\ldots$ using any alternative method.

hence gradient of normal $= \frac{\sqrt{7 - 4x^2}}{4x}$ ***(M1)***

hence gradient of normal at $\left(1, \sqrt{3}\right)$ is $\frac{\sqrt{3}}{4} (= 0.433)$ ***(A1)***

hence equation of normal is $y - \sqrt{3} = \frac{\sqrt{3}}{4}(x - 1)$ ***(M1)A1***

$\left(y = \frac{\sqrt{3}}{4}x + \frac{3\sqrt{3}}{4}\right)\left(y = 0.433x + 1.30\right)$

[6 marks]

(b) Use of $V = \pi \int_0^{\frac{\sqrt{7}}{2}} y^2 dx$

$V = \pi \int_0^{\frac{\sqrt{7}}{2}} \left(7 - 4x^2\right) dx$ ***(M1)(A1)***

Note: Condone absence of limits or incorrect limits for ***M*** mark.
Do not condone absence of or multiples of π.

$= 19.4 \left(= \frac{7\sqrt{7}\pi}{3}\right)$ ***A1***

[3 marks]

28 – (MAT/21_HL_Summer_2017_Q8) – *Differentiation*

(a) area of segment $= \frac{1}{2} \times 0.5^2 \times (\theta - \sin\theta)$ ***M1A1***

$V =$ area of segment $\times 10$

$V = \frac{5}{4}(\theta - \sin\theta)$ ***A1***

[3 marks]

(b) **METHOD 1**

$\frac{dV}{dt} = \frac{5}{4}(1 - \cos\theta)\frac{d\theta}{dt}$ ***M1A1***

$0.0008 = \frac{5}{4}\left(1 - \cos\frac{\pi}{3}\right)\frac{d\theta}{dt}$ ***(M1)***

$\frac{d\theta}{dt} = 0.00128 \text{ (rad } s^{-1})$ ***A1***

METHOD 2

$\frac{d\theta}{dt} = \frac{d\theta}{dV} \times \frac{dV}{dt}$ ***(M1)***

$\frac{dV}{d\theta} = \frac{5}{4}(1 - \cos\theta)$ ***A1***

$\frac{d\theta}{dt} = \frac{4 \times 0.0008}{5\left(1 - \cos\frac{\pi}{3}\right)}$ ***(M1)***

$\frac{d\theta}{dt} = 0.00128 \left(\frac{4}{3125}\right) \text{(rad } s^{-1})$ ***A1***

[4 marks]

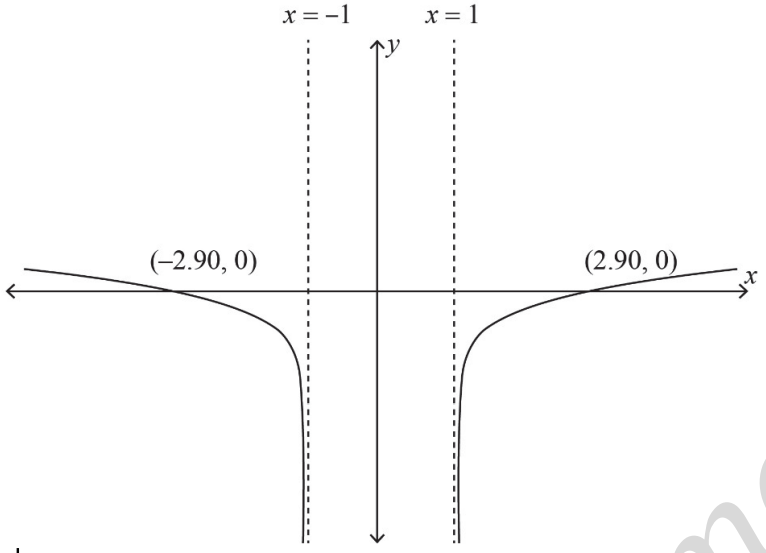
29 – (MAT/21_HL_Summer_2017_Q12) – *Functions - Roots, Graphs, Differentiation*

(a) $x^2 - 1 > 0$ ***(M1)***

$x < -1$ or $x > 1$ ***A1***

[2 marks]

(b)



shape ***A1***

$x = 1$ and $x = -1$ ***A1***

x-intercepts ***A1***

[3 marks]

(c) **EITHER**

f is symmetrical about the y-axis ***R1***

OR

$f(-x) = f(x)$ ***R1***

[1 mark]

(d) **EITHER**

f is not one-to-one function ***R1***

OR

horizontal line cuts twice ***R1***

Note: Accept any equivalent correct statement.

[1 mark]

(e) $x = -1 + \ln\left(\sqrt{y^2 - 1}\right)$ ***M1***

$e^{2x+2} = y^2 - 1$ ***M1***

$g^{-1}(x) = \sqrt{e^{2x+2} + 1},\ x \in \mathbb{R}$ ***A1A1***

[4 marks]

(f) $g'(x) = \dfrac{1}{\sqrt{x^2 - 1}} \times \dfrac{2x}{2\sqrt{x^2 - 1}}$ ***M1A1***

$g'(x) = \dfrac{x}{x^2 - 1}$ ***A1***

[3 marks]

(g) (i) $g'(x) = \dfrac{x}{x^2 - 1} = 0 \Rightarrow x = 0$ ***M1***

which is not in the domain of g (hence no solutions to $g'(x) = 0$) ***R1***

(ii) $\left(g^{-1}\right)'(x) = \dfrac{e^{2x+2}}{\sqrt{e^{2x+2} + 1}}$ ***M1***

as $e^{2x+2} > 0 \Rightarrow \left(g^{-1}\right)'(x) > 0$ so no solutions to $\left(g^{-1}\right)'(x) = 0$ ***R1***

Note: Accept: equation $e^{2x+2} = 0$ has no solutions.

[4 marks]

30 – (MAT/20_HL_Winter_2017_Q10) – *Differentiation, Integration*

(a) (i) attempt to use quotient rule or product rule ***M1***

$$f'(x) = \frac{\sin x\left(\frac{1}{2}x^{-\frac{1}{2}}\right) - \sqrt{x}\cos x}{\sin^2 x}\left(= \frac{1}{2\sqrt{x}\sin x} - \frac{\sqrt{x}\cos x}{\sin^2 x}\right)$$ ***A1A1***

Note: Award ***A1*** for $\frac{1}{2\sqrt{x}\sin x}$ or equivalent and ***A1*** for $-\frac{\sqrt{x}\cos x}{\sin^2 x}$ or equivalent.

setting $f'(x) = 0$ ***M1***

$\frac{\sin x}{2\sqrt{x}} - \sqrt{x}\cos x = 0$

$\frac{\sin x}{2\sqrt{x}} = \sqrt{x}\cos x$ or equivalent ***A1***

$\tan x = 2x$ ***AG***

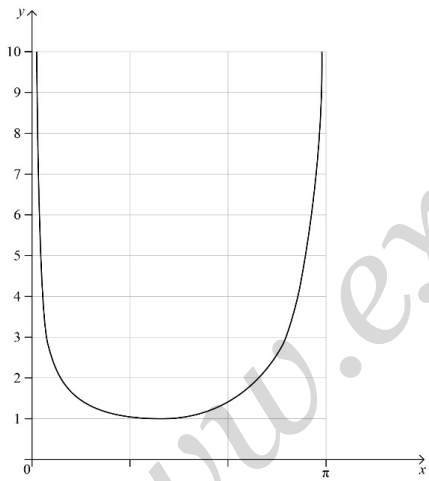
(ii) $x = 1.17$

$0 < x \le 1.17$ ***A1A1***

Note: Award ***A1*** for $0 < x$ and ***A1*** for $x \le 1.17$. Accept $x < 1.17$.

[7 marks]

(b)



concave up curve over correct domain with one minimum point above the *x*-axis. ***A1***

approaches $x = 0$ asymptotically ***A1***

approaches $x = \pi$ asymptotically ***A1***

Note: For the final ***A1*** an asymptote must be seen, and π must be seen on the x-axis or in an equation.

[3 marks]

(c) $f'(x)\left(=\dfrac{\sin x\left(\frac{1}{2}x^{-\frac{1}{2}}\right)-\sqrt{x}\cos x}{\sin^2 x}\right)=1$ ***(A1)***

attempt to solve for x ***(M1)***

$x=1.96$ ***A1***

$y=f(1.96...)$

$=1.51$ ***A1***

[4 marks]

(d) $V=\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\dfrac{x\,\mathrm{d}x}{\sin^2 x}$ ***(M1)(A1)***

Note: ***M1*** is for an integral of the correct squared function (with or without limits and/or π).

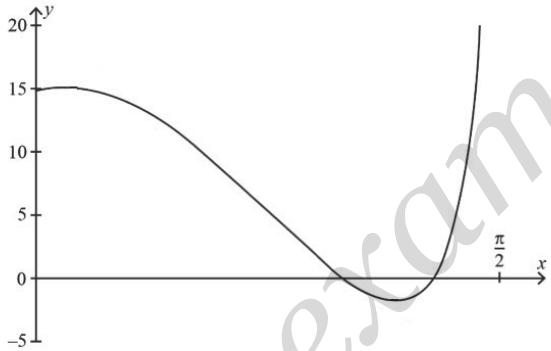
$=2.68(=0.852\pi)$ ***A1***

[3 marks]

31 – (MAT/20_HL_Winter_2017_Q11) – *Trigonometry, Differentiation*

(a) (i) $f'(x)=4\sin x\cos x+14\cos 2x+\sec^2 x$ (or equivalent) ***(M1)A1***

(ii)



A1A1A1A1

Note: Award ***A1*** for correct behaviour at $x=0$, ***A1*** for correct domain and correct behaviour for $x\to\frac{\pi}{2}$, ***A1*** for two clear intersections with x-axis and minimum point, ***A1*** for clear maximum point.

(iii) $x = 0.0736$ ***A1***

$x = 1.13$ ***A1***

[8 marks]

(b) (i) attempt to write $\sin x$ in terms of u only ***(M1)***

$\sin x = \dfrac{u}{\sqrt{1+u^2}}$ ***A1***

(ii) $\cos x = \dfrac{1}{\sqrt{1+u^2}}$ ***(A1)***

attempt to use $\sin 2x = 2\sin x\cos x \left(= 2\dfrac{u}{\sqrt{1+u^2}}\dfrac{1}{\sqrt{1+u^2}} \right)$ ***(M1)***

$\sin 2x = \dfrac{2u}{1+u^2}$ ***A1***

(iii) $2\sin^2 x + 7\sin 2x + \tan x - 9 = 0$

$\dfrac{2u^2}{1+u^2} + \dfrac{14u}{1+u^2} + u - 9\ (= 0)$ ***M1***

$\dfrac{2u^2 + 14u + u\left(1+u^2\right) - 9\left(1+u^2\right)}{1+u^2} = 0$ (or equivalent) ***A1***

$u^3 - 7u^2 + 15u - 9 = 0$ ***AG***

[7 marks]

(c) $u = 1$ or $u = 3$ ***(M1)***

$x = \arctan(1)$ ***A1***

$x = \arctan(3)$ ***A1***

Note: Only accept answers given the required form.

[3 marks]

32 – (MAT/21_HL_Summer_2018_Q9) – *Differentiation, Integration*

(a) differentiating implicitly: ***M1***

$2xy + x^2 \frac{dy}{dx} = -4y^3 \frac{dy}{dx}$ ***A1A1***

Note: Award ***A1*** for each side.

if $\frac{dy}{dx} = 0$ then either $x = 0$ or $y = 0$ ***M1A1***

$x = 0 \Rightarrow$ two solutions for $y\left(y = \pm\sqrt[4]{5}\right)$ ***R1***

$y = 0$ not possible (as $0 \neq 5$) ***R1***

hence exactly two points ***AG***

Note: For a solution that only refers to the graph giving two solutions at $x = 0$ and no solutions for $y = 0$ award ***R1*** only.

[7 marks]

(b) at $(2, 1)$ $4 + 4\frac{dy}{dx} = -4\frac{dy}{dx}$ ***M1***

$\frac{dy}{dx} = -\frac{1}{2}$ ***(A1)***

gradient of normal is 2 ***M1***

$1 = 4 + c$ ***(M1)***

equation of normal is $y = 2x - 3$ ***A1***

[5 marks]

(c) substituting ***(M1)***

$x^2(2x - 3) = 5 - (2x - 3)^4$ or $\left(\frac{y+3}{2}\right)^2 y = 5 - y^4$ ***(A1)***

$x = 0.724$ ***A1***

[3 marks]

(d) recognition of two volumes ***(M1)***

volume 1 $= \pi\int_1^{\sqrt[4]{5}} \frac{5-y^4}{y}\,\mathrm{d}y (= 1.01\pi = 3.178\ldots)$ ***M1A1A1***

Note: Award ***M1*** for attempt to use $\pi\int x^2\mathrm{d}y$, ***A1*** for limits, ***A1*** for $\frac{5-y^4}{y}$. Condone omission of π at this stage.

volume 2

EITHER

$= \frac{1}{3}\pi \times 2^2 \times 4 (= 16.75\ldots)$ ***(M1)(A1)***

OR

$= \pi\int_{-3}^{1}\left(\frac{y+3}{2}\right)^2 \mathrm{d}y \left(= \frac{16\pi}{3} = 16.75\ldots\right)$ ***(M1)(A1)***

THEN

total volume $= 19.9$ ***A1***

[7 marks]

33 – (MAT/22_HL_Summer_2018_Q11) – *Differentiation*

(a) attempt at implicit differentiation ***M1***

$1 + \frac{dy}{dx} + (y + x\frac{dy}{dx})\sin(xy) = 0$ ***A1M1A1***

Note: Award ***A1*** for first two terms. Award ***M1*** for an attempt at chain rule ***A1*** for last term.

$(1 + x\sin(xy))\frac{dy}{dx} = -1 - y\sin(xy)$ or equivalent ***A1***

$\frac{dy}{dx} = -\left(\frac{1 + y\sin(xy)}{1 + x\sin(xy)}\right)$ ***AG***

[5 marks]

(b) (i) **EITHER**

when $xy = -\frac{\pi}{2}$, $\cos xy = 0$ ***M1***

$\Rightarrow x + y = 0$ ***(A1)***

OR

$x - \frac{\pi}{2x} - \cos\left(\frac{-\pi}{2}\right) = 0$ or equivalent ***M1***

$x - \frac{\pi}{2x} = 0$ ***(A1)***

THEN

therefore $x^2 = \frac{\pi}{2}\left(x = \pm\sqrt{\frac{\pi}{2}}\right)(x = \pm 1.25)$ ***A1***

$\mathrm{P}\left(\sqrt{\frac{\pi}{2}}, -\sqrt{\frac{\pi}{2}}\right), \mathrm{Q}\left(-\sqrt{\frac{\pi}{2}}, \sqrt{\frac{\pi}{2}}\right)$ or $P(1.25, -1.25)$, $Q(-1.25, 1.25)$ ***A1***

(ii) $m_1 = -\left(\dfrac{1-\sqrt{\frac{\pi}{2}}\times -1}{1+\sqrt{\frac{\pi}{2}}\times -1}\right)$ ***M1A1***

$m_2 = -\left(\dfrac{1+\sqrt{\frac{\pi}{2}}\times -1}{1-\sqrt{\frac{\pi}{2}}\times -1}\right)$ ***A1***

$m_1 m_2 = 1$ ***AG***

Note: Award ***M1A0A0*** if decimal approximations are used.

Note: No ***FT*** applies.

[7 marks]

(c) equate derivative to -1 ***M1***

$(y-x)\sin(xy) = 0$ ***(A1)***

$y = x,\ \sin(xy) = 0$ ***R1***

in the first case, attempt to solve $2x = \cos\left(x^2\right)$ ***M1***

$(0.486, 0.486)$ ***A1***

in the second case, $\sin(xy) = 0 \Rightarrow xy = 0$ and $x + y = 1$ ***(M1)***

$(0,1), (1,0)$ ***A1***

[7 marks]

34 – (MAT/20_HL_Winter_2018_Q5) – *Differentiation*

METHOD 1

$\frac{f(x+h)-f(x)}{h}$

$= \frac{\left(3(x+h)^3-(x+h)\right)-\left(3x^3-x\right)}{h}$ ***M1***

$= \frac{3\left(x^3+3x^2h+3xh^2+h^3\right)-x-h-3x^3+x}{h}$ ***(A1)***

$= \frac{9x^2h+9xh^2+3h^3-h}{h}$ ***A1***

cancelling h ***M1***

$= 9x^2+9xh+3h^2-1$

then $\lim_{h\to 0}\left(9x^2+9xh+3h^2-1\right)$

$= 9x^2-1$ ***A1***

Note: Final ***A1*** dependent on all previous marks.

METHOD 2

$\frac{f(x+h)-f(x)}{h}$

$= \frac{\left(3(x+h)^3-(x+h)\right)-\left(3x^3-x\right)}{h}$ ***M1***

$= \frac{3\left((x+h)^3-x^3\right)+\left(x-(x+h)\right)}{h}$ ***(A1)***

$= \frac{3h\left((x+h)^2+x(x+h)+x^2\right)-h}{h}$ ***A1***

cancelling h ***M1***

$= 3\left((x+h)^2+x(x+h)+x^2\right)-1$

then $\lim_{h\to 0}\left(3\left((x+h)^2+x(x+h)+x^2\right)-1\right)$

$= 9x^2-1$ ***A1***

Note: Final ***A1*** dependent on all previous marks.

[5 marks]

35 – (MAT/20_HL_Winter_2018_Q9) – *Functions - Roots, Differentiation*

(a) **METHOD 1**

$$f'(x) = \frac{\frac{2(x-3)}{x} - (2\ln x + 1)}{(x-3)^2} \left(= \frac{2(x-3) - x(2\ln x + 1)}{x(x-3)^2} \right)$$ ***(M1)A1A1A1***

Note: Award ***M1*** for attempt at quotient rule, ***A1A1*** for numerator and ***A1*** for denominator.

METHOD 2

$$f(x) = (2\ln x + 1)(x-3)^{-1}$$ ***(A1)***

$$f'(x) = \left(\frac{2}{x}\right)(x-3)^{-1} - (2\ln x + 1)(x-3)^{-2} \left(= \frac{2(x-3) - x(2\ln x + 1)}{x(x-3)^2} \right)$$ ***(M1)A1A1***

Note: Award ***M1*** for attempt at product rule, ***A1*** for first term, ***A1*** for second term.

[4 marks]

(b) finding turning point of $y = f'(x)$ or finding root of $y = f''(x)$ ***(M1)***

$x = 0.899$ ***A1***

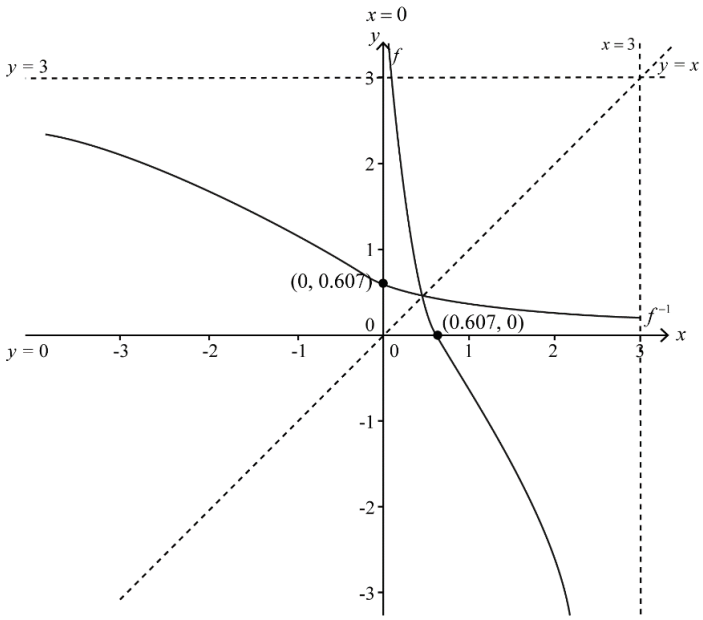
$y = f(0.899048\ldots) = -0.375$ ***(M1)A1***

$(0.899, -0.375)$

Note: Do not accept $x = 0.9$. Accept y-coordinates rounding to -0.37 or -0.375 but not -0.38.

[4 marks]

(c)



(i) smooth curve over the correct domain which does not cross the y-axis and is concave down for $x > 1$ ***A1***

x-intercept at 0.607 ***A1***

equations of asymptotes given as $x = 0$ and $x = 3$ (the latter must be drawn) ***A1A1***

[4 marks]

(ii) attempt to reflect graph of f in $y = x$ ***(M1)***

smooth curve over the correct domain which does not cross the x-axis and is concave down for $y > 1$ ***A1***

y-intercept at 0.607 ***A1***

equations of asymptotes given as $y = 0$ and $y = 3$ (the latter must be drawn) ***A1***

Note: For ***FT*** from (i) to (ii) award max ***M1A0A1A0***.

[4 marks]

(d) solve $f(x) = f^{-1}(x)$ or $f(x) = x$ to get $x = 0.372$ ***(M1)A1***

$0 < x < 0.372$ ***A1***

Note: Do not award ***FT*** marks.

[3 marks]

36 – (MAT/21_HL_Summer_2019_Q1) – *Differentiation*

METHOD 1

equation of tangent is $y = 22.167...\,x - 14.778...$ **OR** $y \quad 7.389... = 22.167...\,(x \quad 1)$ ***(M1)(A1)***

meets the x-axis when $y = 0$

$x = 0.667$

meets x -axis at $(0.667, 0)\left(= \left(\frac{2}{3}, 0\right)\right)$ ***A1A1***

Note: Award ***A1*** for $x = \frac{2}{3}$ or $x = 0.667$ seen and ***A1*** for coordinates $(x, 0)$ given.

37 – (MAT/20_HL_Winter_2019_Q11) – *Differentiation, Integration*

(a) (i) valid attempt to differentiate implicitly ***(M1)***

$4x = 3\sin^2 y \cos y \frac{dy}{dx}$ ***A1A1***

$\frac{dy}{dx} = \frac{4x}{3\sin^2 y \cos y}$ ***A1***

(ii) at $\left(\frac{1}{4}, \frac{5\pi}{6}\right)$, $\frac{dy}{dx} = \frac{4x}{3\sin^2 y \cos y} = \frac{1}{3\left(\frac{1}{2}\right)^2\left(-\frac{\sqrt{3}}{2}\right)}$ ***(M1)***

$\Rightarrow \frac{dy}{dx} = -\frac{8}{3\sqrt{3}}(= -1.54)$ ***A1***

hence equation of tangent is

$y - \frac{5\pi}{6} = -1.54\left(x - \frac{1}{4}\right)$ **OR** $y = -1.54x + 3.00$ ***(M1)A1***

Note: Accept $y = -1.54x + 3$.

