IB Diploma

MATHEMATICS AA

HL P2

2012 — 2022

1 – (MAT/21_HL_Summer_2012_Q6) – *Functions - Roots*

[Maximum mark: 5]

Let $f(x) = \ln x$. The graph of f is transformed into the graph of the function g by a translation of $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, followed by a reflection in the x-axis. Find an expression for $g(x)$, giving your answer as a single logarithm.

2 – (MAT/22_HL_Summer_2014_Q7) – *Functions - Roots, Graphs*

The function f is defined as $f(x) = -3 + \frac{1}{x-2}$, $x \neq 2$.

(a) (i) Sketch the graph of $y = f(x)$, clearly indicating any asymptotes and axes intercepts.

(ii) Write down the equations of any asymptotes and the coordinates of any axes intercepts. *[4]*

(b) Find the inverse function f^{-1}, stating its domain. *[4]*

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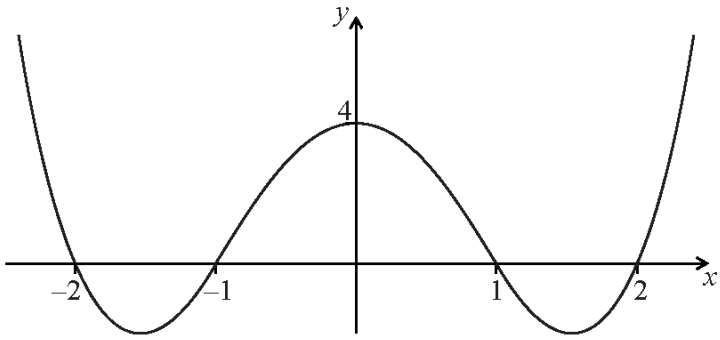
3 – (MAT/21_HL_Summer_2014_Q12) – *Functions - Roots, Graphs*

Let $f(x) = |x| - 1$.

(a) The graph of $y = g(x)$ is drawn below.



(i) Find the value of $(f \circ g)(1)$.

(ii) Find the value of $(f \circ g \circ g)(1)$.

(iii) Sketch the graph of $y = (f \circ g)(x)$. *[5]*

(b) (i) Sketch the graph of $y = f(x)$.

(ii) State the zeros of f. *[3]*

(c) (i) Sketch the graph of $y = (f \circ f)(x)$.

(ii) State the zeros of $f \circ f$. *[3]*

(d) Given that we can denote $\underbrace{f \circ f \circ f \circ \ldots \circ f}_{n \text{ times}}$ as f^n,

(i) find the zeros of f^3;

(ii) find the zeros of f^4;

(iii) deduce the zeros of f^8. *[3]*

(e) The zeros of f^{2n} are $a_1, a_2, a_3, \ldots, a_N$.

(i) State the relation between n and N;

(ii) Find, and simplify, an expression for $\sum_{r=1}^{N} |a_r|$ in terms of n. *[4]*

4 – (MAT/22_HL_Summer_2015_Q3) – *Graphs, Functions - Roots*

(a) Sketch the graph of $y = (x-5)^2 - 2|x-5| - 9$, for $0 \le x \le 10$. [3]

(b) Hence, or otherwise, solve the equation $(x-5)^2 - 2|x-5| - 9 = 0$. [2]

5 – (MAT/21_HL_Summer_2015_Q10) – *Functions - Roots*

A function f is defined by $f(x) = (x+1)(x-1)(x-5)$, $x \in \mathbb{R}$.