[8 marks]

(b) $x = \sqrt{\frac{1}{2}\sin^3 y}$ ***(M1)***

$\int_0^{\pi} \sqrt{\frac{1}{2}\sin^3 y}\, dy$ ***(A1)***

$= 1.24$ ***A1***

[3 marks]

(c) use of volume $= \int \pi x^2 \mathrm{d}y$ ***(M1)***

$= \int_0^{\pi} \frac{1}{2}\pi \sin^3 y \, \mathrm{d}y$ ***A1***

$= \frac{1}{2}\pi \int_0^{\pi} \left(\sin y - \sin y \cos^2 y\right) \mathrm{d}y$

Note: Condone absence of limits up to this point.

reasonable attempt to integrate ***(M1)***

$= \frac{1}{2}\pi \left[-\cos y + \frac{1}{3}\cos^3 y\right]_0^{\pi}$ ***A1A1***

Note: Award ***A1*** for correct limits (not to be awarded if previous ***M1*** has not been awarded) and ***A1*** for correct integrand.

$= \frac{1}{2}\pi\left(1 - \frac{1}{3}\right) - \frac{1}{2}\pi\left(-1 + \frac{1}{3}\right)$ ***A1***

$= \frac{2\pi}{3}$ ***AG***

Note: Do not accept decimal answer equivalent to $\frac{2\pi}{3}$.

[6 marks]

38 – (MAT/20_HL_Winter_2020_Q8) – *Differentiation*

(a) valid attempt to use chain rule or quotient rule ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-10\mathrm{e}^{0.5x}}{\left(3 - 2\mathrm{e}^{0.5x}\right)^2}$ OR $\frac{\mathrm{d}y}{\mathrm{d}x} = -10\mathrm{e}^{0.5x}\left(3 - 2\mathrm{e}^{0.5x}\right)^{-2}$ ***A1A1***

[3 marks]

Note: Award ***A1*** for numerator and ***A1*** for denominator, or ***A1*** for each part if the second alternative given.

(b) valid attempt to use chain rule $\left(\text{eg } \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}\right)$ ***(M1)***

$\frac{\mathrm{d}x}{\mathrm{d}t} = -0.1 \div \frac{-10\mathrm{e}^{-2}}{\left(3 - 2\mathrm{e}^{-2}\right)^2}$ $\left(= -0.1 \div -0.181676...\right)$ or equivalent ***(A1)***

$= 0.550428...$

$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.550 \ (\mathrm{ms}^{-1})$ ***A1***

[3 marks]

39 – (MathAA/21_HL_Summer_2021_Q9) – *Differentiation, Kinematics, Relative Velocity*

(a) $\tan\theta = \dfrac{50}{y-x}$ OR $\cot\theta = \dfrac{y-x}{50}$ ***A1***

$y = x + 50\cot\theta$ ***AG***

Note: $y - x$ may be identified as a length on a diagram, and not written explicitly.

[1 mark]

(b) attempt to differentiate with respect to t ***(M1)***

$\dfrac{dy}{dt} = \dfrac{dx}{dt} - 50(\operatorname{cosec}\theta)^2\dfrac{d\theta}{dt}$ ***A1***

attempt to set speed of B equal to double the speed of A ***(M1)***

$2\dfrac{dx}{dt} = \dfrac{dx}{dt} - 50(\operatorname{cosec}\theta)^2\dfrac{d\theta}{dt}$

$\dfrac{dx}{dt} = -50(\operatorname{cosec}\theta)^2\dfrac{d\theta}{dt}$ ***A1***

$\theta = \arctan 5 (= 1.373\ldots = 78.69\ldots°)$ OR $\operatorname{cosec}^2\theta = 1 + \cot^2\theta = 1 + \left(\dfrac{1}{5}\right)^2 = \dfrac{26}{25}$ ***(A1)***

Note: This ***A1*** can be awarded independently of previous marks.

$\dfrac{dx}{dt} = -50\left(\dfrac{26}{25}\right) \times -0.1$

So the speed of boat A is 5.2 (ms^{-1}) ***A1***

Note: Accept 5.20 from the use of inexact values.

[6 marks]

40 – (MathAA/21_HL_Summer_2021_Q10) – *Exponentials - Logarithms, Differentiation*

(a) attempt to find the point of intersection of the graph of f and the line $y = x$ ***(M1)***

$x = 5.56619...$

$= 5.57$ ***A1***

[2 marks]

(b) $f'(x) = -45e^{-0.5x}$ ***A1***

attempt to set the gradient of f equal to -1 ***(M1)***

$-45e^{-0.5x} = -1$

Q has coordinates $(2\ln 45, 2)$ (accept $(-2\ln\frac{1}{45}, 2)$) ***A1A1***

Note: Award ***A1*** for each value, even if the answer is not given as a coordinate pair.

Do not accept $\frac{\ln\frac{1}{45}}{-0.5}$ or $\frac{\ln 45}{0.5}$ as a final value for x. Do not accept 2.0 or 2.00 as a final value for y.

[4 marks]

(c) attempt to substitute coordinates of Q (in any order) into an appropriate equation ***(M1)***

$y - 2 = -(x - 2\ln 45)$ OR $2 = -2\ln 45 + c$ ***A1***

equation of L is $y = -x + 2\ln 45 + 2$ ***AG***

[2 marks]

(d) (i) $x = \ln 45 + 1 \left(= 4.81\right)$ ***A1***

(ii) appropriate method to find the sum of two areas using integrals of the difference of two functions ***(M1)***

Note: Allow absent or incorrect limits.

$$\int_{4.806...}^{5.566...} \left(x - \left(-x + 2\ln 45 + 2\right)\right) dx + \int_{5.566...}^{7.613...} \left(90e^{-0.5x} - \left(-x + 2\ln 45 + 2\right)\right) dx$$ ***(A1)(A1)***

Note: Award ***A1*** for one correct integral expression including correct limits and integrand.
Award ***A1*** for a second correct integral expression including correct limits and integrand.

$= 1.51965...$
$= 1.52$ ***A1***

[5 marks]

(e) by symmetry 2×1.52 ***(M1)***
$= 3.03930...$
$= 3.04$ ***A1***

Note: Accept any answer that rounds to 3.0 (but do not accept 3).

[2 marks]

41 – (MathAA/22_HL_Summer_2021_Q11) – *Differentiation*

(a) attempt to use $V = \pi\int_a^b \left(f(x)\right)^2 \mathrm{d}x$ ***(M1)***

$$V = \pi\int_0^{\ln 16}\left(\frac{k\mathrm{e}^{\frac{x}{2}}}{1+\mathrm{e}^x}\right)^2 \mathrm{d}x \quad \left(V = k^2\pi\int_0^{\ln 16}\frac{\mathrm{e}^x}{\left(1+\mathrm{e}^x\right)^2}\mathrm{d}x\right)$$

EITHER

applying integration by recognition ***(M1)***

$= k^2\pi\left[-\frac{1}{1+\mathrm{e}^x}\right]_0^{\ln 16}$ ***A3***

OR

$u = 1+\mathrm{e}^x \Rightarrow \mathrm{d}u = \mathrm{e}^x\mathrm{d}x$ ***(A1)***

attempt to express the integral in terms of u ***(M1)***

when $x = 0$, $u = 2$ and when $x = \ln 16$, $u = 17$

$V = k^2\pi\int_2^{17}\frac{1}{u^2}\mathrm{d}u$ ***(A1)***

$= k^2\pi\left[-\frac{1}{u}\right]_2^{17}$ ***A1***

OR

$u = \mathrm{e}^x \Rightarrow \mathrm{d}u = \mathrm{e}^x\mathrm{d}x$ ***(A1)***

attempt to express the integral in terms of u ***(M1)***

when $x = 0$, $u = 1$ and when $x = \ln 16$, $u = 16$

$V = k^2\pi\int_1^{16}\frac{1}{(1+u)^2}\mathrm{d}u$ ***(A1)***

$= k^2\pi\left[-\frac{1}{1+u}\right]_1^{16}$ ***A1***

Note: Accept equivalent working with indefinite integrals and original limits for x.

THEN

$= k^2\pi\left(\frac{1}{2}-\frac{1}{17}\right)$ ***A1***

so the volume of the solid formed is $\frac{15k^2\pi}{34}$ cubic units ***AG***

Note: Award ***(M1)(A0)(M0)(A0)(A0)(A1)*** when $\frac{15}{34}$ is obtained from GDC

[6 marks]

(b) a valid algebraic or graphical attempt to find k ***(M1)***

$k^2 = \frac{300 \times 34}{15\pi}$

$k = 14.7 \left(= 2\sqrt{\frac{170}{\pi}} = \sqrt{\frac{680}{\pi}} \right)$ (as $k \in \mathbb{R}^+$) ***A1***

Note: Candidates may use their GDC numerical solve feature.

[2 marks]

(c) (i) attempting to find $\text{OA} = f(0) = \frac{k}{2}$

with $k = 14.712... \left(= 2\sqrt{\frac{170}{\pi}} = \sqrt{\frac{680}{\pi}} \right)$ ***(M1)***

$\text{OA} = 7.36 \left(= \sqrt{\frac{170}{\pi}} \right)$ ***A1***

(ii) attempting to find $\text{BC} = f(\ln 16) = \frac{4k}{17}$

with $k = 14.712... \left(= 2\sqrt{\frac{170}{\pi}} = \sqrt{\frac{680}{\pi}} \right)$ ***(M1)***

$\text{BC} = 3.46 \left(= \frac{8}{17}\sqrt{\frac{170}{\pi}} = \frac{8\sqrt{10}}{\sqrt{17\pi}} \right)$ ***A1***

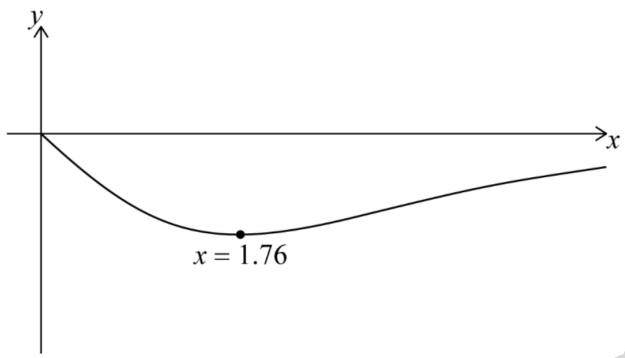
[4 marks]

(d) (i) **EITHER**

recognising to graph $y = f'(x)$ ***(M1)***

Note: Award ***M1*** for attempting to use quotient rule or product rule differentiation.

$$f'(x) = \frac{ke^{\frac{x}{2}}\left(1 - e^{x}\right)}{2\left(1 + e^{x}\right)^{2}}$$



for $x > 0$ graph decreasing to the local minimum ***A1***

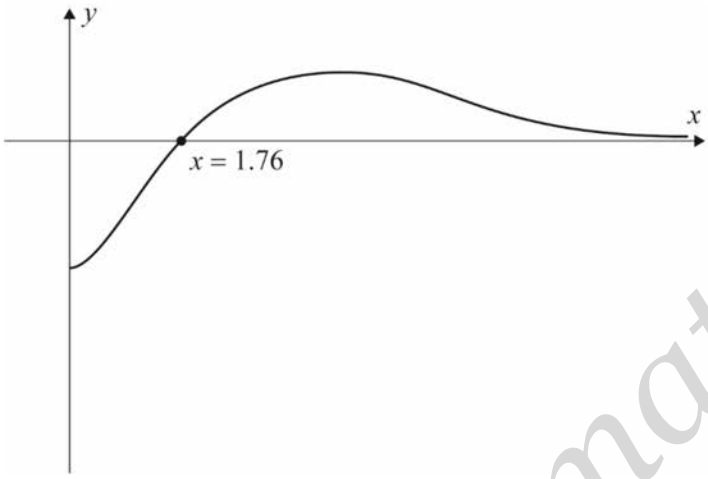
before increasing towards the x-axis ***A1***

OR

recognising to graph $y = f''(x)$ ***(M1)***

Note: Award ***M1*** for attempting to use quotient rule or product rule differentiation.

$$f''(x) = \frac{ke^{\frac{x}{2}}\left(e^{2x} - 6e^{x} + 1\right)}{4\left(1 + e^{x}\right)^{3}}$$



for $x > 0$, graph increasing towards and beyond the x-intercept ***A1***

recognising $f''(x) = 0$ for maximum rate ***(A1)***

THEN

$x = 1.76 \ \left(= \ln\left(2\sqrt{2} + 3\right)\right)$ ***A1***

Note: Only award ***A*** marks if either graph is seen.

(ii) attempting to find $f(1.76...)$ ***(M1)***

the cross-sectional radius at this point is $5.20 \ \left(\sqrt{\frac{85}{\pi}}\right)$ (cm) ***A1***

[6 marks]

42 – (MathAA/22_HL_Summer_2021_Q12) – *Algebra-functions - Roots, Differentiation*

(a) **EITHER**

$$f(-x) = \arcsin\left(\frac{(-x)^2 - 1}{(-x)^2 + 1}\right) = \arcsin\left(\frac{x^2 - 1}{x^2 + 1}\right) = f(x)$$ ***R1***

OR

a sketch graph of $y = f(x)$ with line symmetry in the y-axis indicated ***R1***

THEN

so $f(x)$ is an even function ***AG***

[1 mark]

(b) as $x \to \pm\infty$, $f(x) \to \arcsin 1 \left(\to \frac{\pi}{2}\right)$ ***A1***

so the horizontal asymptote is $y = \frac{\pi}{2}$ ***A1***

[2 marks]

(c) (i) attempting to use the quotient rule to find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2-1}{x^2+1}\right)$ ***M1***

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2-1}{x^2+1}\right)=\frac{2x\left(x^2+1\right)-2x\left(x^2-1\right)}{\left(x^2+1\right)^2}\quad\left(=\frac{4x}{\left(x^2+1\right)^2}\right)$$ ***A1***

attempting to use the chain rule to find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\arcsin\left(\frac{x^2-1}{x^2+1}\right)\right)$ ***M1***

let $u=\frac{x^2-1}{x^2+1}$ and so $y=\arcsin u$ and $\frac{\mathrm{d}y}{\mathrm{d}u}=\frac{1}{\sqrt{1-u^2}}$

$$f'(x)=\frac{1}{\sqrt{1-\left(\frac{x^2-1}{x^2+1}\right)^2}}\times\frac{4x}{\left(x^2+1\right)^2}$$ ***M1***

$$=\frac{4x}{\sqrt{\left(x^2+1\right)^2-\left(x^2-1\right)^2}}\times\frac{1}{\left(x^2+1\right)}$$ ***A1***

$$=\frac{4x}{\sqrt{4x^2}}\times\frac{1}{\left(x^2+1\right)}$$ ***A1***

$$=\frac{2x}{\sqrt{x^2}\left(x^2+1\right)}$$ ***AG***

(ii) $f'(x)=\frac{2x}{|x|\left(x^2+1\right)}$

EITHER

for $x<0$, $|x|=-x$ ***(A1)***

so $f'(x)=-\frac{2}{x^2+1}$ ***A1***

OR

$|x|>0$ and $x^2+1>0$ ***A1***

$2x<0,\ x<0$ ***A1***

THEN

$f'(x)<0$ ***R1***

Note: Award ***R1*** for stating that in $f'(x)$, the numerator is negative, and the denominator is positive.

so f is decreasing for $x<0$ ***AG***

Note: Do not accept a graphical solution.

[9 marks]

(d) $x = \arcsin\left(\frac{y^2-1}{y^2+1}\right)$ ***M1***

$\sin x = \frac{y^2-1}{y^2+1} \Rightarrow y^2 \sin x + \sin x = y^2 - 1$ ***A1***

$y^2 = \frac{1+\sin x}{1-\sin x}$ ***A1***

domain of g is $x \in \mathbb{R}, x \geq 0$ and so the range of g^{-1} must be $y \in \mathbb{R}, y \geq 0$

hence the positive root is taken (or the negative root is rejected) ***R1***

Note: The ***R1*** is dependent on the above ***A1***.

so $\left(g^{-1}(x) =\right)\sqrt{\frac{1+\sin x}{1-\sin x}}$ ***A1***

Note: The final ***A1*** is not dependent on ***R1*** mark.

[5 marks]

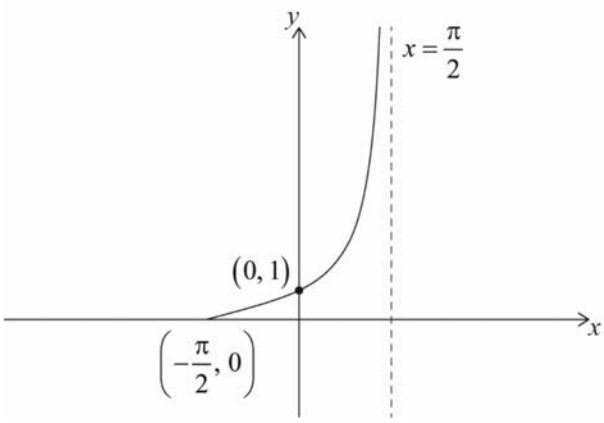
(e) domain is $-\frac{\pi}{2} \leq x < \frac{\pi}{2}$ ***A1***

Note: Accept correct alternative notations, for example, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right[$ or $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Accept $[-1.57, 1.57[$ if correct to 3 s.f.

[1 mark]

(f)



A1A1A1

Note: ***A1*** for correct domain and correct range and y-intercept at $y = 1$

A1 for asymptotic behaviour $x \to \frac{\pi}{2}$

A1 for $x = \frac{\pi}{2}$

Coordinates are not required.
Do not accept $x = 1.57$ or other inexact values.

[3 marks]

43 – (MathAA/20_HL_Winter_2021_Q8) – *Differentiation*

(a) **METHOD 1**

attempts to differentiate implicitly including at least one application of the product rule ***(M1)***

$u = xy$, $v = \ln(xy)$, $\frac{du}{dx} = x\frac{dy}{dx} + y$, $\frac{dv}{dx} = \left(x\frac{dy}{dx} + y\right)\frac{1}{xy}$

$\frac{dy}{dx} = 1 - \left[\frac{xy}{xy}\left(x\frac{dy}{dx} + y\right) + \left(x\frac{dy}{dx} + y\right)\ln(xy)\right]$ ***A1***

Note: Award ***(M1)A1*** for implicitly differentiating $y = x(1 - y\ln(xy))$ and obtaining

$\frac{dy}{dx} = 1 - \left[\frac{xy}{xy}\left(x\frac{dy}{dx} + y\right) + x\frac{dy}{dx}\ln(xy) + y\ln(xy)\right]$.

$\frac{dy}{dx} = 1 - \left[\left(x\frac{dy}{dx} + y\right) + \left(x\frac{dy}{dx} + y\right)\ln(xy)\right]$

$\frac{dy}{dx} = 1 - \left(x\frac{dy}{dx} + y\right)(1 + \ln(xy))$ ***A1***

$\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right)(1 + \ln(xy)) = 1$ ***AG***

METHOD 2

$y = x - xy\ln x - xy\ln y$

attempts to differentiate implicitly including at least one application of the product rule ***(M1)***

$\frac{dy}{dx} = 1 - \left(\frac{xy}{x} + \left(x\frac{dy}{dx} + y\right)\ln x\right) - \left(\frac{xy}{y}\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right)\ln y\right)$ ***A1***

or equivalent to the above, for example

$\frac{dy}{dx} = 1 - \left(x\ln x\frac{dy}{dx} + (1 + \ln x)y\right) - \left(y\ln y + x\left(\ln y\frac{dy}{dx} + \frac{dy}{dx}\right)\right)$

$\frac{dy}{dx} = 1 - x\frac{dy}{dx}(\ln x + \ln y + 1) - y(\ln x + \ln y + 1)$ ***A1***

or equivalent to the above, for example

$\frac{dy}{dx} = 1 - x\frac{dy}{dx}(\ln(xy) + 1) - y(\ln(xy) + 1)$

$\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right)(1 + \ln(xy)) = 1$ ***AG***

METHOD 3

attempt to differentiate implicitly including at least one application of the product rule ***M1***

$u = x\ln(xy),\ v = y,\ \frac{du}{dx} = \ln(xy) + \left(x\frac{dy}{dx} + y\right)\frac{x}{xy},\ \frac{dv}{dx} = \frac{dy}{dx}$

$\frac{dy}{dx} = 1 - \left(x\frac{dy}{dx}\ln(xy) + y\ln(xy) + \frac{xy}{xy}\left(x\frac{dy}{dx} + y\right)\right)$ ***A1***

$\frac{dy}{dx} = 1 - x\frac{dy}{dx}(\ln(xy) + 1) - y(\ln(xy) + 1)$ ***A1***

$\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right)(1 + \ln(xy)) = 1$ ***AG***

METHOD 4

lets $w = xy$ and attempts to find $\frac{dy}{dx}$ where $y = x - w\ln w$ ***M1***

$\frac{dy}{dx} = 1 - \left(\frac{dw}{dx} + \frac{dw}{dx}\ln w\right)\ \left(= 1 - \frac{dw}{dx}(1 + \ln w)\right)$ ***A1***

$\frac{dw}{dx} = x\frac{dy}{dx} + y$ ***A1***

$\frac{dy}{dx} = 1 - \left(x\frac{dy}{dx} + y + \left(x\frac{dy}{dx} + y\right)\ln(xy)\right)\ \left(= 1 - \left(x\frac{dy}{dx} + y\right)(1 + \ln(xy))\right)$

$\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right)(1 + \ln(xy)) = 1$ ***AG***

[3 marks]

(b) **METHOD 1**

substitutes $x=1$ into $y=x-xy\ln(xy)$ ***(M1)***

$y=1-y\ln y \Rightarrow y=1$ ***A1***

substitutes $x=1$ and their non-zero value of y into $\frac{dy}{dx}+\left(x\frac{dy}{dx}+y\right)(1+\ln(xy))=1$ ***(M1)***

$2\frac{dy}{dx}=0\ \left(\frac{dy}{dx}=0\right)$ ***A1***

equation of the tangent is $y=1$ ***A1***

METHOD 2

substitutes $x=1$ into $\frac{dy}{dx}+\left(x\frac{dy}{dx}+y\right)(1+\ln(xy))=1$ ***(M1)***

$\frac{dy}{dx}+\left(\frac{dy}{dx}+y\right)(1+\ln(y))=1$

EITHER

correctly substitutes $\ln y=\frac{1-y}{y}$ into $\frac{dy}{dx}+\left(\frac{dy}{dx}+y\right)(1+\ln(y))=1$ ***A1***

$\frac{dy}{dx}\left(1+\frac{1}{y}\right)=0 \Rightarrow \frac{dy}{dx}=0\ (y=1)$ ***A1***

OR

correctly substitutes $y+y\ln y=1$ into $\frac{dy}{dx}+\left(\frac{dy}{dx}+y\right)(1+\ln(y))=1$ ***A1***

$\frac{dy}{dx}(2+\ln y)=0 \Rightarrow \frac{dy}{dx}=0\ (y=1)$ ***A1***

THEN

substitutes $x=1$ into $y=x-xy\ln(xy)$ ***(M1)***

$y=1-y\ln y \Rightarrow y=1$

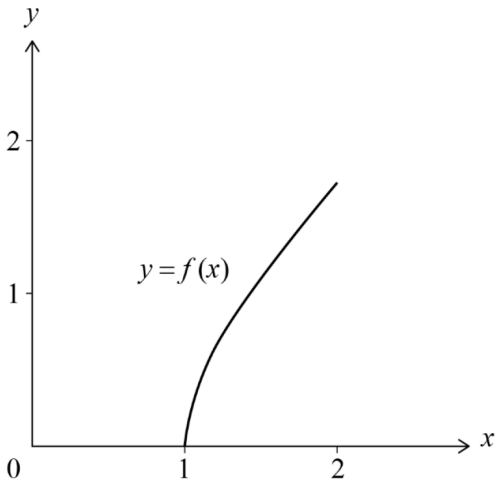
equation of the tangent is $y=1$ ***A1***

[5 marks]

Total [8 marks]

44 – (MathAA/21_HL_Summer_2022_Q10) – *Algebra-functions - Roots, Differentiation, Integration*

(a)



correct shape (concave down) within the given domain $1 \le x \le 2$ ***A1***

$(1,0)$ and $(2,\sqrt{3})(=(2,1.73))$ ***A1***

Note: The coordinates of endpoints may be seen on the graph or marked on the axes.

(b) (i) interchanging x and y (seen anywhere) ***M1***

$x = \sqrt{y^2 - 1}$

$x^2 = y^2 - 1$ ***A1***

$y = \sqrt{x^2 + 1}$ ***A1***

$f^{-1}(x) = \sqrt{x^2 + 1}$ ***AG***

(ii) $0 \le x \le \sqrt{3}$ OR domain $[0,\sqrt{3}](=[0,1.73])$ ***A1***

$1 \le y \le 2$ OR $1 \le f^{-1}(x) \le 2$ OR range $[1,2]$ ***A1***

[5 marks]

(c) (i) attempt to substitute $x=\sqrt{y^2+1}$ into the correct volume formula ***(M1)***

$V=\pi\int_0^h\left(\sqrt{y^2+1}\right)^2\,\mathrm{d}y\ \left(=\pi\int_0^h\left(y^2+1\right)\,\mathrm{d}y\right)$ ***A1***

$=\pi\left[\frac{1}{3}y^3+y\right]_0^h$ ***A1***

$=\pi\left(\frac{1}{3}h^3+h\right)$ ***AG***

Note: Award marks as appropriate for correct work using a different variable e.g.

$\pi\int_0^h\left(\sqrt{x^2+1}\right)^2\,\mathrm{d}x$

(ii) attempt to substitute $h=\sqrt{3}\ (=1.732...)$ into V ***(M1)***

$V=10.8828...$

$V=10.9$ (m³) $\left(=2\sqrt{3}\pi\right)$ (m³) ***A1***

[5 marks]

(d) **METHOD 1**

time $=\frac{10.8828...}{0.4}\left(=\frac{2\sqrt{3}\pi}{0.4}\right)$ ***(M1)***

$=27.207...$

$=27.2\left(=5\sqrt{3}\pi\right)(s)$ ***A1***

[2 marks]

(e) attempt to find the height of the tank when $V = 5.4414...\ \left(= \sqrt{3}\pi\right)$ ***(M1)***

$\pi\left(\frac{1}{3}h^3 + h\right) = 5.4414...\ \left(= \sqrt{3}\pi\right)$

$h = 1.1818...$ ***(A1)***

attempt to use the chain rule or differentiate $V = \pi\left(\frac{1}{3}h^3 + h\right)$ with respect to t ***(M1)***

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{\pi(h^2+1)} \times \frac{\mathrm{d}V}{\mathrm{d}t}$ OR $\frac{\mathrm{d}V}{\mathrm{d}t} = \pi\left(h^2+1\right)\frac{\mathrm{d}h}{\mathrm{d}t}$ ***(A1)***

attempt to substitute **their** h and $\frac{\mathrm{d}V}{\mathrm{d}t} = 0.4$ ***(M1)***

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{0.4}{\pi\left(1.1818...^2+1\right)} = 0.053124...$

$= 0.0531$ (m s^{-1}) ***A1***

[6 marks]

45 – (MathAA/22_HL_Summer_2022_Q10) – *Trigonometry, Differentiation*

(a) (i) 32 (cm) ***A1***

(ii) $h_A(0) = \sin(6) + 27$ ***(M1)***

$= 26.7205...$

$= 26.7$ (cm) ***A1***

[3 marks]

(b) attempts to solve $h_A(t) = h_B(t)$ for t ***(M1)***

$t = 4.0074..., 4.7034..., 5.88332...$

$t = 4.01, 4.70, 5.88$ (weeks) ***A2***

[3 marks]

(c) $h_A(t) - h_B(t) = \sin(2t+6) + t - 5$ ***A1***

EITHER

for $t > 6$, $t - 5 > 1$ ***A1***

and as $\sin(2t+6) \geq -1 \Rightarrow h_A(t) - h_B(t) > 0$ ***R1***

OR

the minimum value of $\sin(2t+6) = -1$ ***R1***

so for $t > 6$, $h_A(t) - h_B(t) = t - 6 > 0$ ***A1***

THEN

hence for $t > 6$, Plant A was always taller than Plant B ***AG***

[3 marks]

(d) recognises that $h_A'(t)$ and $h_B'(t)$ are required ***(M1)***

attempts to solve $h_A'(t) = h_B'(t)$ for t ***(M1)***

$t = 1.18879...$ and $2.23598...$ OR $4.33038...$ and $5.37758...$ OR $7.47197...$ and $8.51917...$ ***(A1)***

Note: Award full marks for $t = \frac{4\pi}{3} - 3, \frac{5\pi}{3} - 3, \left(\frac{7\pi}{3} - 3, \frac{8\pi}{3} - 3, \frac{10\pi}{3} - 3, \frac{11\pi}{3} - 3\right)$.
Award subsequent marks for correct use of these exact values.

$1.18879... < t < 2.23598...$ OR $4.33038... < t < 5.37758...$ OR
$7.47197... < t < 8.51917...$ ***(A1)***

attempts to calculate the total amount of time ***(M1)***

$3(2.2359... - 1.1887...)$ $\left(= 3\left(\left(\frac{5\pi}{3} - 3\right) - \left(\frac{4\pi}{3} - 3\right)\right)\right)$

$= 3.14$ $(= \pi)$ (weeks) ***A1***

[6 marks]

46 – (MathAA/20_HL_Winter_2022_Q2) – *Differentiation, Integration*

(a) $(0.708519...,0.639580...)$

$(0.709,0.640)$ $(x=0.709,\ y=0.640)$ ***A1A1***

[2 marks]

(b) $1.09885...$

$x=1.10$ (accept $(1.10,0)$) ***A1***

[1 mark]

(c) **METHOD 1**

$\int_0^2 |f(x)|dx$ ***(A1)***

$4.61117...$

area = 4.61 ***A2***

METHOD 2

$-\int_{1.09885...}^2 f(x)dx$ OR $\int_{1.09885...}^2 |f(x)|dx$ OR 4.17527... ***(A1)***

$\int_0^{1.09885...} f(x)dx - \int_{1.09885...}^2 f(x)dx$ OR $0.435901... + 4.17527...$ ***(A1)***

$4.61117...$

area = 4.61 ***A1***

[3 marks]

47 – (MathAA/20_HL_Winter_2022_Q8) – *Differentiation*

EITHER

$\left(\frac{\mathrm{d}V}{\mathrm{d}h}=\right)10\pi h-\pi h^2$ ***(A1)***

Note: This ***A1*** may be implied by the value $\frac{\mathrm{d}V}{\mathrm{d}h}=76.5616...$.

attempt to use chain rule to find a relationship between $\frac{\mathrm{d}h}{\mathrm{d}t}$, $\frac{\mathrm{d}V}{\mathrm{d}t}$ and $\frac{\mathrm{d}V}{\mathrm{d}h}$ ***(M1)***

$$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mathrm{d}h}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}\left(=\frac{1}{\left(\frac{\mathrm{d}V}{\mathrm{d}h}\right)}\times\frac{\mathrm{d}V}{\mathrm{d}t}\right)$$

OR

attempt to differentiate $V=5\pi h^2-\frac{1}{3}\pi h^3$ throughout with respect to t ***(M1)***

$\frac{\mathrm{d}V}{\mathrm{d}t}=10\pi h\frac{\mathrm{d}h}{\mathrm{d}t}-\pi h^2\frac{\mathrm{d}h}{\mathrm{d}t}$ ***(A1)***

THEN

$\left(10\pi h - \pi h^2\right)\frac{dh}{dt} = 2$ OR $\frac{dh}{dt} = \frac{2}{10\pi h - \pi h^2}$ ***(A1)***

Note: Award this ***A1*** if the correct expression is seen with their h already substituted.

attempt to solve $200 = 5\pi h^2 - \frac{1}{3}\pi h^3$ ***(M1)***

$h = 4.20648...$ ***(A1)***

Note: This ***(M1)(A1)*** can be awarded independently of all previous marks, and may be implied by the value $\frac{dV}{dh} = 76.5616...$
Ignore extra values of h -3.24 and 14.0.

$\frac{dh}{dt} = 0.0261227...$

$\frac{dh}{dt} = 0.0261\left(\text{cms}^{-1}\right)$ ***A1***

[6 marks]

48 – (MathAA/20_HL_Winter_2022_Q11) – *Algebra-functions - Roots, Differentiation*

(a) attempt to use product rule ***(M1)***

$f''(x) = 3e^{2x} + 2e^{2x}(3x-4)\left(= e^{2x}(6x-5)\right)$ ***A2***

Note: Award ***A1*** for 2 out of 3 of $3e^{2x}$, $6xe^{2x}$ and $-8e^{2x}$ seen or implied.

[3 marks]

(b) $f'(x) = 1$ ***(M1)***

$x = 0.86299...$

$x = 0.863$ ***A1***

$y = -7.92719...$

$y = -7.93$ ***A1***

$(0.863, -7.93)$

[3 marks]

(c) x-intercept is at $\frac{4}{3}(1.33)$ ***(A1)***

attempt to use formula for volume of revolution ***(M1)***

Note: Award ***(M1)*** for an integral involving π and $(f(x))^2$. Condone use of 2π and incorrect or absent limits.

$\pi\int_0^{\frac{4}{3}}\left(e^{2x}(3x-4)\right)^2 dx$ ***(A1)***

Note: This ***(A1)*** can be awarded if the dx is omitted.

$= 164.849...$

$= 165$ ***A1***

[4 marks]

(d) (i) attempt to compose functions in the correct order ***(M1)***

$(f \circ g)(0) = f(g(0)) = f(1)$

$= -7.38905...$

$= -7.39 (= -e^{2})$ ***A1***

(ii) attempt to use the chain rule ***(M1)***

$(f \circ g)'(0) = f'(g(0))g'(0)$

Note: For this ***(M1)*** to be awarded, multiplication of two derivatives should be seen or implied.

$= 2f'(1)(= 2 \times 7.38905...)$ ***(A1)***

$= 14.7781...$

$= 14.8 (= 2e^{2})$ ***A1***

[5 marks]

1 - (MAT/11_HL_Summer_2012_Q6) - *Integration*

(a) $\int_{\frac{1}{6}}^{1} \frac{k}{x} - \frac{1}{x}\,dx = (k-1)[\ln x]_{\frac{1}{6}}^{1}$

Note: Award ***M1*** for $\int \frac{k}{x} - \frac{1}{x}\,dx$ or $\int \frac{1}{x} - \frac{k}{x}\,dx$ and ***A1*** for $(k-1)\ln x$ seen in part (a) or later in part (b).

$= (1-k)\ln\frac{1}{6}$

(b) $\int_{1}^{\sqrt{6}} \frac{k}{x} - \frac{1}{x}\,dx = (k-1)[\ln x]_{1}^{\sqrt{6}}$

Note: Award ***A1*** for correct change of limits.

$= (k-1)\ln\sqrt{6}$

(c) $(1-k)\ln\frac{1}{6} = (k-1)\ln 6$ ***A1***

$(k-1)\ln\sqrt{6} = \frac{1}{2}(k-1)\ln 6$ ***A1***

Note: This simplification could have occurred earlier, and marks should still be awarded.

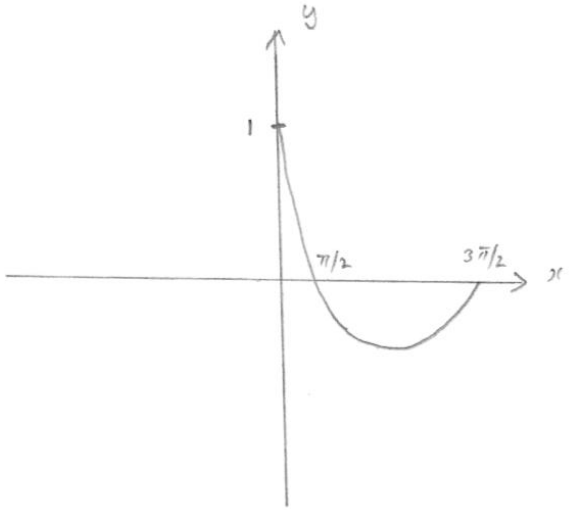
ratio is 2 (or 2:1) ***A1***

2 – (MAT/12_HL_Summer_2012_Q10) – *Functions - Roots, Graphs, Integration*

(a) $e^{-x}\cos x = 0$

$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$

(b)



(c) attempt at integration by parts

EITHER

$$I=\int e^{-x}\cos x\,dx=-e^{-x}\cos x\,dx-\int e^{-x}\sin x\,dx$$

$$\Rightarrow I=-e^{-x}\cos x\,dx-\left[-e^{-x}\sin x+\int e^{-x}\cos x\,dx\right]$$

$$\Rightarrow I=\frac{e^{-x}}{2}(\sin x-\cos x)+C$$

Note: Do not penalize absence of C.

OR

$$I=\int e^{-x}\cos x\,dx=e^{-x}\sin x+\int e^{-x}\sin x\,dx$$

$$\Rightarrow I=e^{-x}\sin x-e^{-x}\cos x-\int e^{-x}\cos x\,dx$$

$$\Rightarrow I=\frac{e^{-x}}{2}(\sin x-\cos x)+C$$

Note: Do not penalize absence of C.

THEN

$$\int_0^{\frac{\pi}{2}} e^{-x}\cos x\,dx=\left[\frac{e^{-x}}{2}(\sin x-\cos x)\right]_0^{\frac{\pi}{2}}=\frac{e^{-\frac{\pi}{2}}}{2}+\frac{1}{2}$$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{-x}\cos x\,dx=\left[\frac{e^{-x}}{2}(\sin x-\cos x)\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}=-\frac{e^{-\frac{3\pi}{2}}}{2}-\frac{e^{-\frac{\pi}{2}}}{2}$$

ratio of $A:B$ is $\dfrac{\frac{e^{-\frac{\pi}{2}}}{2}+\frac{1}{2}}{\frac{e^{-\frac{3\pi}{2}}}{2}+\frac{e^{-\frac{\pi}{2}}}{2}}$

$$=\frac{e^{\frac{3\pi}{2}}\left(e^{-\frac{\pi}{2}}+1\right)}{e^{\frac{3\pi}{2}}\left(e^{-\frac{3\pi}{2}}+e^{-\frac{\pi}{2}}\right)}$$

$$=\frac{e^{\pi}\left(e^{\frac{\pi}{2}}+1\right)}{e^{\pi}+1}$$

3 – (MAT/11_HL_Summer_2012_Q12) – *Differentiation, Integration*

(a) **EITHER**

derivative of $\frac{x}{1-x}$ is $\frac{(1-x)-x(-1)}{(1-x)^2}$

$$f'(x)=\frac{1}{2}\left(\frac{x}{1-x}\right)^{-\frac{1}{2}}\frac{1}{(1-x)^2}$$

$$=\frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}$$

$f'(x)>0$ (for all $0<x<1$) so the function is increasing

OR

$$f(x)=\frac{x^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}}$$

$$f'(x)=\frac{(1-x)^{\frac{1}{2}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)-\frac{1}{2}x^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}(-1)}{1-x}$$

$$=\frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}+\frac{1}{2}x^{\frac{1}{2}}(1-x)^{-\frac{3}{2}}$$

$$=\frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}[1-x+x]$$

$$=\frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}$$

$f'(x)>0$ (for all $0<x<1$) so the function is increasing

(b) $f'(x)=\frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}$

$$\Rightarrow f''(x)=-\frac{1}{4}x^{-\frac{3}{2}}(1-x)^{-\frac{3}{2}}+\frac{3}{4}x^{-\frac{1}{2}}(1-x)^{-\frac{5}{2}}$$

$$=-\frac{1}{4}x^{-\frac{3}{2}}(1-x)^{-\frac{5}{2}}[1-4x]$$

$f''(x)=0\Rightarrow x=\frac{1}{4}$

$f''(x)$ changes sign at $x=\frac{1}{4}$ hence there is a point of inflexion

$x=\frac{1}{4}\Rightarrow y=\frac{1}{\sqrt{3}}$

the coordinates are $\left(\frac{1}{4},\frac{1}{\sqrt{3}}\right)$

(c) $x = \sin^2\theta \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}\theta} = 2\sin\theta\cos\theta$

$$\int\sqrt{\frac{x}{1-x}}\,\mathrm{d}x = \int\sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}\;2\sin\theta\cos\theta\,\mathrm{d}\theta$$
$$= \int 2\sin^2\theta\,\mathrm{d}\theta$$
$$= \int 1-\cos 2\theta\,\mathrm{d}\theta$$
$$= \theta - \frac{1}{2}\sin 2\theta + c$$

$\theta = \arcsin\sqrt{x}$

$\dfrac{1}{2}\sin 2\theta = \sin\theta\cos\theta = \sqrt{x}\sqrt{1-x} = \sqrt{x-x^2}$

hence $\displaystyle\int\sqrt{\frac{x}{1-x}}\,\mathrm{d}x = \arcsin\sqrt{x} - \sqrt{x-x^2} + c$

4 - (MAT/12_HL_Summer_2013_Q1) - *Integration*

$$\left[\frac{1}{3}(x-2)^3 + \ln x - \frac{1}{\pi}\cos\pi x\right]_{(1)}^{(2)}$$

Note: Accept $\dfrac{1}{3}x^3 - 2x^2 + 4x$ in place of $\dfrac{1}{3}(x-2)^3$.

$$= \left(0 + \ln 2 - \frac{1}{\pi}\cos 2\pi\right) - \left(-\frac{1}{3} + \ln 1 - \frac{1}{\pi}\cos\pi\right)$$
$$= \frac{1}{3} + \ln 2 - \frac{2}{\pi}$$

5 - (MAT/11_HL_Summer_2013_Q10) - *Integration*

(a) $\sin\left(\pi x^{-1}\right)=0 \quad \frac{\pi}{x}=\pi, 2\pi\ (...)$

$x=1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}$

(b) $\left[\cos(\pi x^{-1})\right]_{\frac{1}{n+1}}^{\frac{1}{n}}$

$=\cos(\pi n)-\cos(\pi(n+1))$

$=2$ when n is even and $=-2$ when n is odd

(c) $\int_{0.1}^{1}\left|\pi x^{-2}\sin(\pi x^{-1})\right|\mathrm{d}x=2+2+...+2=18$

6 – (MAT/11_HL_Summer_2013_Q12) – *Quadratics, Graphs, Integration*

(a) $4(x-0.5)^2+4$

Note: ***A1*** for two correct parameters, ***A2*** for all three correct.

(b) translation $\begin{pmatrix}0.5\\0\end{pmatrix}$ (allow "0.5 to the right")

stretch parallel to y-axis, scale factor 4 (allow vertical stretch or similar)

translation $\begin{pmatrix}0\\4\end{pmatrix}$ (allow "4 up")

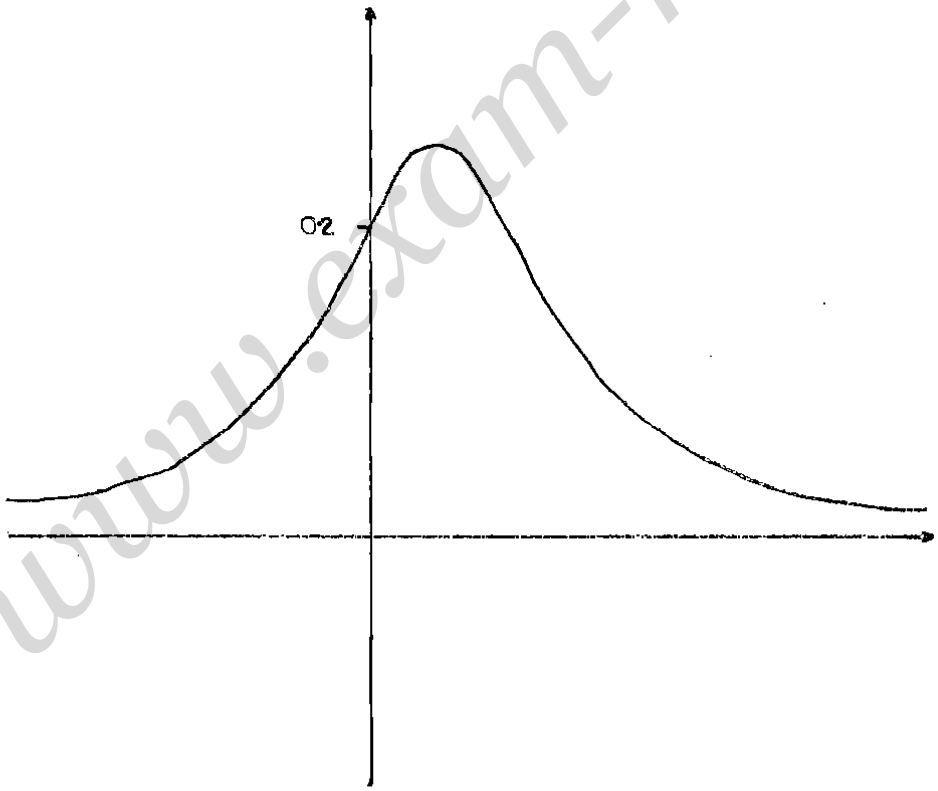
Note: All transformations must state magnitude and direction.

Note: First two transformations can be in either order.

It could be a stretch followed by a single translation

of $\begin{pmatrix}0.5\\4\end{pmatrix}$. If the vertical translation is before the stretch it is $\begin{pmatrix}0\\1\end{pmatrix}$.

(c)



general shape (including asymptote and single maximum in first quadrant),

intercept $\left(0,\frac{1}{5}\right)$ **or** maximum $\left(\frac{1}{2},\frac{1}{4}\right)$ shown

(d) $0 < f(x) \le \frac{1}{4}$

Note: ***A1*** for $\le \frac{1}{4}$, ***A1*** for $0 <$.

(e) let $u = x - \frac{1}{2}$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 1$ (or $\mathrm{d}u = \mathrm{d}x$)

$$\int \frac{1}{4x^2 - 4x + 5}\mathrm{d}x = \int \frac{1}{4\left(x - \frac{1}{2}\right)^2 + 4}\mathrm{d}x$$

$$\int \frac{1}{4u^2 + 4}\mathrm{d}u = \frac{1}{4}\int \frac{1}{u^2 + 1}\mathrm{d}u$$

Note: If following through an incorrect answer to part (a), do not award final ***A1*** mark.