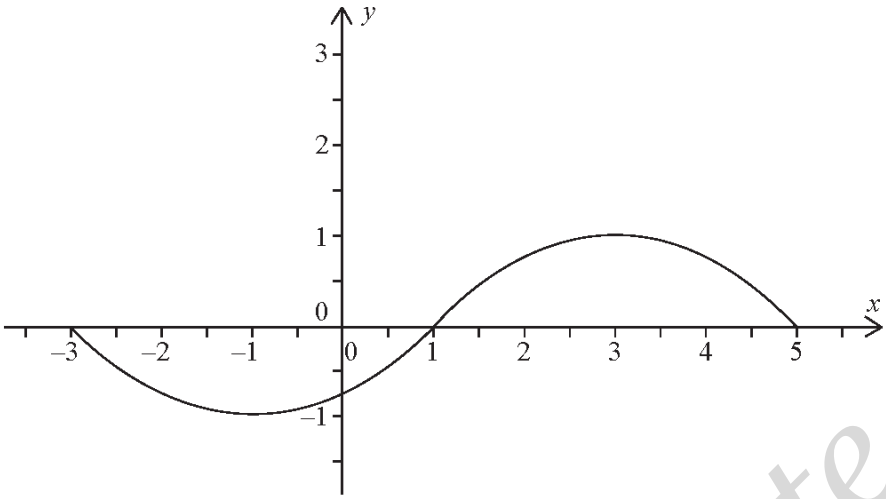
(a) Find the values of x for which $f(x) < |f(x)|$. [3]

A function g is defined by $g(x) = x^2 + x - 6$, $x \in \mathbb{R}$.

(b) Find the values of x for which $g(x) < \dfrac{1}{g(x)}$. [7]

6 – (MAT/20_HL_Winter_2015_Q12) – *Functions - Roots*

The following graph represents a function $y = f(x)$, where $-3 \le x \le 5$.
The function has a maximum at $(3, 1)$ and a minimum at $(-1, -1)$.



(a) The functions u and v are defined as $u(x) = x - 3$, $v(x) = 2x$ where $x \in \mathbb{R}$.

(i) State the range of the function $u \circ f$.

(ii) State the range of the function $u \circ v \circ f$.

(iii) Find the largest possible domain of the function $f \circ v \circ u$. [7]

(b) (i) Explain why f does not have an inverse.

(ii) The domain of f is restricted to define a function g so that it has an inverse g^{-1}.
State the largest possible domain of g.

(iii) Sketch a graph of $y = g^{-1}(x)$, showing clearly the y-intercept and stating the coordinates of the endpoints. [6]

Consider the function defined by $h(x) = \dfrac{2x-5}{x+d}$, $x \ne -d$ and $d \in \mathbb{R}$.

(c) (i) Find an expression for the inverse function $h^{-1}(x)$.

(ii) Find the value of d such that h is a self-inverse function.

For this value of d, there is a function k such that $h \circ k(x) = \dfrac{2x}{x+1}$, $x \ne -1$.

(iii) Find $k(x)$. [8]

7 – (MAT/21_HL_Summer_2016_Q2) – *Quadratics, Functions - Roots*

(a) Express $x^2 + 4x - 2$ in the form $(x + a)^2 + b$ where $a, b \in \mathbb{Z}$. [2]

(b) If $f(x) = x + 2$ and $(g \circ f)(x) = x^2 + 4x - 2$ write down $g(x)$. [2]

8 – (MAT/22_HL_Summer_2016_Q5) – *Functions - Roots*

[Maximum mark: 6]

The function f is defined as $f(x) = \sqrt{\frac{1-x}{1+x}}, -1 < x \leq 1$.

Find the inverse function, f^{-1} stating its domain and range.

9 – (MAT/21_HL_Summer_2016_Q11) – *Functions - Roots*

Let $f(x) = x^4 + 0.2x^3 - 5.8x^2 - x + 4$, $x \in \mathbb{R}$.

(a) Find the solutions of $f(x) > 0$. [3]

(b) For the curve $y = f(x)$.

(i) Find the coordinates of both local minimum points.

(ii) Find the x-coordinates of the points of inflexion. [5]

The domain of f is now restricted to $[0, a]$.

(c) (i) Write down the largest value of a for which f has an inverse. Give your answer correct to 3 significant figures.

(ii) For this value of a sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes, showing clearly the coordinates of the end points of each curve.

(iii) Solve $f^{-1}(x) = 1$. [6]

Let $g(x) = 2\sin(x-1) - 3$, $-\frac{\pi}{2} + 1 \le x \le \frac{\pi}{2} + 1$.

(d) (i) Find an expression for $g^{-1}(x)$, stating the domain.