(f) $\int_1^{3.5} \frac{1}{4x^2 - 4x + 5}\mathrm{d}x = \frac{1}{4}\int_{0.5}^{3} \frac{1}{u^2 + 1}\mathrm{d}u$

Note: ***A1*** for correct change of limits. Award also if they do not change limits but go back to x values when substituting the limit (even if there is an error in the integral).

$$\frac{1}{4}\left[\arctan(u)\right]_{0.5}^{3}$$

$$\frac{1}{4}\left(\arctan(3) - \arctan\left(\frac{1}{2}\right)\right)$$

let the integral $= I$

$$\tan 4I = \tan\left(\arctan(3) - \arctan\left(\frac{1}{2}\right)\right)$$

$$\frac{3 - 0.5}{1 + 3 \times 0.5} = \frac{2.5}{2.5} = 1$$

$$4I = \frac{\pi}{4} \quad \Rightarrow I = \frac{\pi}{16}$$

7 – (MAT/10_HL_Winter_2013_Q12) – *Complex Numbers, Trigonometry, Integration*

(a) $z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$
$= \cos n\theta + \cos n\theta + i\sin n\theta - i\sin n\theta$
$= 2\cos n\theta$

(b) $\left(z + z^{-1}\right)^4 = z^4 + 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z^2}\right) + 4z\left(\frac{1}{z^3}\right) + \frac{1}{z^4}$

Note: Accept $\left(z + z^{-1}\right)^4 = 16\cos^4\theta$.

(c) **METHOD 1**

$\left(z + z^{-1}\right)^4 = \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$

$(2\cos\theta)^4 = 2\cos 4\theta + 8\cos 2\theta + 6$

Note: Award ***A1*** for RHS, ***A1*** for LHS independent of the ***M1***.

$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

$\left(\text{or } p = \frac{1}{8},\ q = \frac{1}{2},\ r = \frac{3}{8}\right)$

METHOD 2

$\cos^4\theta = \left(\frac{\cos 2\theta + 1}{2}\right)^2$

$= \frac{1}{4}(\cos^2 2\theta + 2\cos 2\theta + 1)$

$= \frac{1}{4}\left(\frac{\cos 4\theta + 1}{2} + 2\cos 2\theta + 1\right)$

$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

$\left(\text{or } p = \frac{1}{8},\ q = \frac{1}{2},\ r = \frac{3}{8}\right)$

(d) $\left(z+z^{-1}\right)^6 = z^6 + 6z^5\left(\frac{1}{z}\right) + 15z^4\left(\frac{1}{z^2}\right) + 20z^3\left(\frac{1}{z^3}\right) + 15z^2\left(\frac{1}{z^4}\right) + 6z\left(\frac{1}{z^5}\right) + \frac{1}{z^6}$

$\left(z+z^{-1}\right)^6 = \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$

$(2\cos\theta)^6 = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$

Note: Award ***A1*** for RHS, ***A1*** for LHS, independent of the ***M1***.

$\cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$

Note: Accept a purely trigonometric solution as for (c).

(e) $\int_0^{\frac{\pi}{2}} \cos^6\theta \, d\theta = \int_0^{\frac{\pi}{2}} \left(\frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}\right) d\theta$

$= \left[\frac{1}{192}\sin 6\theta + \frac{3}{64}\sin 4\theta + \frac{15}{64}\sin 2\theta + \frac{5}{16}\theta\right]_0^{\frac{\pi}{2}}$

$= \frac{5\pi}{32}$

(f) $V = \pi\int_0^{\frac{\pi}{2}} \sin^2 x \cos^4 x \, dx$

$= \pi\int_0^{\frac{\pi}{2}} \cos^4 x \, dx - \pi\int_0^{\frac{\pi}{2}} \cos^6 x \, dx$

$\int_0^{\frac{\pi}{2}} \cos^4 x \, dx = \frac{3\pi}{16}$

$V = \frac{3\pi^2}{16} - \frac{5\pi^2}{32} = \frac{\pi^2}{32}$

(g) (i) constant term $= \binom{2k}{k} = \frac{(2k)!}{k!k!} = \frac{(2k)!}{(k!)^2}$ (accept C_k^{2k})

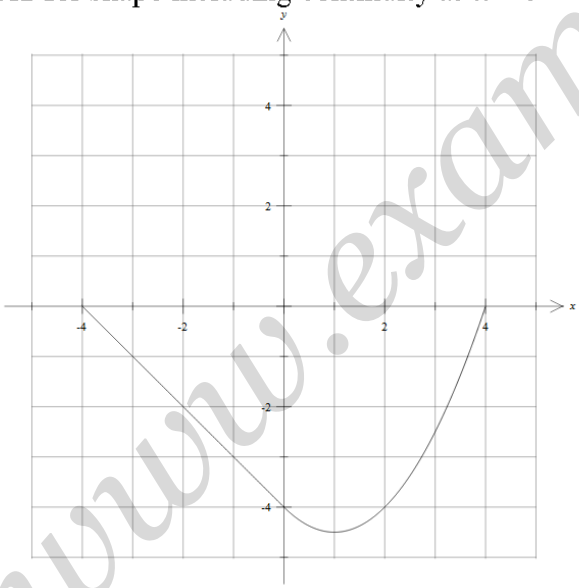
(ii) $2^{2k}\int_0^{\frac{\pi}{2}} \cos^{2k}\theta \mathrm{d}\theta = \frac{(2k)!}{(k!)^2}\frac{\pi}{2}$

$\int_0^{\frac{\pi}{2}} \cos^{2k}\theta \mathrm{d}\theta = \frac{(2k)!\pi}{2^{2k+1}(k!)^2}$ $\left(\text{or } \frac{\binom{2k}{k}\pi}{2^{2k+1}}\right)$

8 – (MAT/11_HL_Summer_2014_Q6) – *Integration, Graphs*

(a) $x = 1$

(b) *A1* for point $(-4, 0)$
A1 for $(0, -4)$
A1 for min at $x = 1$ in approximately the correct place
A1 for $(4, 0)$
A1 for shape including continuity at $x = 0$



9 – (MAT/12_HL_Summer_2014_Q10) – *Integration*

$x = a\sec\theta$

$\frac{\mathrm{d}x}{\mathrm{d}\theta} = a\sec\theta\tan\theta$

new limits:

$x = a\sqrt{2} \Rightarrow \theta = \frac{\pi}{4}$ and $x = 2a \Rightarrow \theta = \frac{\pi}{3}$

$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{a\sec\theta\tan\theta}{a^3\sec^3\theta\sqrt{a^2\sec^2\theta - a^2}}\,\mathrm{d}\theta$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos^2\theta}{a^3}\mathrm{d}\theta$

using $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$

$\frac{1}{2a^3}\left[\frac{1}{2}\sin 2\theta + \theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ or equivalent

$= \frac{1}{4a^3}\left(\frac{\sqrt{3}}{2} + \frac{2\pi}{3} - 1 - \frac{\pi}{2}\right)$ or equivalent

$= \frac{1}{24a^3}\left(3\sqrt{3} + \pi - 6\right)$

10 – (MAT/11_HL_Summer_2014_Q11) – *Differentiation, Integration*

(a) $f'(x)=\dfrac{x\times\frac{1}{x}-\ln x}{x^2}$

$=\dfrac{1-\ln x}{x^2}$

(b) $\dfrac{1-\ln x}{x^2}=0$ has solution $x=\mathrm{e}$

$y=\dfrac{1}{\mathrm{e}}$

hence maximum at the point $\left(\mathrm{e},\dfrac{1}{\mathrm{e}}\right)$

(c) $f''(x)=\dfrac{x^2\left(-\frac{1}{x}\right)-2x(1-\ln x)}{x^4}$

$=\dfrac{2\ln x-3}{x^3}$

Note: The ***M1A1*** should be awarded if the correct working appears in part (b).

point of inflexion where $f''(x)=0$

so $x=\mathrm{e}^{\frac{3}{2}}$, $y=\dfrac{3}{2}\mathrm{e}^{\frac{-3}{2}}$

C has coordinates $\left(\mathrm{e}^{\frac{3}{2}},\dfrac{3}{2}\mathrm{e}^{\frac{-3}{2}}\right)$

(d) $f(1)=0$

$f'(1)=1$

$y=x+c$

through $(1,0)$

equation is $y=x-1$

(e) **METHOD 1**

$$\text{area} = \int_1^{e} x - 1 - \frac{\ln x}{x}\,dx$$

Note: Award ***M1*** for integration of difference between line and curve, ***A1*** for correct limits, ***A1*** for correct expressions in either order.

$$\int \frac{\ln x}{x}\,dx = \frac{(\ln x)^2}{2}(+c)$$

$$\int (x-1)\,dx = \frac{x^2}{2} - x(+c)$$

$$= \left[\frac{1}{2}x^2 - x - \frac{1}{2}(\ln x)^2\right]_1^{e}$$

$$= \left(\frac{1}{2}e^2 - e - \frac{1}{2}\right) - \left(\frac{1}{2} - 1\right)$$

$$= \frac{1}{2}e^2 - e$$

METHOD 2

$$\text{area} = \text{area of triangle} - \int_1^{e} \frac{\ln x}{x}\,dx$$

Note: ***A1*** is for correct integral with limits and is dependent on the ***M1***.

$$\int \frac{\ln x}{x}\,dx = \frac{(\ln x)^2}{2}(+c)$$

$$\text{area of triangle} = \frac{1}{2}(e-1)(e-1)$$

$$\frac{1}{2}(e-1)(e-1) - \left(\frac{1}{2}\right) = \frac{1}{2}e^2 - e$$

11 - (MAT/12_HL_Summer_2014_Q13) - *Differentiation, Integration*

(a) $f'(x)=\dfrac{\left(x^2+1\right)-2x(x+1)}{\left(x^2+1\right)^2}\left(=\dfrac{-x^2-2x+1}{\left(x^2+1\right)^2}\right)$

(b) $\dfrac{-x^2-2x+1}{\left(x^2+1\right)^2}=0$

$x=-1\pm\sqrt{2}$

(c) $f''(x)=\dfrac{(-2x-2)(x^2+1)^2-2(2x)(x^2+1)(-x^2-2x+1)}{\left(x^2+1\right)^4}$

Note: Award *A1* for $(-2x-2)\left(x^2+1\right)^2$ or equivalent.

Note: Award *A1* for $-2(2x)\left(x^2+1\right)\left(-x^2-2x+1\right)$ or equivalent.

$$=\frac{(-2x-2)\left(x^2+1\right)-4x\left(-x^2-2x+1\right)}{\left(x^2+1\right)^3}$$

$$=\frac{2x^3+6x^2-6x-2}{\left(x^2+1\right)^3}$$

$$\left(=\frac{2\left(x^3+3x^2-3x-1\right)}{\left(x^2+1\right)^3}\right)$$

(d) recognition that $(x-1)$ is a factor

$(x-1)\left(x^2+bx+c\right)=\left(x^3+3x^2-3x-1\right)$

$\Rightarrow x^2+4x+1=0$

$x=-2\pm\sqrt{3}$

Note: Allow long division / synthetic division.

(e) $\displaystyle\int_{-1}^{0}\frac{x+1}{x^2+1}\,\mathrm{d}x$

$$\int\frac{x+1}{x^2+1}\,\mathrm{d}x=\int\frac{x}{x^2+1}\,\mathrm{d}x+\int\frac{1}{x^2+1}\,\mathrm{d}x$$

$$=\frac{1}{2}\ln\left(x^2+1\right)+\arctan(x)$$

$$=\left[\frac{1}{2}\ln\left(x^2+1\right)+\arctan(x)\right]_{-1}^{0}=\frac{1}{2}\ln 1+\arctan 0-\frac{1}{2}\ln 2-\arctan(-1)$$

$$=\frac{\pi}{4}-\ln\sqrt{2}$$

12 – (MAT/10_HL_Winter_2014_Q6) – *Integration*

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2\sqrt{x}}$$

$$\mathrm{d}x = 2(u-1)\,\mathrm{d}u$$

Note: Award the ***A1*** for any correct relationship between dx and du.

$$\int \frac{\sqrt{x}}{1+\sqrt{x}}\,\mathrm{d}x = 2\int \frac{(u-1)^2}{u}\,\mathrm{d}u$$

Note: Award the ***M1*** for an attempt at substitution resulting in an integral only involving u.

$$= 2\int u - 2 + \frac{1}{u}\,\mathrm{d}u$$

$$= u^2 - 4u + 2\ln u\,(+C)$$

$$= x - 2\sqrt{x} - 3 + 2\ln\left(1+\sqrt{x}\right)(+C)$$

13 – (MAT/10_HL_Winter_2014_Q11) – *Functions - Roots, Differentiation, Integration*

(a) (i) $x = e^{3y+1}$

Note: The ***M1*** is for switching variables and can be awarded at any stage. Further marks do not rely on this mark being awarded.

taking the natural logarithm of both sides and attempting to transpose

$\left(f^{-1}(x)\right) = \frac{1}{3}(\ln x - 1)$

(ii) $x \in \mathbb{R}^+$ or equivalent, for example $x > 0$.

(b) $\ln x = \frac{1}{3}(\ln x - 1) \Rightarrow \ln x - \frac{1}{3}\ln x = -\frac{1}{3}$ (or equivalent)

$\ln x = -\frac{1}{2}$ (or equivalent)

$x = e^{-\frac{1}{2}}$

coordinates of P are $\left(e^{-\frac{1}{2}}, -\frac{1}{2}\right)$

(c) coordinates of Q are $(1, 0)$ seen anywhere

$\frac{dy}{dx} = \frac{1}{x}$

at Q, $\frac{dy}{dx} = 1$

$y = x - 1$

(d) let the required area be A

$$A = \int_1^e x - 1\,\mathrm{d}x - \int_1^e \ln x\,\mathrm{d}x$$

Note: The ***M1*** is for a difference of integrals. Condone absence of limits here.

attempting to use integration by parts to find $\int \ln x\,\mathrm{d}x$

$$= \left[\frac{x^2}{2} - x\right]_1^{\mathrm{e}} - [x\ln x - x]_1^{\mathrm{e}}$$

Note: Award ***A1*** for $\frac{x^2}{2} - x$ and ***A1*** for $x\ln x - x$.

Note: The second ***M1*** and second ***A1*** are independent of the first ***M1*** and the first ***A1***.

$$= \frac{\mathrm{e}^2}{2} - \mathrm{e} - \frac{1}{2}\left(= \frac{\mathrm{e}^2 - 2\mathrm{e} - 1}{2}\right)$$

(e) (i) **METHOD 1**

consider for example $h(x) = x - 1 - \ln x$

$h(1) = 0$ and $h'(x) = 1 - \frac{1}{x}$

as $h'(x) \geq 0$ for $x \geq 1$, then $h(x) \geq 0$ for $x \geq 1$

as $h'(x) \leq 0$ for $0 < x \leq 1$, then $h(x) \geq 0$ for $0 < x \leq 1$

so $g(x) \leq x - 1,\ x \in \mathbb{R}^+$

METHOD 2

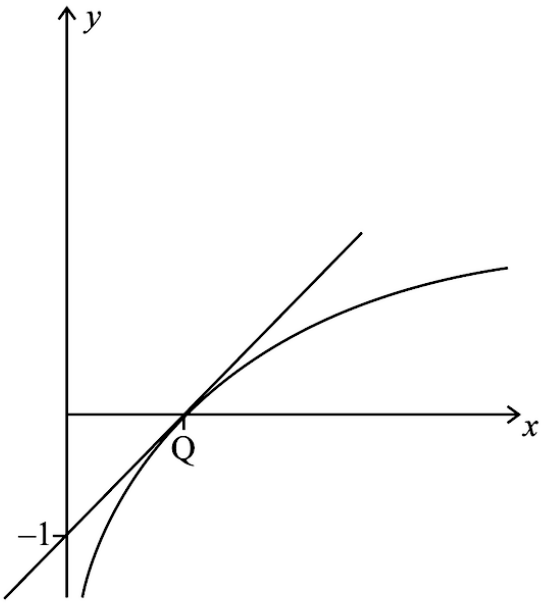
$g''(x) = -\frac{1}{x^2}$

$g''(x) < 0$ (concave down) for $x \in \mathbb{R}^+$

the graph of $y = g(x)$ is below its tangent ($y = x - 1$ at $x = 1$)

so $g(x) \leq x - 1,\ x \in \mathbb{R}^+$

METHOD 3



clear correct graphs of $y = x - 1$ and $\ln x$ for $x > 0$
statement to the effect that the graph of $\ln x$ is below the graph of its tangent at $x = 1$

(ii) replacing x by $\frac{1}{x}$ to obtain $\ln\left(\frac{1}{x}\right) \le \frac{1}{x} - 1\left(= \frac{1-x}{x}\right)$

$-\ln x \le \frac{1}{x} - 1\left(= \frac{1-x}{x}\right)$

$\ln x \ge 1 - \frac{1}{x}\left(= \frac{x-1}{x}\right)$

so $\frac{x-1}{x} \le g(x),\ x \in \mathbb{R}^+$

14 – (MAT/10_HL_Winter_2014_Q13) – *Complex Numbers, Trigonometry, Integration*

(a) (i) METHOD 1

$$(1+\mathrm{i}\tan\theta)^n+(1-\mathrm{i}\tan\theta)^n=(1+\mathrm{i}\frac{\sin\theta}{\cos\theta})^n+(1-\mathrm{i}\frac{\sin\theta}{\cos\theta})^n$$

$$=\left(\frac{\cos\theta+i\sin\theta}{\cos\theta}\right)^n+\left(\frac{\cos\theta-i\sin\theta}{\cos\theta}\right)^n$$

by de Moivre's theorem

$$\left(\frac{\cos\theta+i\sin\theta}{\cos\theta}\right)^n=\frac{\cos n\theta+i\sin n\theta}{\cos^n\theta}$$

recognition that $\cos\theta-i\sin\theta$ is the complex conjugate of $\cos\theta+i\sin\theta$

use of the fact that the operation of complex conjugation commutes with the operation of raising to an integer power:

$$\left(\frac{\cos\theta-i\sin\theta}{\cos\theta}\right)^n=\frac{\cos n\theta-i\sin n\theta}{\cos^n\theta}$$

$$(1+\mathrm{i}\tan\theta)^n+(1-\mathrm{i}\tan\theta)^n=\frac{2\cos n\theta}{\cos^n\theta}$$

METHOD 2

$$(1+\mathrm{i}\tan\theta)^n+(1-\mathrm{i}\tan\theta)^n=(1+\mathrm{i}\tan\theta)^n+(1+i\tan(-\theta))^n$$

$$=\frac{(\cos\theta+i\sin\theta)^n}{\cos^n\theta}+\frac{\left(\cos(-\theta)+i\sin(-\theta)\right)^n}{\cos^n\theta}$$

Note: Award ***M1*** for converting to cosine and sine terms.

use of de Moivre's theorem

$$=\frac{1}{\cos^n\theta}\left(\cos n\theta+\mathrm{i}\sin n\theta+\cos(-n\theta)+\mathrm{i}\sin(-n\theta)\right)$$

$$=\frac{2\cos n\theta}{\cos^n\theta}\text{ as }\cos(-n\theta)=\cos n\theta\text{ and }\sin(-n\theta)=-\sin n\theta$$

(ii) $\left(1+\mathrm{i}\tan\frac{3\pi}{8}\right)^4+\left(1-\mathrm{i}\tan\frac{3\pi}{8}\right)^4=\dfrac{2\cos\left(4\times\frac{3\pi}{8}\right)}{\cos^4\frac{3\pi}{8}}$

$=\dfrac{2\cos\frac{3\pi}{2}}{\cos^4\frac{3\pi}{8}}$

$=0$ as $\cos\frac{3\pi}{2}=0$

Note: The above working could involve theta and the solution of $\cos(4\theta)=0$.

so $\mathrm{i}\tan\frac{3\pi}{8}$ is a root of the equation

(iii) either $-\mathrm{i}\tan\frac{3\pi}{8}$ or $-\mathrm{i}\tan\frac{\pi}{8}$ or $\mathrm{i}\tan\frac{\pi}{8}$

Note: Accept $\mathrm{i}\tan\frac{5\pi}{8}$ or $\mathrm{i}\tan\frac{7\pi}{8}$.

Accept $-\left(1+\sqrt{2}\right)\mathrm{i}$ or $\left(1-\sqrt{2}\right)\mathrm{i}$ or $\left(-1+\sqrt{2}\right)\mathrm{i}$.

(b) (i) $\tan\frac{\pi}{4}=\dfrac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$

$\tan^2\frac{\pi}{8}+2\tan\frac{\pi}{8}-1=0$

let $t=\tan\frac{\pi}{8}$

attempting to solve $t^2+2t-1=0$ for t

$t=-1\pm\sqrt{2}$

$\frac{\pi}{8}$ is a first quadrant angle and tan is positive in this quadrant, so

$\tan\frac{\pi}{8}>0$

$\tan\frac{\pi}{8}=\sqrt{2}-1$

(ii) $\cos 4x = 2\cos^2 2x - 1$

$= 2\left(2\cos^2 x - 1\right)^2 - 1$

$= 2\left(4\cos^4 x - 4\cos^2 x + 1\right) - 1$

$= 8\cos^4 x - 8\cos^2 x + 1$

Note: Accept equivalent complex number derivation.

(iii) $\int_0^{\frac{\pi}{8}} \frac{2\cos 4x}{\cos^2 x}\,\mathrm{d}x = 2\int_0^{\frac{\pi}{8}} \frac{8\cos^4 x - 8\cos^2 x + 1}{\cos^2 x}\,\mathrm{d}x$

$= 2\int_0^{\frac{\pi}{8}} 8\cos^2 x - 8 + \sec^2 x\,\mathrm{d}x$

Note: The ***M1*** is for an integrand involving no fractions.

use of $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$

$= 2\int_0^{\frac{\pi}{8}} 4\cos 2x - 4 + \sec^2 x\,\mathrm{d}x$

$= \left[4\sin 2x - 8x + 2\tan x\right]_0^{\frac{\pi}{8}}$

$= 4\sqrt{2} - \pi - 2$ (or equivalent)

15 - (MAT/11_HL_Summer_2015_Q3) - *Integration*

(a) $\int\left(1 + \tan^2 x\right)\mathrm{d}x = \int \sec^2 x\,\mathrm{d}x = \tan x\,(+c)$

(b) $\int \sin^2 x\,\mathrm{d}x = \int \frac{1 - \cos 2x}{2}\,\mathrm{d}x$

$= \frac{x}{2} - \frac{\sin 2x}{4}\,(+c)$

16 – (MAT/12_HL_Summer_2015_Q5) – *Integration*

any attempt at integration by parts

$$u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = x^3 \Rightarrow v = \frac{x^4}{4}$$

$$= \left[\frac{x^4}{4}\ln x\right]_1^2 - \int_1^2 \frac{x^3}{4}\,\mathrm{d}x$$

Note: Condone absence of limits at this stage.

$$= \left[\frac{x^4}{4}\ln x\right]_1^2 - \left[\frac{x^4}{16}\right]_1^2$$

Note: Condone absence of limits at this stage.

$$= 4\ln 2 - \left(1 - \frac{1}{16}\right)$$

$$= 4\ln 2 - \frac{15}{16}$$

17 - (MAT/11_HL_Summer_2015_Q6) - *Functions - Roots, Integration*

(a) $f: x \to y = \dfrac{3x-2}{2x-1}$ $\quad f^{-1}: y \to x$

$y = \dfrac{3x-2}{2x-1} \Rightarrow 3x-2 = 2xy - y$

$\Rightarrow 3x - 2xy = -y + 2$

$x(3-2y) = 2-y$

$x = \dfrac{2-y}{3-2y}$

$(f^{-1}(y) = \dfrac{2-y}{3-2y})$

$f^{-1}(x) = \dfrac{2-x}{3-2x} \qquad \left(x \neq \dfrac{3}{2}\right)$

Note: x and y might be interchanged earlier.

Note: First ***M1*** is for interchange of variables second ***M1*** for manipulation

Note: Final answer must be a function of x

(b) $\dfrac{3x-2}{2x-1} = A + \dfrac{B}{2x-1} \Rightarrow 3x-2 = A(2x-1)+B$

equating coefficients $3 = 2A$ and $-2 = -A + B$

$A = \dfrac{3}{2}$ and $B = -\dfrac{1}{2}$

Note: Could also be done by division or substitution of values.

(c) $\int f(x)\,\mathrm{d}x = \dfrac{3}{2}x - \dfrac{1}{4}\ln\left|2x-1\right| + c$

18 – (MAT/11_HL_Summer_2015_Q8) – *Integration*

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x$$

EITHER

integral is $\int \frac{\mathrm{e}^x}{\left(\mathrm{e}^x+3\right)^2+2^2}\mathrm{d}x$

$$= \int \frac{1}{u^2+2^2}\mathrm{d}u$$

Note: Award ***M1*** only if the integral has completely changed to one in u.

Note: $\mathrm{d}u$ needed for final ***A1***

OR

$\mathrm{e}^x = u-3$

integral is $\int \frac{1}{(u-3)^2+6(u-3)+13}\mathrm{d}u$

Note: Award ***M1*** only if the integral has completely changed to one in u.

$$= \int \frac{1}{u^2+2^2}\mathrm{d}u$$

Note: In both solutions the two method marks are independent.

THEN

$$= \frac{1}{2}\arctan\left(\frac{u}{2}\right)(+c)$$

$$= \frac{1}{2}\arctan\left(\frac{\mathrm{e}^x+3}{2}\right)(+c)$$

19 – (MAT/12_HL_Summer_2015_Q8) – *Integration*

EITHER

$x = \arctan t$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{1+t^2}$$

OR

$t = \tan x$

$\frac{dt}{dx} = \sec^2 x$

$= 1 + \tan^2 x$

$= 1 + t^2$

THEN

$$\sin x = \frac{t}{\sqrt{1+t^2}}$$

Note: This ***A1*** is independent of the first two marks

$$\int \frac{\mathrm{d}x}{1+\sin^2 x} = \int \frac{\frac{\mathrm{d}t}{1+t^2}}{1+\left(\frac{t}{\sqrt{1+t^2}}\right)^2}$$

Note: Award ***M1*** for attempting to obtain integral in terms of t and $\mathrm{d}t$

$$= \int \frac{\mathrm{d}t}{\left(1+t^2\right)+t^2} = \int \frac{\mathrm{d}t}{1+2t^2}$$

$$= \frac{1}{2}\int \frac{\mathrm{d}t}{\frac{1}{2}+t^2} = \frac{1}{2} \times \frac{1}{\frac{1}{\sqrt{2}}} \arctan\left(\frac{t}{\frac{1}{\sqrt{2}}}\right)$$

$$= \frac{\sqrt{2}}{2} \arctan\left(\sqrt{2}\tan x\right)(+c)$$

20 – (MAT/12_HL_Summer_2015_Q11) – *Functions - Roots, Differentiation, Integration*

(a) $g \circ f(x) = \dfrac{\tan x + 1}{\tan x - 1}$

$x \neq \dfrac{\pi}{4}$, $0 \leq x < \dfrac{\pi}{2}$

(b) $\dfrac{\tan x + 1}{\tan x - 1} = \dfrac{\dfrac{\sin x}{\cos x} + 1}{\dfrac{\sin x}{\cos x} - 1}$

$= \dfrac{\sin x + \cos x}{\sin x - \cos x}$

(c) **METHOD 1**

$$\frac{dy}{dx} = \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$\frac{dy}{dx} = \frac{\left(2\sin x\cos x - \cos^2 x - \sin^2 x\right) - \left(2\sin x\cos x + \cos^2 x + \sin^2 x\right)}{\cos^2 x + \sin^2 x - 2\sin x\cos x}$$

$$= \frac{-2}{1 - \sin 2x}$$

Substitute $\dfrac{\pi}{6}$ into any formula for $\dfrac{dy}{dx}$

$$\frac{-2}{1 - \sin\frac{\pi}{3}}$$

$$= \frac{-2}{1 - \frac{\sqrt{3}}{2}}$$

$$= \frac{-4}{2 - \sqrt{3}}$$

$$= \frac{-4}{2 - \sqrt{3}}\left(\frac{2 + \sqrt{3}}{2 + \sqrt{3}}\right)$$

$$= \frac{-8 - 4\sqrt{3}}{1} = -8 - 4\sqrt{3}$$

METHOD 2

$$\frac{dy}{dx} = \frac{(\tan x - 1)\sec^2 x - (\tan x + 1)\sec^2 x}{(\tan x - 1)^2}$$

$$= \frac{-2\sec^2 x}{(\tan x - 1)^2}$$

$$= \frac{-2\sec^2 \frac{\pi}{6}}{\left(\tan\frac{\pi}{6} - 1\right)^2} = \frac{-2\left(\frac{4}{3}\right)}{\left(\frac{1}{\sqrt{3}} - 1\right)^2} = \frac{-8}{\left(1 - \sqrt{3}\right)^2}$$

Note: Award ***M1*** for substitution of $\frac{\pi}{6}$.

$$\frac{-8}{\left(1 - \sqrt{3}\right)^2} = \frac{-8}{\left(4 - 2\sqrt{3}\right)}\frac{\left(4 + 2\sqrt{3}\right)}{\left(4 + 2\sqrt{3}\right)} = -8 - 4\sqrt{3}$$

(d) Area $= \left| \int_0^{\frac{\pi}{6}} \frac{\sin x + \cos x}{\sin x - \cos x} \mathrm{d}x \right|$

$= \left| \left[\ln \left| \sin x - \cos x \right| \right]_0^{\frac{\pi}{6}} \right|$

Note: Condone absence of limits and absence of modulus signs at this stage.

$= \left| \ln \left| \sin \frac{\pi}{6} - \cos \frac{\pi}{6} \right| - \ln \left| \sin 0 - \cos 0 \right| \right|$

$= \left| \ln \left| \frac{1}{2} - \frac{\sqrt{3}}{2} \right| - 0 \right|$

$= \left| \ln \left(\frac{\sqrt{3} - 1}{2} \right) \right|$

$= - \ln \left(\frac{\sqrt{3} - 1}{2} \right) = \ln \left(\frac{2}{\sqrt{3} - 1} \right)$

$= \ln \left(\frac{2}{\sqrt{3} - 1} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1} \right)$

$= \ln \left(\sqrt{3} + 1 \right)$

21 - (MAT/10_HL_Winter_2015_Q2) - *Integration*

attempt to integrate one factor and differentiate the other, leading to a sum of two terms ***M1***

$\int x \sin x \, \mathrm{d}x = x(-\cos x) + \int \cos x \, \mathrm{d}x$ ***(A1)(A1)***

$= -x \cos x + \sin x + c$ ***A1***

22 – (MAT/10_HL_Winter_2015_Q5) – *Integration*

METHOD 1

$$\int_{e}^{e^2} \frac{dx}{x \ln x} = \left[\ln(\ln x)\right]_{e}^{e^2}$$

$$= \ln\left(\ln e^2\right) - \ln(\ln e) \ (= \ln 2 - \ln 1)$$

$$= \ln 2$$

METHOD 2

$$u = \ln x, \ \frac{du}{dx} = \frac{1}{x}$$

$$= \int_{1}^{2} \frac{du}{u}$$

$= \left[\ln u\right]_{1}^{2}$ or equivalent in x $(= \ln 2 - \ln 1)$

$$= \ln 2$$

23 – (MAT/10_HL_Winter_2015_Q12) – *Differentiation, Integration*

(a) $f(-x) = (-x)\sqrt{1-(-x)^2}$

$= -x\sqrt{1-x^2}$

$= -f(x)$

hence f is odd

(b) $f'(x) = x \cdot \frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}} \cdot -2x + \left(1-x^2\right)^{\frac{1}{2}}$

(c) $f'(x) = \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} \left(= \frac{1-2x^2}{\sqrt{1-x^2}} \right)$

Note: This may be seen in part (b).

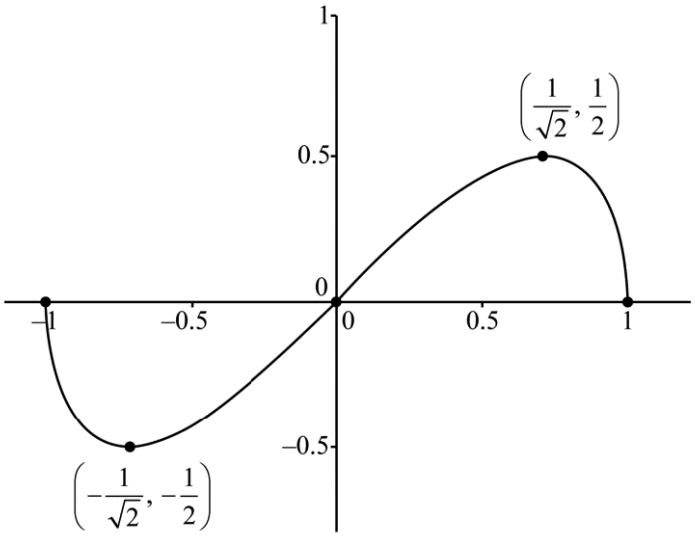
$f'(x) = 0 \Rightarrow 1 - 2x^2 = 0$

$x = \pm\frac{1}{\sqrt{2}}$

(d) y-coordinates of the Max Min Points are $y = \pm\frac{1}{2}$

so range of $f(x)$ is $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(e)



Shape: The graph of an odd function, on the given domain, s-shaped, where the max(min) is the right(left) of $0.5(-0.5)$
x-intercepts
turning points

(f) area $= \int_0^1 x\sqrt{1-x^2}\,\mathrm{d}x$

attempt at "backwards chain rule" or substitution

$$= -\frac{1}{2}\int_0^1 (-2x)\sqrt{1-x^2}\,\mathrm{d}x$$

$$= \left[\frac{2}{3}\left(1-x^2\right)^{\frac{3}{2}} \cdot -\frac{1}{2}\right]_0^1$$

$$= \left[-\frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}}\right]_0^1$$

$$= 0 - \left(-\frac{1}{3}\right) = \frac{1}{3}$$

(g) $\int_{-1}^1 \left| x\sqrt{1-x^2} \right| \mathrm{d}x > 0$

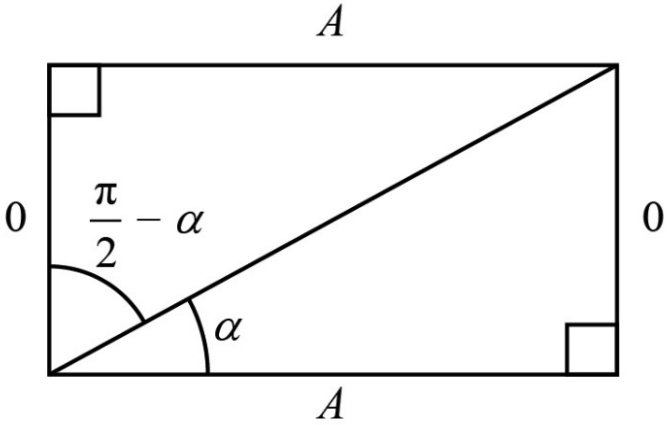
$$\left| \int_{-1}^1 x\sqrt{1-x^2}\,\mathrm{d}x \right| = 0$$

so $\int_{-1}^1 \left| x\sqrt{1-x^2} \right| \mathrm{d}x > \left| \int_{-1}^1 x\sqrt{1-x^2}\,\mathrm{d}x \right| = 0$

24 – (MAT/12_HL_Summer_2016_Q3) – *Trigonometry, Integration*

(a) **EITHER**

use of a diagram and trig ratios
eg,



$\tan\alpha = \frac{O}{A} \Rightarrow \cot\alpha = \frac{A}{O}$

from diagram, $\tan\left(\frac{\pi}{2} - \alpha\right) = \frac{A}{O}$

OR

use of $\tan\left(\frac{\pi}{2} - \alpha\right) = \frac{\sin\left(\frac{\pi}{2} - \alpha\right)}{\cos\left(\frac{\pi}{2} - \alpha\right)} = \frac{\cos\alpha}{\sin\alpha}$

THEN

$\cot\alpha = \tan\left(\frac{\pi}{2} - \alpha\right)$

(b) $\int_{\tan\alpha}^{\cot\alpha} \frac{1}{1+x^2}\,dx = [\arctan x]_{\tan\alpha}^{\cot\alpha}$

Note: Limits (or absence of such) may be ignored at this stage.

$= \arctan(\cot\alpha) - \arctan(\tan\alpha)$

$= \frac{\pi}{2} - \alpha - \alpha$

$= \frac{\pi}{2} - 2\alpha$

25 – (MAT/11_HL_Summer_2016_Q13) – *Integration*

(a) $a = 1$

(b) $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$

$\int \frac{(\ln x)^2}{x}\mathrm{d}x = \int u^2 \mathrm{d}u$

area $= \left[\frac{1}{3}u^3\right]_0^1$ or $\left[\frac{1}{3}(\ln x)^3\right]_1^{\mathrm{e}}$

$= \frac{1}{3}$

(c) (i) $I_0 = \left[-\frac{1}{x}\right]_1^{\mathrm{e}}$

$= 1 - \frac{1}{\mathrm{e}}$

(ii) use of integration by parts

$I_n = \left[-\frac{1}{x}(\ln x)^n\right]_1^{\mathrm{e}} + \int_1^{\mathrm{e}} \frac{n(\ln x)^{n-1}}{x^2}\mathrm{d}x$

$= -\frac{1}{\mathrm{e}} + nI_{n-1}$

(iii) $I_1 = -\frac{1}{e} + 1 \times I_0$

$= 1 - \frac{2}{e}$

(d) volume $= \pi \int_1^e \frac{(\ln x)^4}{x^2} dx \left(= \pi I_4\right)$

EITHER

$I_4 = -\frac{1}{e} + 4I_3$

$= -\frac{1}{e} + 4\left(-\frac{1}{e} + 3I_2\right)$

$= -\frac{5}{e} + 12I_2 = -\frac{5}{e} + 12\left(-\frac{1}{e} + 2I_1\right)$

OR

using parts $\int_1^e \frac{(\ln x)^4}{x^2} dx = -\frac{1}{e} + 4\int_1^e \frac{(\ln x)^3}{x^2} dx$

$= -\frac{1}{e} + 4\left(-\frac{1}{e} + 3\int_1^e \frac{(\ln x)^2}{x^2} dx\right)$

THEN

$= -\frac{17}{e} + 24\left(1 - \frac{2}{e}\right) = 24 - \frac{65}{e}$

volume $= \pi\left(24 - \frac{65}{e}\right)$

26 – (MAT/12_HL_Summer_2017_Q6) – *Integration*

(a) let $x = \tan\theta$

$\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = \sec^2\theta$ ***(A1)***

$\int \frac{1}{\left(x^2+1\right)^2}\mathrm{d}x = \int \frac{\sec^2\theta}{\left(\tan^2\theta+1\right)^2}\mathrm{d}\theta$ ***M1***

Note: The method mark is for an attempt to substitute for both x and $\mathrm{d}x$.

$= \int \frac{1}{\sec^2\theta}\mathrm{d}\theta$ (or equivalent) ***A1***

when $x = 0,\ \theta = 0$ and when $x = 1,\ \theta = \frac{\pi}{4}$ ***M1***

$= \int_0^{\frac{\pi}{4}} \cos^2\theta \mathrm{d}\theta$ ***AG***

[4 marks]

(b) $\left(\int_0^1 \frac{1}{\left(x^2+1\right)^2}\mathrm{d}x = \int_0^{\frac{\pi}{4}} \cos^2\theta \mathrm{d}\theta\right) = \frac{1}{2}\int_0^{\frac{\pi}{4}}(1+\cos 2\theta)\mathrm{d}\theta$ ***M1***

$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}$ ***A1***

$= \frac{\pi}{8} + \frac{1}{4}$ ***A1***

[3 marks]

27 – (MAT/11_HL_Summer_2017_Q9) – *Integration*

attempt at integration by parts with $u = \arcsin x$ and $v' = 1$ ***M1***

$\int \arcsin x\ \mathrm{d}x = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\mathrm{d}x$ ***A1A1***

Note: Award ***A1*** for $x\arcsin x$ and ***A1*** for $-\int \frac{x}{\sqrt{1-x^2}}\mathrm{d}x$.

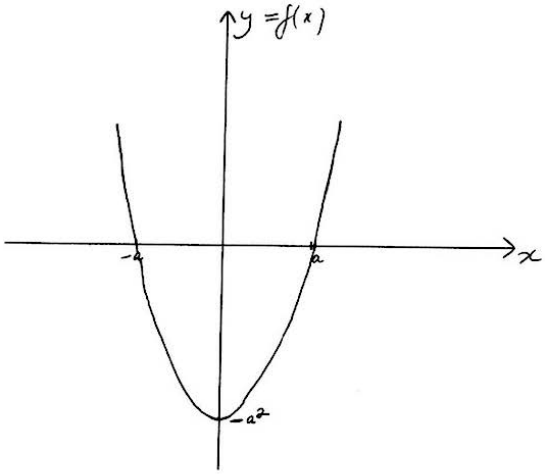
solving $\int \frac{x}{\sqrt{1-x^2}}\mathrm{d}x$ by substitution with $u = 1 - x^2$ or inspection ***(M1)***

$\int \arcsin x\ \mathrm{d}x = x\arcsin x + \sqrt{1-x^2} + c$ ***A1***

[5 marks]

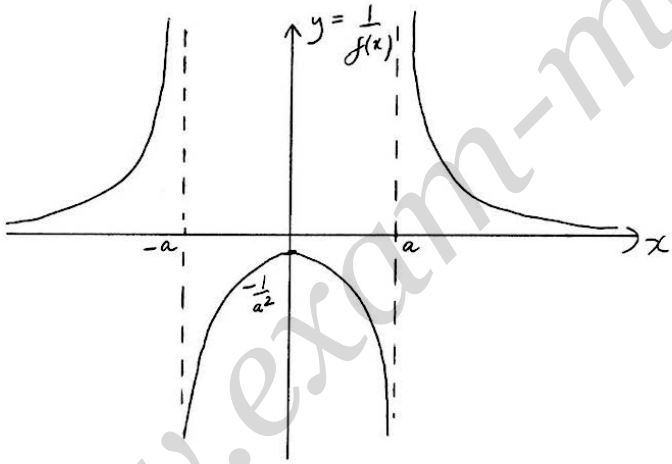
28 – (MAT/12_HL_Summer_2017_Q9) – *Functions - Roots, Differentiation, Integration*

(a) (i)



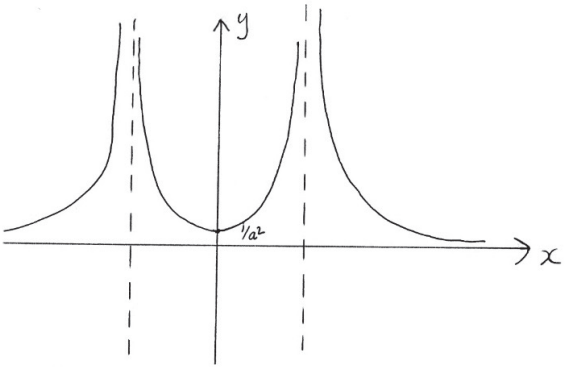
A1 for correct shape
A1 for correct x and y intercepts and minimum point

(ii)



A1 for correct shape
A1 for correct vertical asymptotes
A1 for correct implied horizontal asymptote
A1 for correct maximum point

(iii)



A1 for reflecting negative branch from (ii) in the x-axis
A1 for correctly labelled minimum point

[8 marks]

(b) **EITHER**

attempt at integration by parts ***(M1)***

$\int (x^2 - a^2)\cos x\,\mathrm{d}x = (x^2 - a^2)\sin x - \int 2x\sin x\,\mathrm{d}x$ ***A1A1***

$= (x^2 - a^2)\sin x - 2\left[-x\cos x + \int \cos x\,\mathrm{d}x\right]$ ***A1***

$= (x^2 - a^2)\sin x + 2x\cos x - 2\sin x + c$ ***A1***

OR

$\int (x^2 - a^2)\cos x\,\mathrm{d}x = \int x^2\cos x\,\mathrm{d}x - \int a^2\cos x\,\mathrm{d}x$

attempt at integration by parts ***(M1)***

$\int x^2\cos x\,\mathrm{d}x = x^2\sin x - \int 2x\sin x\,\mathrm{d}x$ ***A1A1***

$= x^2\sin x - 2\left[-x\cos x + \int \cos x\,\mathrm{d}x\right]$ ***A1***

$= x^2\sin x + 2x\cos x - 2\sin x$

$-\int a^2\cos x\,\mathrm{d}x = -a^2\sin x$

$\int (x^2 - a^2)\cos x\,\mathrm{d}x = (x^2 - a^2)\sin x + 2x\cos x - 2\sin x + c$ ***A1***

[5 marks]

(c) $g(x) = x\left(x^2 - a^2\right)^{\frac{1}{2}}$

$g'(x) = \left(x^2 - a^2\right)^{\frac{1}{2}} + \frac{1}{2}x\left(x^2 - a^2\right)^{-\frac{1}{2}}(2x)$ ***M1A1A1***

Note: Method mark is for differentiating the product. Award ***A1*** for each correct term.