(ii) Solve $\left(f^{-1} \circ g\right)(x) < 1$. [8]

10 – (MAT/20_HL_Winter_2016_Q5) – *Functions - Roots, Graphs*

Consider the function f defined by $f(x) = 3x \arccos(x)$ where $-1 \le x \le 1$.

(a) Sketch the graph of f indicating clearly any intercepts with the axes and the coordinates of any local maximum or minimum points. [3]

(b) State the range of f. [2]

(c) Solve the inequality $|3x \arccos(x)| > 1$. [4]

11 - (MAT/22_HL_Summer_2017_Q11) - *Functions - Roots, Graphs*

It is given that $f(x) = 3x^4 + ax^3 + bx^2 - 7x - 4$ where a and b are positive integers.

(a) Given that $x^2 - 1$ is a factor of $f(x)$ find the value of a and the value of b. [4]

(b) Factorize $f(x)$ into a product of linear factors. [3]

(c) Sketch the graph of $y = f(x)$, labelling the maximum and minimum points and the x and y intercepts. [3]

(d) Using your graph state the range of values of c for which $f(x) = c$ has exactly two distinct real roots. [3]

12 - (MAT/21_HL_Summer_2018_Q2) - *Functions - Roots*

[Maximum mark: 6]

The equation $x^2 - 5x - 7 = 0$ has roots α and β. The equation $x^2 + px + q = 0$ has roots $\alpha + 1$ and $\beta + 1$. Find the value of p and the value of q.

13 – (MAT/22_HL_Summer_2018_Q10) – *Functions - Roots*

Consider the expression $f(x) = \tan\left(x + \frac{\pi}{4}\right)\cot\left(\frac{\pi}{4} - x\right)$.

(a) (i) Sketch the graph of $y = f(x)$ for $-\frac{5\pi}{8} \le x \le \frac{\pi}{8}$.

(ii) With reference to your graph, explain why f is a function on the given domain.

(iii) Explain why f has no inverse on the given domain.

(iv) Explain why f is not a function for $-\frac{3\pi}{4} \le x \le \frac{\pi}{4}$. [5]

The expression $f(x)$ can be written as $g(t)$ where $t = \tan x$.

(b) Show that $g(t) = \left(\frac{1+t}{1-t}\right)^2$. [3]

(c) Sketch the graph of $y = g(t)$ for $t \le 0$. Give the coordinates of any intercepts and the equations of any asymptotes. [3]

(d) Let α, β be the roots of $g(t) = k$, where $0 < k < 1$.

(i) Find α and β in terms of k.

(ii) Show that $\alpha + \beta < -2$. [7]

14 – (MathAA/22_HL_Summer_2021_Q4) – *Algebra-functions - Roots*

[Maximum mark: 6]

The functions f and g are defined for $x \in \mathbb{R}$ by $f(x) = 6x^2 - 12x + 1$ and $g(x) = -x + c$, where $c \in \mathbb{R}$.

(a) Find the range of f. [2]

(b) Given that $(g \circ f)(x) \leq 0$ for all $x \in \mathbb{R}$, determine the set of possible values for c. [4]

15 – (MathAA/21_HL_Summer_2021_Q11) – *Algebra-functions - Roots, Graphs*

[Maximum mark: 20]

The function f is defined by $f(x) = \dfrac{3x+2}{4x^2-1}$, for $x \in \mathbb{R}$, $x \neq p$, $x \neq q$.

(a) Find the value of p and the value of q. [2]

(b) Find an expression for $f'(x)$. [3]

The graph of $y = f(x)$ has exactly one point of inflexion.

(c) Find the x-coordinate of the point of inflexion. [2]

(d) Sketch the graph of $y = f(x)$ for $-3 \leq x \leq 3$, showing the values of any axes intercepts, the coordinates of any local maxima and local minima, and giving the equations of any asymptotes. [5]

The function g is defined by $g(x) = \dfrac{4x^2-1}{3x+2}$, for $x \in \mathbb{R}$, $x \neq -\dfrac{2}{3}$.