$g'(x) = \left(x^2 - a^2\right)^{\frac{1}{2}} + x^2\left(x^2 - a^2\right)^{-\frac{1}{2}}$

both parts of the expression are positive hence $g'(x)$ is positive ***R1***

and therefore g is an increasing function (for $|x| > a$) ***AG***

[4 marks]

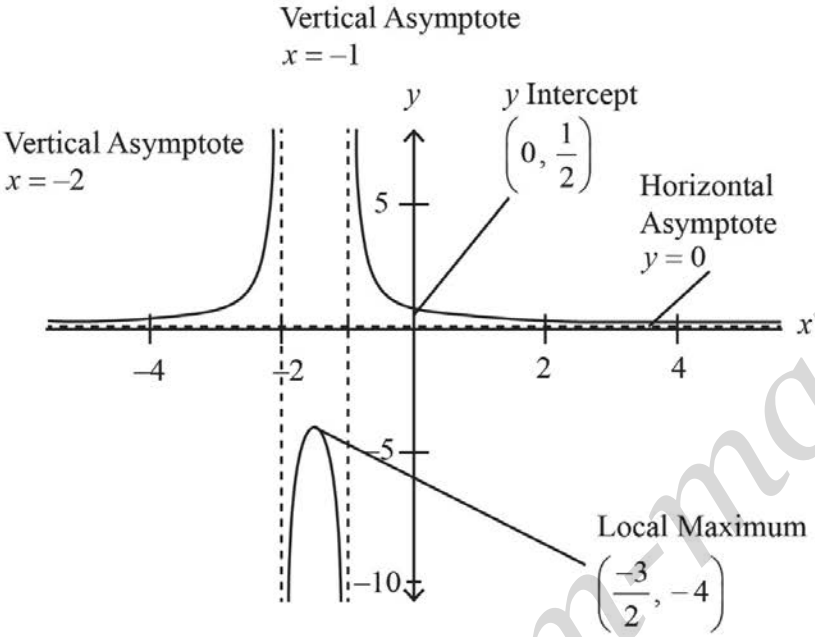
29 – (MAT/11_HL_Summer_2017_Q11) – *Functions - Roots, Graphs, Integration*

(a) (i) $x^2 + 3x + 2 = \left(x + \frac{3}{2}\right)^2 - \frac{1}{4}$ ***A1***

(ii) $x^2 + 3x + 2 = (x + 2)(x + 1)$ ***A1***

[2 marks]

(b)



A1 for the shape
A1 for the equation $y = 0$
A1 for asymptotes $x = -2$ and $x = -1$
A1 for coordinates $\left(-\frac{3}{2}, -4\right)$
A1 y-intercept $\left(0, \frac{1}{2}\right)$

[5 marks]

(c) $\frac{1}{x+1} - \frac{1}{x+2} = \frac{(x+2) - (x+1)}{(x+1)(x+2)}$ ***M1***

$= \frac{1}{x^2 + 3x + 2}$ ***AG***

[1 mark]

(d) $\int_0^1 \frac{1}{x+1} - \frac{1}{x+2}\,\mathrm{d}x$

$= \left[\ln(x+1) - \ln(x+2)\right]_0^1$ ***A1***

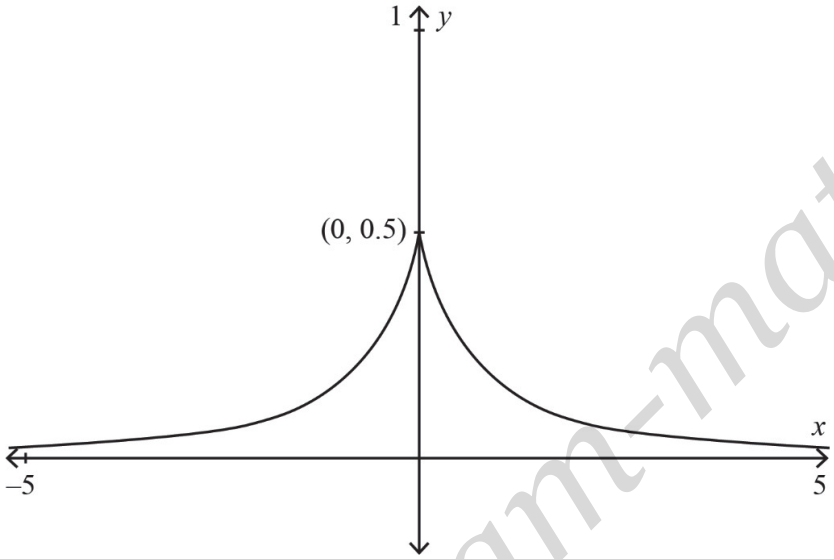
$= \ln 2 - \ln 3 - \ln 1 + \ln 2$ ***M1***

$= \ln\left(\frac{4}{3}\right)$ ***M1A1***

$\therefore\ p = \frac{4}{3}$

[4 marks]

(e)



symmetry about the y-axis ***M1***
correct shape ***A1***

Note: Allow ***FT*** from part (b).

[2 marks]

(f) $2\int_0^1 f(x)\,\mathrm{d}x$ ***(M1)(A1)***

$= 2\ln\left(\frac{4}{3}\right)$ ***A1***

Note: Do not award ***FT*** from part (e).

[3 marks]

30 – (MAT/11_HL_Summer_2018_Q4) – *Integration*

(a) $\int_{-2}^{0} f(x)\mathrm{d}x = 10 - 12 = -2$ ***(M1)(A1)***

$\int_{-2}^{0} 2\,\mathrm{d}x = [2x]_{-2}^{0} = 4$ ***A1***

$\int_{-2}^{0} (f(x)+2)\,\mathrm{d}x = 2$ ***A1***

[4 marks]

(b) $\int_{-2}^{0} f(x+2)\mathrm{d}x = \int_{0}^{2} f(x)\mathrm{d}x$ ***(M1)***

$= 12$ ***A1***

[2 marks]

31 – (MAT/12_HL_Summer_2018_Q6) – *Differentiation, Integration*

(a) (i) attempt at product rule ***M1***

$f'(x) = -\mathrm{e}^{-x}\sin x + \mathrm{e}^{-x}\cos x$ ***A1***

(ii) $g'(x) = -\mathrm{e}^{-x}\cos x - \mathrm{e}^{-x}\sin x$ ***A1***

[3 marks]

(b) **METHOD 1**

Attempt to add $f'(x)$ and $g'(x)$ ***(M1)***

$f'(x) + g'(x) = -2\mathrm{e}^{-x}\sin x$ ***A1***

$\int_{0}^{\pi} \mathrm{e}^{-x}\sin x\,\mathrm{d}x = \left[-\frac{\mathrm{e}^{-x}}{2}(\sin x + \cos x)\right]_{0}^{\pi}$ (or equivalent) ***A1***

Note: Condone absence of limits.

$= \frac{1}{2}\left(1 + \mathrm{e}^{-\pi}\right)$ ***A1***

METHOD 2

$I = \int \mathrm{e}^{-x}\sin x\mathrm{d}x$

$= -\mathrm{e}^{-x}\cos x - \int \mathrm{e}^{-x}\cos x\mathrm{d}x$ **OR** $= -\mathrm{e}^{-x}\sin x + \int \mathrm{e}^{-x}\cos x\mathrm{d}x$ ***M1A1***

$= -\mathrm{e}^{-x}\sin x - \mathrm{e}^{-x}\cos x - \int \mathrm{e}^{-x}\sin x\mathrm{d}x$

$I = -\frac{1}{2}\mathrm{e}^{-x}(\sin x + \cos x)$ ***A1***

$\int_{0}^{\pi} \mathrm{e}^{-x}\sin x\mathrm{d}x = \frac{1}{2}\left(1 + \mathrm{e}^{-\pi}\right)$ ***A1***

[4 marks]

32 – (MAT/11_HL_Summer_2018_Q7) – *Differentiation, Integration*

(a) $y = \arccos\left(\frac{x}{2}\right) \Rightarrow \frac{dy}{dx} = -\frac{1}{2\sqrt{1-\left(\frac{x}{2}\right)^2}} \left(= -\frac{1}{\sqrt{4-x^2}}\right)$ ***M1A1***

Note: ***M1*** is for use of the chain rule.

[2 marks]

(b) attempt at integration by parts ***M1***

$u = \arccos\left(\frac{x}{2}\right) \Rightarrow \frac{du}{dx} = -\frac{1}{\sqrt{4-x^2}}$

$\frac{dv}{dx} = 1 \Rightarrow v = x$ ***(A1)***

$\int_0^1 \arccos\left(\frac{x}{2}\right) dx = \left[x\arccos\left(\frac{x}{2}\right)\right]_0^1 + \int_0^1 \frac{x}{\sqrt{4-x^2}} dx$ ***A1***

using integration by substitution or inspection ***(M1)***

$\left[x\arccos\left(\frac{x}{2}\right)\right]_0^1 + \left[-\left(4-x^2\right)^{\frac{1}{2}}\right]_0^1$ ***A1***

Note: Award ***A1*** for $-\left(4-x^2\right)^{\frac{1}{2}}$ or equivalent.

Note: Condone lack of limits to this point.

attempt to substitute limits into their integral ***M1***

$= \frac{\pi}{3} - \sqrt{3} + 2$ ***A1***

[7 marks]

33 – (MAT/12_HL_Summer_2018_Q8) – *Integration*

(a) $\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ (accept $du = \frac{1}{2}x^{-\frac{1}{2}}dx$ or equivalent) ***A1***

substitution, leading to an integrand in terms of u ***M1***

$\int \frac{2u\,du}{u^3+u}$ or equivalent ***A1***

$= 2\arctan(\sqrt{x})(+c)$ ***A1***

[4 marks]

(b) $\frac{1}{2}\int_1^9 \frac{dx}{x^{\frac{3}{2}} + x^{\frac{1}{2}}} = \arctan 3 - \arctan 1$ ***A1***

$\tan(\arctan 3 - \arctan 1) = \frac{3-1}{1+3\times 1}$ ***(M1)***

$\tan(\arctan 3 - \arctan 1) = \frac{1}{2}$

$\arctan 3 - \arctan 1 = \arctan\frac{1}{2}$ ***A1***

[3 marks]

34 – (MAT/12_HL_Summer_2018_Q11) – *Exponentials - Logarithms, Integration*

(a) **METHOD 1**

$\log_{r^2} x = \dfrac{\log_r x}{\log_r r^2} \left(= \dfrac{\log_r x}{2\log_r r} \right)$ ***M1A1***

$= \dfrac{\log_r x}{2}$ ***AG***

[2 marks]

METHOD 2

$\log_{r^2} x = \dfrac{1}{\log_x r^2}$ ***M1***

$= \dfrac{1}{2\log_x r}$ ***A1***

$= \dfrac{\log_r x}{2}$ ***AG***

[2 marks]

(b) **METHOD 1**

$\log_2 y + \log_4 x + \log_4 2x = 0$

$\log_2 y + \log_4 2x^2 = 0$ ***M1***

$\log_2 y + \dfrac{1}{2}\log_2 2x^2 = 0$ ***M1***

$\log_2 y = -\dfrac{1}{2}\log_2 2x^2$

$\log_2 y = \log_2\left(\dfrac{1}{\sqrt{2}x}\right)$ ***M1A1***

$y = \dfrac{1}{\sqrt{2}}x^{-1}$ ***A1***

Note: For the final ***A*** mark, y must be expressed in the form px^q.

[5 marks]

METHOD 2

$\log_2 y + \log_4 x + \log_4 2x = 0$

$\log_2 y + \frac{1}{2}\log_2 x + \frac{1}{2}\log_2 2x = 0$ ***M1***

$\log_2 y + \log_2 x^{\frac{1}{2}} + \log_2 (2x)^{\frac{1}{2}} = 0$ ***M1***

$\log_2 \left(\sqrt{2}xy\right) = 0$ ***M1***

$\sqrt{2}xy = 1$ ***A1***

$y = \frac{1}{\sqrt{2}}x^{-1}$ ***A1***

Note: For the final ***A*** mark, y must be expressed in the form px^q.

[5 marks]

(c) the area of R is $\int_1^{\alpha} \frac{1}{\sqrt{2}}x^{-1}\mathrm{d}x$ ***M1***

$= \left[\frac{1}{\sqrt{2}}\ln x\right]_1^{\alpha}$ ***A1***

$= \frac{1}{\sqrt{2}}\ln \alpha$ ***A1***

$\frac{1}{\sqrt{2}}\ln \alpha = \sqrt{2}$ ***M1***

$\alpha = \mathrm{e}^2$ ***A1***

Note: Only follow through from part (b) if y is in the form $y = px^q$.

[5 marks]

35 – (MAT/10_HL_Winter_2018_Q10) – *Integration*

(a) **METHOD 1**

attempt at integration by parts with $u = e^x$, $\frac{dv}{dx} = \cos 2x$ ***M1***

$\int e^x \cos 2x\,dx = \frac{e^x}{2}\sin 2x\,dx - \int \frac{e^x}{2}\sin 2x\,dx$ ***A1***

$= \frac{e^x}{2}\sin 2x - \frac{1}{2}\left(-\frac{e^x}{2}\cos 2x + \int \frac{e^x}{2}\cos 2x\right)$ ***M1A1***

$= \frac{e^x}{2}\sin 2x + \frac{e^x}{4}\cos 2x - \frac{1}{4}\int e^x \cos 2x\,dx$

$\therefore \frac{5}{4}\int e^x \cos 2x\,dx = \frac{e^x}{2}\sin 2x + \frac{e^x}{4}\cos 2x$ ***M1***

$\int e^x \cos 2x\,dx = \frac{2e^x}{5}\sin 2x + \frac{e^x}{5}\cos 2x (+c)$ ***AG***

METHOD 2

attempt at integration by parts with $u = \cos 2x$, $\frac{dv}{dx} = e^x$ ***M1***

$\int e^x \cos 2x\,dx = e^x \cos 2x + 2\int e^x \sin 2x\,dx$ ***A1***

$= e^x \cos 2x + 2\left(e^x \sin 2x - 2\int e^x \cos 2x\,dx\right)$ ***M1A1***

$= e^x \cos 2x + 2e^x \sin 2x - 4\int e^x \cos 2x\,dx$

$\therefore 5\int e^x \cos 2x\,dx = e^x \cos 2x + 2e^x \sin 2x$ ***M1***

$\int e^x \cos 2x\,dx = \frac{2e^x}{5}\sin 2x + \frac{e^x}{5}\cos 2x (+c)$ ***AG***

METHOD 3

attempt at use of table ***M1***

eg

$\cos 2x$	e^x
$-2\sin 2x$	e^x
$-4\cos 2x$	e^x

A1A1

Note: ***A1*** for first 2 lines correct, ***A1*** for third line correct.

$\int e^x \cos 2x\,dx = e^x \cos 2x + 2e^x \sin 2x - 4\int e^x \cos 2x\,dx$ ***M1***

$\therefore 5\int e^x \cos 2x\,dx = = e^x \cos 2x + 2e^x \sin 2x$ ***M1***

$\int e^x \cos 2x\,dx = \frac{2e^x}{5}\sin 2x + \frac{e^x}{5}\cos 2x (+c)$ ***AG***

[5 marks]

(b) $\int e^x \cos^2 x dx = \int \frac{e^x}{2}(\cos 2x + 1)\,dx$ ***M1A1***

$= \frac{1}{2}\left(\frac{2e^x}{5}\sin 2x + \frac{e^x}{5}\cos 2x\right) + \frac{e^x}{2}$ ***A1***

$= \frac{e^x}{5}\sin 2x + \frac{e^x}{10}\cos 2x + \frac{e^x}{2}\ (+c)$ ***AG***

Note: Do not accept solutions where the RHS is differentiated.

[3 marks]

(c) $f'(x) = e^x \cos^2 x - 2e^x \sin x \cos x$ ***M1A1***

Note: Award ***M1*** for an attempt at both the product rule and the chain rule.

$e^x \cos x(\cos x - 2\sin x) = 0$ ***(M1)***

Note: Award ***M1*** for an attempt to factorise $\cos x$ or divide by $\cos x\left(\cos x \neq 0\right)$.

discount $\cos x = 0$ (as this would also be a zero of the function)

$\Rightarrow \cos x - 2\sin x = 0$

$\Rightarrow \tan x = \frac{1}{2}$ ***(M1)***

$\Rightarrow x = \arctan\left(\frac{1}{2}\right)$ (at A) and $x = \pi + \arctan\left(\frac{1}{2}\right)$ (at C) ***A1A1***

Note: Award ***A1*** for each correct answer. If extra values are seen award ***A1A0***.

[6 marks]

(d) $\cos x = 0 \Rightarrow x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ ***A1***

Note: The ***A1*** may be awarded for work seen in part (c).

$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(e^x \cos^2 x\right) dx = \left[\frac{e^x}{5}\sin 2x + \frac{e^x}{10}\cos 2x + \frac{e^x}{2}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$ ***M1***

$= \left(-\frac{e^{\frac{3\pi}{2}}}{10} + \frac{e^{\frac{3\pi}{2}}}{2}\right) - \left(-\frac{e^{\frac{\pi}{2}}}{10} + \frac{e^{\frac{\pi}{2}}}{2}\right)\left(= \frac{2e^{\frac{3\pi}{2}}}{5} - \frac{2e^{\frac{\pi}{2}}}{5}\right)$ ***M1(A1)A1***

Note: Award ***M1*** for substitution of the end points and subtracting, ***(A1)*** for $\sin 3\pi = \sin \pi = 0$ and $\cos 3\pi = \cos \pi = -1$ and ***A1*** for a completely correct answer.

[5 marks]

36 – (MAT/12_HL_Summer_2019_Q4) – *Integration*

$u = \sin x \Rightarrow \mathrm{d}u = \cos x \mathrm{d}x$ ***(A1)***

valid attempt to write integral in terms of u and $\mathrm{d}u$ ***M1***

$\int \frac{\cos^3 x \ \mathrm{d}x}{\sqrt{\sin x}} = \int \frac{\left(1-u^2\right) \ \mathrm{d}u}{\sqrt{u}}$ ***A1***

$= \int \left(u^{-\frac{1}{2}} - u^{\frac{3}{2}} \right) \mathrm{d}u$

$= 2u^{\frac{1}{2}} - \frac{2u^{\frac{5}{2}}}{5} (+c)$ ***(A1)***

$= 2\sqrt{\sin x} - \frac{2\left(\sqrt{\sin x}\right)^5}{5} (+c)$ or equivalent ***A1***

[5 marks]

37 – (MAT/11_HL_Summer_2019_Q8) – *Integration*

(a) 3 ***A1***

[1 mark]

(b) attempt to use definite integral of $f'(x)$ ***(M1)***

$\int_0^1 f'(x)\mathrm{d}x = 0.5$

$f(1) - f(0) = 0.5$ ***(A1)***

$f(1) = 0.5 + 3$

$= 3.5$ ***A1***

[3 marks]

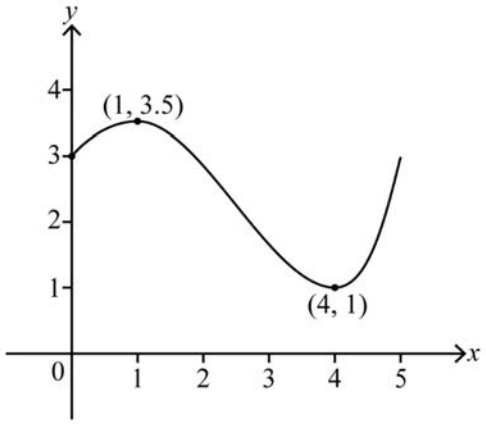
(c) $\int_1^4 f'(x)\mathrm{d}x = -2.5$ ***(A1)***

Note: ***(A1)*** is for -2.5.

$f(4) - f(1) = -2.5$
$f(4) = 3.5 - 2.5$
$= 1$ ***A1***

[2 marks]

(d)



A1A1A1

A1 for correct shape over approximately the correct domain

A1 for maximum and minimum (coordinates or horizontal lines from 3.5 and 1 are required),
A1 for y-intercept at 3

[3 marks]

38 – (MAT/11_HL_Summer_2019_Q9) – *Trigonometry, Integration*

(a) $(\sin x+\cos x)^2=\sin^2 x+2\sin x\cos x+\cos^2 x$ ***M1A1***

Note: Do not award the ***M1*** for just $\sin^2 x+\cos^2 x$.

Note: Do not award ***A1*** if correct expression is followed by incorrect working.

$=1+\sin 2x$ ***AG***

[2 marks]

(b) $\sec 2x+\tan 2x=\dfrac{1}{\cos 2x}+\dfrac{\sin 2x}{\cos 2x}$ ***M1***

Note: ***M1*** is for an attempt to change both terms into sine and cosine forms (with the same argument) or both terms into functions of $\tan x$.

$=\dfrac{1+\sin 2x}{\cos 2x}$

$=\dfrac{(\sin x+\cos x)^2}{\cos^2 x-\sin^2 x}$ ***A1A1***

Note: Award ***A1*** for numerator, ***A1*** for denominator.

$=\dfrac{(\sin x+\cos x)^2}{(\cos x-\sin x)(\cos x+\sin x)}$ ***M1***

$=\dfrac{\cos x+\sin x}{\cos x-\sin x}$ ***AG***

Note: Apply MS in reverse if candidates have worked from RHS to LHS.

Note: Alternative method using $\tan 2x$ and $\sec 2x$ in terms of $\tan x$.

[4 marks]

(c) **METHOD 1**

$\int_0^{\frac{\pi}{6}}\left(\dfrac{\cos x+\sin x}{\cos x-\sin x}\right)\mathrm{d}x$ ***A1***

Note: Award ***A1*** for correct expression with or without limits.

EITHER

$=\left[-\ln(\cos x-\sin x)\right]_0^{\frac{\pi}{6}}$ or $\left[\ln(\cos x-\sin x)\right]_{\frac{\pi}{6}}^0$ ***(M1)A1A1***

Note: Award ***M1*** for integration by inspection or substitution, ***A1*** for $\ln(\cos x-\sin x)$, ***A1*** for completely correct expression including limits.

$=-\ln\left(\cos\dfrac{\pi}{6}-\sin\dfrac{\pi}{6}\right)+\ln(\cos 0-\sin 0)$ ***M1***

Note: Award ***M1*** for substitution of limits into their integral and subtraction.

$=-\ln\left(\dfrac{\sqrt{3}}{2}-\dfrac{1}{2}\right)$ ***(A1)***

OR

let $u = \cos x - \sin x$ ***M1***

$\frac{du}{dx} = -\sin x - \cos x = -(\sin x + \cos x)$

$-\int_1^{\frac{\sqrt{3}}{2}-\frac{1}{2}} \left(\frac{1}{u}\right) du$ ***A1A1***

Note: Award ***A1*** for correct limits even if seen later, ***A1*** for integral.

$= \left[-\ln u\right]_1^{\frac{\sqrt{3}}{2}-\frac{1}{2}}$ or $\left[\ln u\right]_{\frac{\sqrt{3}}{2}-\frac{1}{2}}^1$ ***A1***

$= -\ln\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)(+\ln 1)$ ***M1***

THEN

$= \ln\left(\frac{2}{\sqrt{3}-1}\right)$ ***M1***

Note: Award ***M1*** for both putting the expression over a common denominator and for correct use of law of logarithms.

$= \ln\left(1+\sqrt{3}\right)$ ***(M1)A1***

[9 marks]

39 – (MAT/12_HL_Summer_2019_Q9) – *Differentiation, Integration*

(a) $3\cos 2x = 4 - 11\cos x$

attempt to form a quadratic in $\cos x$ ***M1***

$3\left(2\cos^2 x - 1\right) = 4 - 11\cos x$ ***A1***

$\left(6\cos^2 x + 11\cos x - 7 = 0\right)$

valid attempt to solve their quadratic ***M1***

$(3\cos x + 7)(2\cos x - 1) = 0$

$\cos x = \frac{1}{2}$ ***A1***

$x = \frac{\pi}{3}, \frac{5\pi}{3}$ ***A1A1***

Note: Ignore any "extra" solutions.

[6 marks]

(b) consider $(\pm)\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (4 - 11\cos x - 3\cos 2x)\,\mathrm{d}x$ ***M1***

$= (\pm)\left[4x - 11\sin x - \frac{3}{2}\sin 2x\right]_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$ ***A1***

Note: Ignore lack of or incorrect limits at this stage.

attempt to substitute their limits into their integral ***M1***

$= \frac{20\pi}{3} - 11\sin\frac{5\pi}{3} - \frac{3}{2}\sin\frac{10\pi}{3} - \left(\frac{4\pi}{3} - 11\sin\frac{\pi}{3} - \frac{3}{2}\sin\frac{2\pi}{3}\right)$

$= \frac{16\pi}{3} + \frac{11\sqrt{3}}{2} + \frac{3\sqrt{3}}{4} + \frac{11\sqrt{3}}{2} + \frac{3\sqrt{3}}{4}$

$= \frac{16\pi}{3} + \frac{25\sqrt{3}}{2}$ ***A1A1***

[5 marks]

(c) attempt to differentiate both functions and equate ***M1***

$-6\sin 2x = 11\sin x$ ***A1***

attempt to solve for x ***M1***

$11\sin x + 12\sin x\cos x = 0$

$\sin x(11 + 12\cos x) = 0$

$\cos x = -\frac{11}{12}$ $\left(\text{or } \sin x = 0\right)$ ***A1***

$\Rightarrow y = 4 - 11\left(-\frac{11}{12}\right)$ ***M1***

$y = \frac{169}{12}\left(= 14\frac{1}{12}\right)$ ***A1***

[6 marks]

40 – (MAT/12_HL_Summer_2019_Q10) – *Integration, Probability*

(a) mode is 0 ***A1***

[1 mark]

(b) (i) attempt at integration by parts ***(M1)***

$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}, dv = dx$

$= x\arcsin x - \int \frac{x\,dx}{\sqrt{1-x^2}}$ ***A1***

$= x\arcsin x + \sqrt{1-x^2}\,(+c)$ ***A1***

(ii) $\int_0^1 (\pi - \arcsin x)\,dx = \left[\pi x - x\arcsin x - \sqrt{1-x^2}\right]_0^1$ ***A1***

$= \left(\pi - \frac{\pi}{2} - 0\right) - (0 - 0 - 1) = \frac{\pi}{2} + 1$

$= \frac{\pi + 2}{2}$ ***A1***

$\int_0^1 k(\pi - \arcsin x)\,dx = 1$ ***(M1)***

Note: This line can be seen (or implied) anywhere.

Note: Do not allow ***FT A*** marks from bi to bii.

$k\left(\frac{\pi+2}{2}\right) = 1$

$\Rightarrow k = \frac{2}{2+\pi}$ ***AG***

[6 marks]

(c) (i) attempt to use product rule to differentiate ***M1***

$\frac{dy}{dx} = x\arcsin x + \frac{x^2}{2\sqrt{1-x^2}} - \frac{1}{4\sqrt{1-x^2}} - \frac{x^2}{4\sqrt{1-x^2}} + \frac{\sqrt{1-x^2}}{4}$ ***A2***

Note: Award ***A2*** for all terms correct, ***A1*** for 4 correct terms.

$= x\arcsin x + \frac{2x^2}{4\sqrt{1-x^2}} - \frac{1}{4\sqrt{1-x^2}} - \frac{x^2}{4\sqrt{1-x^2}} + \frac{1-x^2}{4\sqrt{1-x^2}}$ ***A1***

Note: Award ***A1*** for equivalent combination of correct terms over a common denominator.

$= x\arcsin x$ ***AG***

(ii) $\mathrm{E}(X) = k\int_0^1 x(\pi - \arcsin x)\,\mathrm{d}x$ ***M1***

$= k\int_0^1 (\pi x - x\arcsin x)\,\mathrm{d}x$

$= k\left[\frac{\pi x^2}{2} - \frac{x^2}{2}\arcsin x + \frac{1}{4}\arcsin x - \frac{x}{4}\sqrt{1-x^2}\right]_0^1$ ***A1A1***

Note: Award ***A1*** for first term, ***A1*** for next 3 terms.

$= k\left[\left(\frac{\pi}{2} - \frac{\pi}{4} + \frac{\pi}{8}\right) - (0)\right]$ ***A1***

$= \left(\frac{2}{2+\pi}\right)\frac{3\pi}{8}$ ***A1***

$= \frac{3\pi}{4(\pi+2)}$ ***AG***

[9 marks]

41 – (MAT/10_HL_Winter_2019_Q2) – *Integration*

$\frac{1}{2}\mathrm{e}^{2x}$ seen ***(A1)***

attempt at using limits in an integrated expression $\left(\left[\frac{1}{2}\mathrm{e}^{2x}\right]_0^{\ln k} = \frac{1}{2}\mathrm{e}^{2\ln k} - \frac{1}{2}\mathrm{e}^0\right)$ ***(M1)***

$= \frac{1}{2}\mathrm{e}^{\ln k^2} - \frac{1}{2}\mathrm{e}^0$ ***(A1)***

Setting their equation $= 12$ ***M1***

Note: their equation must be an integrated expression with limits substituted.

$\frac{1}{2}k^2 - \frac{1}{2} = 12$ ***A1***

$(k^2 = 25 \Rightarrow)\, k = 5$ ***A1***

Note: Do not award final ***A1*** for $k = \pm 5$.

[6 marks]

42 – (MAT/10_HL_Winter_2019_Q7) – *Integration*

(a) attempt to complete the square or multiplication and equating coefficients ***(M1)***

$2x - x^2 = -(x-1)^2 + 1$ ***A1***

$a = -1,\ h = 1,\ k = 1$

[2 marks]

(b) use of their identity from part (a) $\left(\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{\sqrt{1-(x-1)^2}}\,\mathrm{d}x\right)$ ***(M1)***

$= \left[\arcsin(x-1)\right]_{\frac{1}{2}}^{\frac{3}{2}}$ or $\left[\arcsin(u)\right]_{-\frac{1}{2}}^{\frac{1}{2}}$ ***A1***

Note: Condone lack of, or incorrect limits up to this point.

$= \arcsin\left(\frac{1}{2}\right) - \arcsin\left(-\frac{1}{2}\right)$ ***(M1)***

$= \frac{\pi}{6} - \left(-\frac{\pi}{6}\right)$ ***(A1)***

$= \frac{\pi}{3}$ ***A1***

[5 marks]

43 – (MAT/10_HL_Winter_2019_Q10) – *Differentiation, Integration*

(a) (i) attempt to use quotient rule (or equivalent) ***(M1)***

$$f'(x) = \frac{(x^2-1)(2)-(2x-4)(2x)}{(x^2-1)^2}$$ ***A1***

$$= \frac{-2x^2+8x-2}{(x^2-1)^2}$$

(ii) $f'(x) = 0$

simplifying numerator (may be seen in part (i)) ***(M1)***

$\Rightarrow x^2 - 4x + 1 = 0$ or equivalent quadratic equation ***A1***

EITHER

use of quadratic formula

$\Rightarrow x = \frac{4 \pm \sqrt{12}}{2}$ ***A1***

OR

use of completing the square

$(x-2)^2 = 3$ ***A1***

THEN

$x = 2 - \sqrt{3}$ (since $2 + \sqrt{3}$ is outside the domain) ***AG***

Note: Do not condone verification that $x = 2 - \sqrt{3} \Rightarrow f'(x) = 0$.
Do not award the final ***A1*** as follow through from part (i).

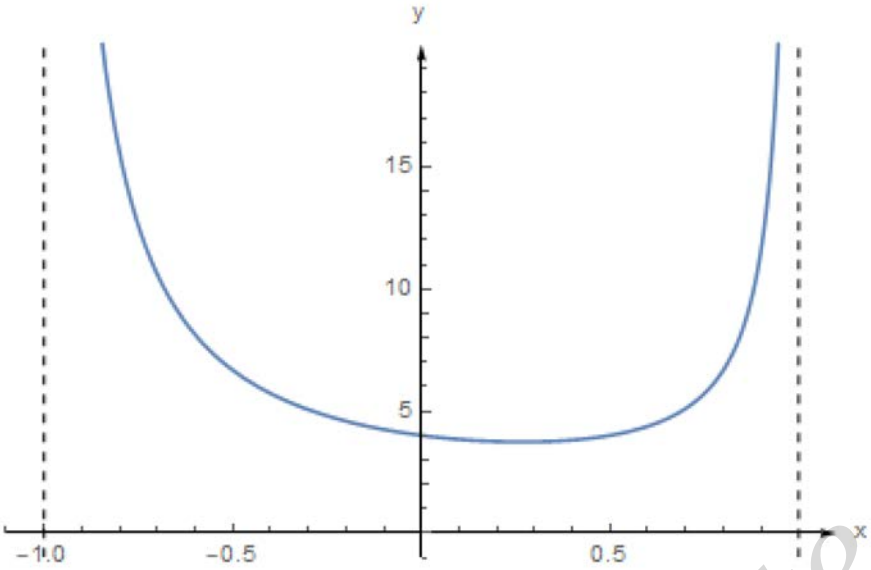
[5 marks]

(b) (i) $(0, 4)$ ***A1***

(ii) $2x - 4 = 0 \Rightarrow x = 2$ ***A1***

outside the domain ***R1***

(iii)



A1A1

award ***A1*** for concave up curve over correct domain with one minimum point in the first quadrant
award ***A1*** for approaching $x = \pm 1$ asymptotically

[5 marks]

(c) valid attempt to combine fractions (using common denominator) ***M1***

$$\frac{3(x-1)-(x+1)}{(x+1)(x-1)}$$ ***A1***

$$= \frac{3x-3-x-1}{x^2-1}$$

$$= \frac{2x-4}{x^2-1}$$ ***AG***

[2 marks]

(d)

$f(x) = 4 \Rightarrow 2x - 4 = 4x^2 - 4$ ***M1***

$(x = 0$ or$)$ $x = \frac{1}{2}$ ***A1***

area under the curve is $\int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x$ ***M1***

$= \int_0^{\frac{1}{2}} \frac{3}{x+1} - \frac{1}{x-1}\,\mathrm{d}x$

Note: Ignore absence of, or incorrect limits up to this point.

$= \left[3\ln|x+1| - \ln|x-1|\right]_0^{\frac{1}{2}}$ ***A1***

$= 3\ln\frac{3}{2} - \ln\frac{1}{2}(-0)$

$= \ln\frac{27}{4}$ ***A1***

area is $2 - \int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x$ or $\int_0^{\frac{1}{2}} 4\,\mathrm{d}x - \int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x$ ***M1***

$= 2 - \ln\frac{27}{4}$

$= \ln\frac{4\mathrm{e}^2}{27}$ ***A1***

$\left(\Rightarrow v = \frac{4\mathrm{e}^2}{27}\right)$

[7 marks]

44 – (MAT/10_HL_Winter_2020_Q9) – *Integration*

$x = \tan u \Rightarrow \frac{dx}{du} = \sec^2 u$ OR $u = \arctan x \Rightarrow \frac{du}{dx} = \frac{1}{1+x^2}$ ***A1***

attempt to write the integral in terms of u ***M1***

$$\int_0^{\frac{\pi}{4}} \frac{\tan^2 u \sec^2 u \, du}{\left(1+\tan^2 u\right)^3}$$

$$\int_0^{\frac{\pi}{4}} \frac{\tan^2 u \sec^2 u \, du}{\left(\sec^2 u\right)^3}$$ ***(A1)***

$$= \int_0^{\frac{\pi}{4}} \sin^2 u \cos^2 u \, du$$ ***A1***

$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} \sin^2 2u \, du$$ ***M1***

$$= \frac{1}{8}\int_0^{\frac{\pi}{4}} \left(1-\cos 4u\right) du$$ ***M1***

$$= \frac{1}{8}\left[u - \frac{\sin 4u}{4}\right]_0^{\frac{\pi}{4}}$$ ***A1***

$$= \frac{1}{8}\left[\frac{\pi}{4} - \frac{\sin \pi}{4} - 0 - 0\right]$$

$$= \frac{\pi}{32}$$ ***A1***

45 – (MAT/10_HL_Winter_2020_Q12) – *Graphs, Integration*

(a) $x = k$ ***A1***

[1 mark]

(b) $y = k$ ***A1***

[1 mark]

(c) **METHOD 1**

$$(f \circ f)(x) = \frac{k\left(\frac{kx-5}{x-k}\right)-5}{\left(\frac{kx-5}{x-k}\right)-k}$$ ***M1***

$$= \frac{k(kx-5)-5(x-k)}{kx-5-k(x-k)}$$ ***A1***

$$= \frac{k^2x-5k-5x+5k}{kx-5-kx+k^2}$$

$$= \frac{k^2x-5x}{k^2-5}$$ ***A1***

$$= \frac{x(k^2-5)}{k^2-5}$$

$$= x$$

$(f \circ f)(x) = x$, (hence f is self-inverse) ***R1***

Note: The statement $f(f(x)) = x$ could be seen anywhere in the candidate's working to award ***R1***.

[4 marks]

METHOD 2

$f(x)=\dfrac{kx-5}{x-k}$

$x=\dfrac{ky-5}{y-k}$ ***M1***

Note: Interchanging x and y can be done at any stage.

$x(y-k)=ky-5$ ***A1***

$xy-xk=ky-5$

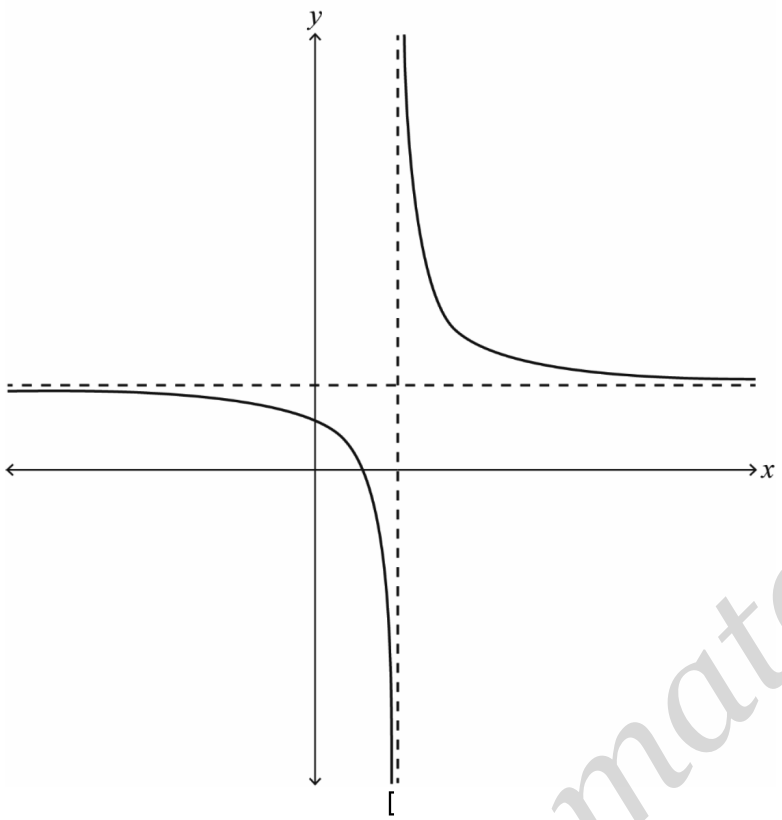
$xy-ky=xk-5$

$y(x-k)=kx-5$ ***A1***

$y=f^{-1}(x)=\dfrac{kx-5}{x-k}$ (hence f is self-inverse.) ***R1***

[4 marks]

(d)



attempt to draw both branches of a rectangular hyperbola ***M1***

$x = 3$ and $y = 3$ ***A1***

$\left(0, \frac{5}{3}\right)$ and $\left(\frac{5}{3}, 0\right)$ ***A1***

[3 marks]

(e) **METHOD 1**

$$\text{volume} = \pi\int_5^7 \left(\frac{3x-5}{x-3}\right)^2 \mathrm{d}x$$ ***(M1)***

EITHER

attempt to express $\frac{3x-5}{x-3}$ in the form $p+\frac{q}{x-3}$ ***M1***

$$\frac{3x-5}{x-3} = 3+\frac{4}{x-3}$$ ***A1***

OR

attempt to expand $\left(\frac{3x-5}{x-3}\right)^2$ or $(3x-5)^2$ and divide out ***M1***

$$\left(\frac{3x-5}{x-3}\right)^2 = 9+\frac{24x-56}{(x-3)^2}$$ ***A1***

THEN

$$\left(\frac{3x-5}{x-3}\right)^2 = 9+\frac{24}{x-3}+\frac{16}{(x-3)^2}$$ ***A1***

$$\text{volume} = \pi\int_5^7 \left(9+\frac{24}{x-3}+\frac{16}{(x-3)^2}\right)\mathrm{d}x$$

$$= \pi\left[9x+24\ln(x-3)-\frac{16}{x-3}\right]_5^7$$ ***A1***

$$= \pi\left[(63+24\ln 4-4)-(45+24\ln 2-8)\right]$$

$$= \pi(22+24\ln 2)$$ ***A1***

[6 marks]

METHOD 2

$$\text{volume} = \pi\int_5^7 \left(\frac{3x-5}{x-3}\right)^2 \mathrm{d}x$$ ***(M1)***

substituting $u = x-3 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$ ***A1***

$3x-5 = 3(u+3)-5 = 3u+4$

$$\text{volume} = \pi\int_2^4 \left(\frac{3u+4}{u}\right)^2 \mathrm{d}u$$ ***M1***

$$= \pi\int_2^4 9 + \frac{16}{u^2} + \frac{24}{u}\,\mathrm{d}u$$ ***A1***

$$= \pi\left[9u - \frac{16}{u} + 24\ln u\right]_2^4$$ ***A1***

Note: Ignore absence of or incorrect limits seen up to this point.

$$= \pi(22 + 24\ln 2)$$ ***A1***

[6 marks]

46 – (MathAA/12_HL_Summer_2021_Q9) – *Integration*

$u = \sin x \Rightarrow \mathrm{d}u = \cos x \ \mathrm{d}x$ (or equivalent) ***A1***

$= \int \frac{u}{u^2 - u - 2} \mathrm{d}u$ ***A1***

attempt to use partial fractions ***M1***

$$\left(\frac{u}{(u+1)(u-2)} \equiv \frac{A}{u+1} + \frac{B}{u-2} \Rightarrow u \equiv A(u-2) + B(u+1) \right)$$

Valid attempt to solve for A and B ***(M1)***

$A = \frac{1}{3}$ and $B = \frac{2}{3}$ ***A1***

$$\frac{u}{(u+1)(u-2)} \equiv \frac{1}{3(u+1)} + \frac{2}{3(u-2)}$$

$\int \left(\frac{1}{3(u+1)} + \frac{2}{3(u-2)} \right) \mathrm{d}u = \frac{1}{3} \ln|u+1| + \frac{2}{3} \ln|u-2| (+C)$ (or equivalent) ***A1***

Note: Condone the absence of $+C$ or lack of moduli here but not in the final answer.

$= \frac{1}{3} \ln|\sin x + 1| + \frac{2}{3} \ln|\sin x - 2| + C$ ***A1***

Note: Condone further simplification of the correct answer.

[7 marks]

47 – (MathAA/10_HL_Winter_2021_Q1) – *Integration*

METHOD 1

recognition that $y = \int \cos\left(x - \frac{\pi}{4}\right) \mathrm{d}x$ ***(M1)***

$y = \sin\left(x - \frac{\pi}{4}\right)(+c)$ ***(A1)***

substitute both x and y values into their integrated expression including c ***(M1)***

$2 = \sin\frac{\pi}{2} + c$

$c = 1$

$y = \sin\left(x - \frac{\pi}{4}\right) + 1$ ***A1***

[4 marks]

METHOD 2

$\int_{2}^{y} \mathrm{d}y = \int_{\frac{3\pi}{4}}^{x} \cos\left(x - \frac{\pi}{4}\right)\mathrm{d}x$ ***(M1)(A1)***

$y - 2 = \sin\left(x - \frac{\pi}{4}\right) - \sin\frac{\pi}{2}$ ***A1***

$y = \sin\left(x - \frac{\pi}{4}\right) + 1$ ***A1***

[4 marks]

48 – (MathAA/11_HL_Summer_2022_Q1) – *Integration*

$\int \frac{3\sqrt{x}-5}{\sqrt{x}}\mathrm{d}x = \int\left(3-5x^{-\frac{1}{2}}\right)\mathrm{d}x$ ***(A1)***

$\int \frac{3\sqrt{x}-5}{\sqrt{x}}\mathrm{d}x = 3x-10x^{\frac{1}{2}}\ (+c)$ ***A1A1***

substituting limits into their integrated function and subtracting ***(M1)***

$3(9)-10(9)^{\frac{1}{2}}-\left(3(1)-10(1)^{\frac{1}{2}}\right)$ OR $27-10\times 3-(3-10)$

$=4$ ***A1***

49 – (MathAA/11_HL_Summer_2022_Q12) – *Sequences - Series, Differentiation, Integration*

(a) **METHOD 1**

recognition of both known series ***(M1)***

$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots$ and $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots$

attempt to multiply the two series up to and including x^3 term ***(M1)***

$e^x \sin x = \left(1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots\right)\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots\right)$

$= x-\frac{x^3}{3!}+x^2+\frac{x^3}{2!}+\ldots$ ***(A1)***

$e^x \sin x = x+x^2+\frac{1}{3}x^3+\ldots$ ***A1***

[4 marks]

METHOD 2

$f(x) = e^x \sin x$

$f'(x) = e^x \cos x + e^x \sin x$ ***A1***

$f''(x) = e^x \cos x - e^x \sin x + e^x \sin x + e^x \cos x \ \left(= 2e^x \cos x\right)$

$f'''(x) = 2e^x \cos x - 2e^x \sin x$

$f''(x) = 2e^x \cos x$ and $f'''(x) = 2e^x\left(\cos x - \sin x\right)$ ***A1***

substitute $x=0$ into f or its derivatives to obtain Maclaurin series ***(M1)***

$e^x \sin x = 0+\frac{x}{1!}\times 1+\frac{x^2}{2!}\times 2+\frac{x^3}{3!}\times 2+\ldots$

$e^x \sin x = x+x^2+\frac{1}{3}x^3+\ldots$ ***A1***

[4 marks]

(b) $e^{x^2}\sin\left(x^2\right) = x^2 + x^4 + \frac{1}{3}x^6 + \ldots$ ***(A1)***

substituting their expression and attempt to integrate ***M1***

$\int_0^1 e^{x^2}\sin(x^2)\,dx \approx \int_0^1\left(x^2 + x^4 + \frac{1}{3}x^6\right)dx$

Note: Condone absence of limits up to this stage.

$= \left[\frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{21}\right]_0^1$ ***A1***

$= \frac{61}{105}$ ***A1***

[4 marks]

(c) (i) attempt to use product rule at least once ***M1***

$g'(x) = e^x\cos x - e^x\sin x$ ***A1***

$g''(x) = e^x\cos x - e^x\sin x - e^x\sin x - e^x\cos x\left(= -2e^x\sin x\right)$ ***A1***

EITHER

$2\left(g'(x) - g(x)\right) = 2\left(e^x\cos x - e^x\sin x - e^x\cos x\right) = -2e^x\sin x$ ***A1***

OR

$g''(x) = 2\left(e^x\cos x - e^x\sin x - e^x\cos x\right)$ ***A1***

THEN

$g''(x) = 2\left(g'(x) - g(x)\right)$ ***AG***

Note: Accept working with each side separately to obtain $-2e^x\sin x$.

(ii) $g'''(x) = 2\left(g''(x) - g'(x)\right)$ ***A1***

$g^{(4)}(x) = 2\left(g'''(x) - g''(x)\right)$ ***AG***

Note: Accept working with each side separately to obtain $-4e^x\cos x$.