(e) Find the equations of all the asymptotes on the graph of $y = g(x)$. [4]

(f) By considering the graph of $y = g(x) - f(x)$, or otherwise, solve $f(x) < g(x)$ for $x \in \mathbb{R}$. [4]

16 – (MathAA/22_HL_Summer_2021_Q12) – *Algebra-functions - Roots, Differentiation*

[Maximum mark: 21]

A function f is defined by $f(x) = \arcsin\left(\frac{x^2-1}{x^2+1}\right)$, $x \in \mathbb{R}$.

(a) Show that f is an even function. [1]

(b) By considering limits, show that the graph of $y = f(x)$ has a horizontal asymptote and state its equation. [2]

(c) (i) Show that $f'(x) = \frac{2x}{\sqrt{x^2}\left(x^2+1\right)}$ for $x \in \mathbb{R}$, $x \neq 0$.

(ii) By using the expression for $f'(x)$ and the result $\sqrt{x^2} = |x|$, show that f is decreasing for $x < 0$. [9]

A function g is defined by $g(x) = \arcsin\left(\frac{x^2-1}{x^2+1}\right)$, $x \in \mathbb{R}$, $x \geq 0$.

(d) Find an expression for $g^{-1}(x)$, justifying your answer. [5]

(e) State the domain of g^{-1}. [1]

(f) Sketch the graph of $y = g^{-1}(x)$, clearly indicating any asymptotes with their equations and stating the values of any axes intercepts. [3]

17 – (MathAA/20_HL_Winter_2021_Q10) – *Algebra-functions - Roots, Graphs, Integration*

[Maximum mark: 18]

Consider the function $f(x) = \dfrac{x^2 - x - 12}{2x - 15}$, $x \in \mathbb{R}$, $x \neq \dfrac{15}{2}$.

(a) Find the coordinates where the graph of f crosses the

(i) x-axis;

(ii) y-axis. [3]

(b) Write down the equation of the vertical asymptote of the graph of f. [1]

(c) The oblique asymptote of the graph of f can be written as $y = ax + b$ where $a, b \in \mathbb{Z}$.

Find the value of a and the value of b. [4]

(d) Sketch the graph of f for $-30 \leq x \leq 30$, clearly indicating the points of intersection with each axis and any asymptotes. [3]

(e) (i) Express $\dfrac{1}{f(x)}$ in partial fractions.

(ii) Hence find the exact value of $\displaystyle\int_0^3 \frac{1}{f(x)}\,\mathrm{d}x$, expressing your answer as a single logarithm. [7]

18 – (MathAA/21_HL_Summer_2022_Q6) – *Algebra-functions - Roots*

[Maximum mark: 6]

Consider the function $f(x) = 2^x - \frac{1}{2^x}$, $x \in \mathbb{R}$.

(a) Show that f is an odd function. [2]

The function g is given by $g(x) = \frac{x-1}{x^2 - 2x - 3}$, where $x \in \mathbb{R}$, $x \neq -1$, $x \neq 3$.

(b) Solve the inequality $f(x) \geq g(x)$. [4]

19 – (MathAA/21_HL_Summer_2022_Q10) – *Algebra-functions - Roots, Differentiation, Integration*

[Maximum mark: 20]

Consider the function $f(x)=\sqrt{x^2-1}$, where $1 \le x \le 2$.

(a) Sketch the curve $y=f(x)$, clearly indicating the coordinates of the endpoints. [2]

(b) (i) Show that the inverse function of f is given by $f^{-1}(x)=\sqrt{x^2+1}$.

(ii) State the domain and range of f^{-1}. [5]

The curve $y=f(x)$ is rotated 2π about the y-axis to form a solid of revolution that is used to model a water container.

(c) (i) Show that the volume, $V\,\mathrm{m}^3$, of water in the container when it is filled to a height of h metres is given by $V=\pi\left(\frac{1}{3}h^3+h\right)$.

(ii) Hence, determine the maximum volume of the container. [5]

At $t=0$, the container is empty. Water is then added to the container at a constant rate of $0.4\,\mathrm{m}^3\,\mathrm{s}^{-1}$.