[5 marks]

(d) attempt to substitute $x = 0$ into a derivative *(M1)*

$g(0) = 1,\ g'(0) = 1,\ g''(0) = 0$ *A1*

$g'''(0) = -2,\ g^{(4)}(0) = -4$ *(A1)*

attempt to substitute into Maclaurin formula *(M1)*

$g(x) = 1 + x - \frac{2}{3!}x^3 - \frac{4}{4!}x^4 + \ldots \left(= 1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 + \ldots \right)$ *A1*

Note: Do not award any marks for approaches that do not use the part (c) result.

[5 marks]

(e) **METHOD 1**

$\lim_{x \to 0} \frac{e^x \cos x - 1 - x}{x^3} = \lim_{x \to 0} \frac{\left(1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 + \ldots\right) - 1 - x}{x^3}$ *M1*

$= \lim_{x \to 0} \left(-\frac{1}{3} - \frac{1}{6}x + \ldots \right)$ *(A1)*

$= -\frac{1}{3}$ *A1*

Note: Condone the omission of $+\ldots$ in their working.

METHOD 2

$\lim_{x\to 0}\frac{e^x\cos x-1-x}{x^3}=\frac{0}{0}$ indeterminate form, attempt to apply l'Hôpital's rule ***M1***

$=\lim_{x\to 0}\frac{e^x\cos x-e^x\sin x-1}{3x^2}\left(=\lim_{x\to 0}\frac{g'(x)-1}{3x^2}\right)$

$=\frac{0}{0}$, using l'Hôpital's rule again

$=\lim_{x\to 0}\frac{-2e^x\sin x}{6x}\left(=\lim_{x\to 0}\frac{g''(x)}{6x}\right)$

$=\frac{0}{0}$, using l'Hôpital's rule again

$=\lim_{x\to 0}\frac{-2e^x\sin x-2e^x\cos x}{6}\left(=\lim_{x\to 0}\frac{g'''(x)}{6}\right)$ ***A1***

$=-\frac{1}{3}$ ***A1***

[3 marks]

50 – (MathAA/12_HL_Summer_2022_Q7) – *Integration*

METHOD 1

$u = \sec x \Rightarrow \mathrm{d}u = \sec x \tan x \ \mathrm{d}x$ ***(A1)***

attempts to express the integral in terms of u ***M1***

$\int_1^2 u^{n-1} \mathrm{d}u$ ***A1***

$= \frac{1}{n}\left[u^n\right]_1^2 \quad \left(= \frac{1}{n}\left[\sec^n x\right]_0^{\frac{\pi}{3}}\right)$ ***A1***

Note: Condone the absence of or incorrect limits up to this point.

$= \frac{2^n - 1^n}{n}$ ***M1***

$= \frac{2^n - 1}{n}$ ***A1***

Note: Award ***M1*** for correct substitution of **their** limits for u into their antiderivative for u (or given limits for x into their antiderivative for x).

METHOD 2

$\int \sec^n x \tan x \ \mathrm{d}x = \int \sec^{n-1} x \sec x \tan x \ \mathrm{d}x$ ***(A1)***

applies integration by inspection ***(M1)***

$= \frac{1}{n}\left[\sec^n x\right]_0^{\frac{\pi}{3}}$ ***A2***

Note: Award ***A2*** if the limits are not stated.

$= \frac{1}{n}\left(\sec^n \frac{\pi}{3} - \sec^n 0\right)$ ***M1***

Note: Award ***M1*** for correct substitution into their antiderivative.

$= \frac{2^n - 1}{n}$ ***A1***

[6 marks]

51 – (MathAA/10_HL_Winter_2022_Q4) – *Integration*

recognizing need to integrate ***(M1)***

$\int \frac{6x}{x^2+1}\mathrm{d}x$ OR $u = x^2+1$ OR $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$

$\int \frac{3}{u}\mathrm{d}u$ OR $3\int \frac{2x}{x^2+1}\mathrm{d}x$ ***(A1)***

$= 3\ln\left(x^2+1\right)(+c)$ or $3\ln u(+c)$ ***A1***

correct substitution of $x = 1$ and $f(x) = 5$ or $x = 1$ and $u = 2$ into equation

using **their** integrated expression (must involve *c*) ***(M1)***

$5 = 3\ln 2 + c$

$f(x) = 3\ln\left(x^2+1\right) + 5 - 3\ln 2$ $\left(= 3\ln\left(x^2+1\right) + 5 - \ln 8 = 3\ln\left(\frac{x^2+1}{2}\right) + 5\right)$ (or equivalent) ***A1***

Note: Accept the use of the modulus sign in working and the final answer.

[5 marks]

1 - (MAT/11_HL_Summer_2009_Q13) - *Complex Numbers, Differential Equations*

Part A

(a) $|z| = z$, $\arg(z) = 0$
so $L(z) = \ln z$

(b) (i) $L(-1) = \ln 1 + \mathrm{i}\pi = \mathrm{i}\pi$

(ii) $L(1-\mathrm{i}) = \ln\sqrt{2} + \mathrm{i}\dfrac{7\pi}{4}$

(iii) $L(-1+\mathrm{i}) = \ln\sqrt{2} + \mathrm{i}\dfrac{3\pi}{4}$

(c) for comparing the product of two of the above results with the third
for stating the result $-1+\mathrm{i} = -1\times(1-\mathrm{i})$ and $L(-1+\mathrm{i}) \neq L(-1) + L(1-\mathrm{i})$
hence, the property $L(z_1 z_2) = L(z_1) + L(z_2)$
does not hold for all values of z_1 and z_2

Part B

(a) from $f(x+y)=f(x)f(y)$

for $x=y=0$

we have $f(0+0)=f(0)f(0) \Leftrightarrow f(0)=\left(f(0)\right)^2$

as $f(0)\neq 0$, this implies that $f(0)=1$

(b) **METHOD 1**

from $f(x+y)=f(x)f(y)$

for $y=-x$, we have $f(x-x)=f(x)f(-x) \Leftrightarrow f(0)=f(x)f(-x)$

as $f(0)\neq 0$ this implies that $f(x)\neq 0$

METHOD 2

suppose that, for a value of x, $f(x)=0$

from $f(x+y)=f(x)f(y)$

for $y=-x$, we have $f(x-x)=f(x)f(-x) \Leftrightarrow f(0)=f(x)f(-x)$

substituting $f(x)$ by 0 gives $f(0)=0$ which contradicts part (a)

therefore $f(x)\neq 0$ for all x.

(c) by the definition of derivative

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) \\ &= \lim_{h\to 0}\left(\frac{f(x)f(h)-f(x)f(0)}{h}\right) \\ &= \lim_{h\to 0}\left(\frac{f(h)-f(0)}{h}\right)f(x) \\ &= f'(0)f(x) \quad \left(=k\,f(x)\right) \end{aligned}$$

(d) $\int \frac{f'(x)}{f(x)}\,\mathrm{d}x=\int k\,\mathrm{d}x \Rightarrow \ln f(x)=kx+C$

$\ln f(0)=C \Rightarrow C=0$

$f(x)=\mathrm{e}^{kx}$

2 – (MAT/10_HL_Winter_2009_Q8) – *Differential Equations*

(a) $\frac{\mathrm{d}y}{\mathrm{d}t} = ky\cos(kt)$

$\frac{\mathrm{d}y}{y} = k\cos(kt)\,\mathrm{d}t$

$\int \frac{\mathrm{d}y}{y} = \int k\cos(kt)\,\mathrm{d}t$

$\ln y = \sin(kt) + c$

$y = A\mathrm{e}^{\sin(kt)}$

$t = 0 \Rightarrow y_0 = A$

$\Rightarrow y = y_0\mathrm{e}^{\sin kt}$

(b) $-1 \le \sin kt \le 1$

$y_0\mathrm{e}^{-1} \le y \le y_0\mathrm{e}^{1}$

so the ratio is $\frac{1}{\mathrm{e}} : \mathrm{e}$ or $1 : \mathrm{e}^2$

3 – (MAT/12_HL_Summer_2010_Q14) – *Differential Equations, Trigonometry*

(a) the differential equation is separable and can be written as

$\int -y^{-2}\,\mathrm{d}y = \int \cos^2 x\,\mathrm{d}x$ (or equivalent)

$= \int \frac{1+\cos 2x}{2}\mathrm{d}x$

$\frac{1}{y} = \frac{1}{2}x + \frac{1}{4}\sin 2x (+C)$

when $x = 0,\ y = 1$

$C = 1$

$y = \dfrac{1}{\frac{1}{2}x + \frac{1}{4}\sin 2x + 1}$

(b) (i) recognizing use of $(1+\tan x)^2$

$(1+\tan x)^2 = 1 + 2\tan x + \tan^2 x \geq 1 + \tan^2 x = \sec^2 x$

(since all terms are positive)

$(1+\tan x)^2 \geq \sec^2 x$

$\sec^2 x = 1 + \tan^2 x \geq 1$

$\Rightarrow (1+\tan x)^2 \geq \sec^2 x \geq 1$

since all terms are positive, taking square root gives

$1 \leq \sec x \leq 1 + \tan x$

(ii) $\int_0^{\frac{\pi}{4}} \mathrm{d}x \leq \int_0^{\frac{\pi}{4}} \sec x\,\mathrm{d}x \leq \int_0^{\frac{\pi}{4}} 1 + \tan x\,\mathrm{d}x$

$x\Big|_0^{\frac{\pi}{4}} \leq \int_0^{\frac{\pi}{4}} \sec x\,\mathrm{d}x \leq x - \ln\cos x\Big|_0^{\frac{\pi}{4}}$

$\frac{\pi}{4} \leq \int_0^{\frac{\pi}{4}} \sec x\,\mathrm{d}x \leq \frac{\pi}{4} - \ln\frac{1}{\sqrt{2}}$

$\frac{\pi}{4} \leq \int_0^{\frac{\pi}{4}} \sec x\,\mathrm{d}x \leq \frac{\pi}{4} + \frac{1}{2}\ln 2$

4 – (MAT/10_HL_Winter_2011_Q13) – *Differential Equations*

(a) $\frac{dy}{dx} = \frac{e}{\ln e}(2+2) = 4e$

at $(2, e)$ the tangent line is $y - e = 4e(x-2)$

hence $y = 4ex - 7e$

(b) $\frac{dy}{dx} = \frac{y}{\ln y}(x+2) \Rightarrow \frac{\ln y}{y}dy = (x+2)\,dx$

$\int \frac{\ln y}{y}dy = \int (x+2)\,dx$

using substitution $u = \ln y$; $du = \frac{1}{y}dy$

$\Rightarrow \int \frac{\ln y}{y}dy = \int u\,du = \frac{1}{2}u^2$

$\Rightarrow \frac{(\ln y)^2}{2} = \frac{x^2}{2} + 2x + c$

at $(2, e)$, $\frac{(\ln e)^2}{2} = 6 + c$

$\Rightarrow c = -\frac{11}{2}$

$\Rightarrow \frac{(\ln y)^2}{2} = \frac{x^2}{2} + 2x - \frac{11}{2} \Rightarrow (\ln y)^2 = x^2 + 4x - 11$

$\ln y = \pm\sqrt{x^2 + 4x - 11} \Rightarrow y = e^{\pm\sqrt{x^2+4x-11}}$

since $y > 1$, $f(x) = e^{\sqrt{x^2+4x-11}}$

Note: ***M1*** for attempt to make y the subject.

(c) **EITHER**

$x^2 + 4x - 11 > 0$

using the quadratic formula

critical values are $\frac{-4 \pm \sqrt{60}}{2}$ $\left(= -2 \pm \sqrt{15}\right)$

using a sign diagram or algebraic solution

$x < -2 - \sqrt{15}$; $x > -2 + \sqrt{15}$

OR

$x^2 + 4x - 11 > 0$

by methods of completing the square

$(x+2)^2 > 15$

$\Rightarrow x + 2 < -\sqrt{15}$ or $x + 2 > \sqrt{15}$

$x < -2 - \sqrt{15}$; $x > -2 + \sqrt{15}$

(d) $f(x) = f'(x) \Rightarrow f(x) = \dfrac{f(x)}{\ln f(x)}(x+2)$

$\Rightarrow \ln\left(f(x)\right) = x+2 \quad \left(\Rightarrow x+2 = \sqrt{x^2+4x-11}\right)$

$\Rightarrow (x+2)^2 = x^2+4x-11 \Rightarrow x^2+4x+4 = x^2+4x-11$

$\Rightarrow 4 = -11$, hence $f(x) \neq f'(x)$

5 – (MathAA/10_HL_Winter_2021_Q8) – *Differential Equations*

$\dfrac{dy}{dx} + \dfrac{2y}{x} = \dfrac{\ln 2x}{x^2}$ ***(M1)***

attempt to find integrating factor ***(M1)***

$\left(e^{\int \frac{2}{x}dx} = e^{2\ln x}\right) = x^2$ ***(A1)***

$x^2\dfrac{dy}{dx} + 2xy = \ln 2x$

$\dfrac{d}{dx}\left(x^2 y\right) = \ln 2x$

$x^2 y = \int \ln 2x \, dx$

attempt to use integration by parts ***(M1)***

$x^2 y = x\ln 2x - x\left(+c\right)$ ***A1***

$y = \dfrac{\ln 2x}{x} - \dfrac{1}{x} + \dfrac{c}{x^2}$

substituting $x = \dfrac{1}{2}, y = 4$ into an integrated equation involving c ***M1***

$4 = 0 - 2 + 4c$

$\Rightarrow c = \dfrac{3}{2}$

$y = \dfrac{\ln 2x}{x} - \dfrac{1}{x} + \dfrac{3}{2x^2}$ ***A1***

[7 marks]

6 – (MathAA/10_HL_Winter_2022_Q9) – *Differential Equations*

(a) $\frac{dy}{dx} = \frac{y^2 - 2x^2}{xy}$

let $y = vx$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ ***(A1)***

$\Rightarrow v + x\frac{dv}{dx} = \frac{v^2x^2 - 2x^2}{vx^2}$ ***(M1)***

$\Rightarrow x\frac{dv}{dx} = \frac{-2}{v}$ ***(A1)***

$\Rightarrow \int v\,dv = -\int \frac{2}{x}\,dx$ ***M1***

$\Rightarrow \frac{v^2}{2} = -2\ln|x|(+c)$ ***A1***

Note: Condone the absence of the modulus sign up to this point.

$\Rightarrow \frac{y^2}{2x^2} = -2\ln|x| + c$ ***A1***

attempt to substitute $x = 1, y = 2$ into their integrated expression to find c ***M1***

$\Rightarrow 2 = -2\ln|1| + c \Rightarrow c = 2$

$\Rightarrow \frac{y^2}{2x^2} = -2\ln|x| + 2$

$\Rightarrow y^2 = 2x^2(-2\ln|x| + 2)\left(= 4x^2(1 - \ln|x|)\right)$ ***A1***

[8 marks]

(b) attempt to set $\frac{dy}{dx} = 0$ in the differential equation ***(M1)***

$y = \sqrt{2}x$ and $y = -\sqrt{2}x$ or $m = \pm\sqrt{2}$ ***A1***

[2 marks]

1 – (MAT/11_HL_Summer_2014_Q8) – *Kinematics*

$$\frac{dv}{ds} = 2s^{-3}$$

Note: Award ***M1*** for $2s^{-3}$ and ***A1*** for the whole expression.

$$a = v\frac{dv}{ds}$$

$$a = -\frac{1}{s^2} \times \frac{2}{s^3}\left(= -\frac{2}{s^5}\right)$$

when $s = \frac{1}{2}$, $a = -\frac{2}{(0.5)^5}\ (= -64)\ (\text{m s}^{-2})$

2 – (MAT/12_HL_Summer_2017_Q4) – *Kinematics*

(a) $s = t + \cos 2t$

$\frac{ds}{dt} = 1 - 2\sin 2t$ ***M1A1***

$= 0$ ***M1***

$\Rightarrow \sin 2t = \frac{1}{2}$

$t_1 = \frac{\pi}{12}(s),\ t_2 = \frac{5\pi}{12}(s)$ ***A1A1***

Note: Award ***A0A0*** if answers are given in degrees.

[5 marks]

(b) $s = \frac{\pi}{12} + \cos\frac{\pi}{6}\left(s = \frac{\pi}{12} + \frac{\sqrt{3}}{2}(m)\right)$ ***A1A1***

[2 marks]

3 – (MAT/10_HL_Winter_2017_Q5) – *Kinematics*

$$s = \int_0^{\frac{1}{2}} 10te^{-2t}\,dt$$

attempt at integration by parts ***M1***

$$= \left[-5te^{-2t}\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} -5e^{-2t}\,dt$$ ***A1***

$$= \left[-5te^{-2t} - \frac{5}{2}e^{-2t}\right]_0^{\frac{1}{2}}$$ ***(A1)***

Note: Condone absence of limits (or incorrect limits) and missing factor of 10 up to this point.

$$s = \int_0^{\frac{1}{2}} 10te^{-2t}\,dt$$ ***(M1)***

$$= -5e^{-1} + \frac{5}{2}\left(= \frac{-5}{e} + \frac{5}{2}\right)\left(= \frac{5e-10}{2e}\right)$$ ***A1***

[5 marks]

4 – (MathAA/12_HL_Summer_2021_Q11) – *Kinematics, Relative Velocity*

(a) $\frac{dv}{dt} = -(1+v)$ ***(A1)***

$\int 1\,dt = \int -\frac{1}{1+v}\,dv$ (or equivalent / use of integrating factor) ***M1***

$t = -\ln(1+v)(+C)$ ***A1***

EITHER

attempt to find C with initial conditions $t = 0, v = v_0$ ***M1***

$C = \ln(1+v_0)$

$t = \ln(1+v_0) - \ln(1+v)$

$t = \ln\left(\frac{1+v_0}{1+v}\right) \Rightarrow e^t = \frac{1+v_0}{1+v}$ ***A1***

$e^t(1+v) = 1+v_0$

$1+v = (1+v_0)e^{-t}$ ***A1***

$v(t) = (1+v_0)e^{-t} - 1$ ***AG***

OR

$t - C = -\ln(1+v) \Rightarrow e^{t-C} = \dfrac{1}{(1+v)}$

Attempt to find C with initial conditions $t = 0, v = v_0$ ***M1***

$e^{-C} = \dfrac{1}{(1+v_0)} \Rightarrow C = \ln(1+v_0)$

$t - \ln(1+v_0) = -\ln(1+v) \Rightarrow t = \ln(1+v_0) - \ln(1+v)$

$t = \ln\left(\dfrac{1+v_0}{1+v}\right) \Rightarrow e^t = \dfrac{1+v_0}{1+v}$ ***A1***

$e^t(1+v) = 1+v_0$

$1+v = (1+v_0)e^{-t}$ ***A1***

$v(t) = (1+v_0)e^{-t} - 1$ ***AG***

OR

$t - C = -\ln(1+v) \Rightarrow e^{-t+C} = 1+v$ ***A1***

$ke^{-t} - 1 = v$

Attempt to find k with initial conditions $t = 0, v = v_0$ ***M1***

$k = 1+v_0$

$e^{-t}(1+v_0) = 1+v$ ***A1***

$v(t) = (1+v_0)e^{-t} - 1$ ***AG***

Note: condone use of modulus within the ln function(s)

[6 marks]

(b) (i) recognition that when $t = T, v = 0$ ***M1***

$(1+v_0)e^{-T} - 1 = 0 \Rightarrow e^{-T} = \frac{1}{1+v_0}$ ***A1***

$e^T = 1 + v_0$ ***AG***

Note: Award ***M1A0*** for substituting $v_0 = e^T - 1$ into v and showing that $v = 0$.

(ii) $s(t) = \int v(t)\ dt\ \left(= \int \left((1+v_0)e^{-t} - 1\right)dt\right)$ ***(M1)***

$= -(1+v_0)e^{-t} - t(+D)$ ***A1***

($t = 0, s = 0$ so) $D = 1 + v_0$ ***A1***

$s(t) = -(1+v_0)e^{-t} - t + 1 + v_0$

at $s_{\max}$, $e^T = 1 + v_0 \Rightarrow T = \ln(1+v_0)$

Substituting into $s(t)\left(= -(1+v_0)e^{-t} - t + 1 + v_0\right)$ ***M1***

$s_{\max} = -(1+v_0)\left(\frac{1}{1+v_0}\right) - \ln(1+v_0) + v_0 + 1$ ***A1***

$\left(s_{\max} = v_0 - \ln(1+v_0)\right)$

[7 marks]

(c) **METHOD 1**

$v(T-k) = (1+v_0)e^{-T}e^k - 1$ ***(M1)***

$= (1+v_0)\left(\frac{1}{1+v_0}\right)e^k - 1$ ***A1***

$= e^k - 1$ ***AG***

[2 marks]

METHOD 2

$v(T-k) = (1+v_0)e^{-(T-k)} - 1$

$= e^T e^{-(T-k)} - 1$ ***M1***

$= e^{T-T+k} - 1$ ***A1***

$= e^k - 1$ ***AG***

[2 marks]

(d) **METHOD 1**

$v(T+k)=(1+v_0)e^{-T}e^{-k}-1$ ***(A1)***

$=e^{-k}-1$ ***A1***

[2 marks]

METHOD 2

$v(T+k)=(1+v_0)e^{-(T+k)}-1$ ***(A1)***

$=e^{T}e^{-(T+k)}-1$

$=e^{T-T-k}-1$

$=e^{-k}-1$ ***A1***

[2 marks]

(e) **METHOD 1**

$v(T-k)+v(T+k)=e^{k}+e^{-k}-2$ ***A1***

attempt to express as a square ***M1***

$=\left(e^{\frac{k}{2}}-e^{-\frac{k}{2}}\right)^2 (\geq 0)$ ***A1***

so $v(T-k)+v(T+k)\geq 0$ ***AG***

[3 marks]

METHOD 2

$v(T-k)+v(T+k)=e^{k}+e^{-k}-2$ ***A1***

Attempt to solve $\frac{d}{dk}(e^{k}+e^{-k})=0$ $(\Rightarrow k=0)$ ***M1***

minimum value of 2, (when $k=0$), hence $e^{k}+e^{-k}\geq 2$ ***R1***

so $v(T-k)+v(T+k)\geq 0$ ***AG***

[3 marks]

1 – (MathAA/10_HL_Winter_2022_Q2) – *Geometry*

(a) (i) attempt to find midpoint of A and B ***(M1)***

centre $(-1, 3, -2)$ (accept vector notation and/or missing brackets) ***A1***

(ii) attempt to find AB or half of AB or distance between the centre and A (or B) ***(M1)***

$\frac{\sqrt{4^2+2^2+4^2}}{2}$ or $\sqrt{2^2+1^2+2^2}$

$= 3$ ***A1***

[4 marks]

(b) attempt to find the distance between their centre and V
(the perpendicular height of the cone) ***(M1)***

$\sqrt{0^2+4^2+2^2}$ OR $\sqrt{(\text{their slant height})^2-(\text{their radius})^2}$

$= \sqrt{20}\left(= 2\sqrt{5}\right)$ ***(A1)***

Volume $= \frac{1}{3}\pi 3^2\sqrt{20}$

$= 3\pi\sqrt{20}\left(= 6\pi\sqrt{5}\right)$ ***A1***

[3 marks]