(d) Find the time it takes to fill the container to its maximum volume. [2]

(e) Find the rate of change of the height of the water when the container is filled to half its maximum volume. [6]

20 – (MathAA/22_HL_Summer_2022_Q7) – *Algebra-functions - Roots*

[Maximum mark: 8]

Consider $\lim_{x \to 0} \frac{\arctan(\cos x) - k}{x^2}$, where $k \in \mathbb{R}$.

(a) Show that a finite limit only exists for $k = \frac{\pi}{4}$. [2]

(b) Using l'Hôpital's rule, show algebraically that the value of the limit is $-\frac{1}{4}$. [6]

21 – (MathAA/20_HL_Winter_2022_Q11) – *Algebra-functions - Roots, Differentiation*

[Maximum mark: 15]

The function f is defined by $f(x) = e^{-2x}(3x - 4)$, where $x \in \mathbb{R}$.

(a) Find $f'(x)$. [3]

(b) Hence or otherwise, find the coordinates of the point on the graph of $y = f(x)$ where the tangent is parallel to the line $y = x$. [3]

The region enclosed by the curve $y = f(x)$, the x-axis and the y-axis is rotated through 2π radians about the x-axis to form a solid of revolution.

(c) Find the volume of this solid. [4]

Consider a function g such that $g(0) = 1$ and $g'(0) = 2$.

(d) Find the value of

(i) $(f \circ g)(0)$;

(ii) $(f \circ g)'(0)$. [5]

1 – (MAT/20_HL_Winter_2012_Q2) – *Quadratics*

[Maximum mark: 4]

Show that the quadratic equation $x^2 - (5-k)\,x - (k+2) = 0$ has two distinct real roots for all real values of k.

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2 – (MAT/21_HL_Summer_2013_Q9) – *Quadratics*

(a) Prove that the equation $3x^2 + 2kx + k - 1 = 0$ has two distinct real roots for all values of $k \in \mathbb{R}$. *[4 marks]*

(b) Find the value of k for which the two roots of the equation are closest together. *[3 marks]*

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3 – (MAT/21_HL_Summer_2016_Q2) – *Quadratics, Functions - Roots*

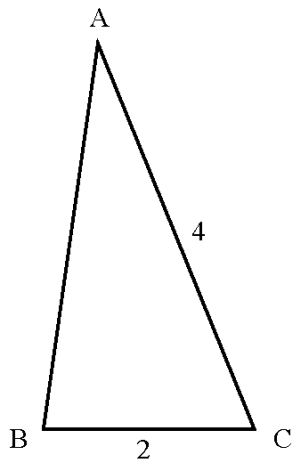
(a) Express $x^2 + 4x - 2$ in the form $(x + a)^2 + b$ where $a, b \in \mathbb{Z}$. [2]

(b) If $f(x) = x + 2$ and $(g \circ f)(x) = x^2 + 4x - 2$ write down $g(x)$. [2]

4 – (MAT/22_HL_Summer_2017_Q4) – *Quadratics, Trigonometry*

(a) Find the set of values of k that satisfy the inequality $k^2 - k - 12 < 0$. [2]

(b) The triangle ABC is shown in the following diagram. Given that $\cos B < \frac{1}{4}$, find the range of possible values for AB. [4]



5 - (MAT/20_HL_Winter_2017_Q7) - *Quadratics*

[Maximum mark: 5]

In the quadratic equation $7x^2 - 8x + p = 0$, $(p \in \mathbb{Q})$, one root is three times the other root.
Find the value of p.

6 – (MathAA/20_HL_Winter_2021_Q6) – *Quadratics*

[Maximum mark: 8]

(a) Prove the identity $(p+q)^3 - 3pq(p+q) = p^3 + q^3$. [2]

The equation $2x^2 - 5x + 1 = 0$ has two real roots, α and β.

Consider the equation $x^2 + mx + n = 0$, where $m, n \in \mathbb{Z}$ and which has roots $\frac{1}{\alpha^3}$ and $\frac{1}{\beta^3}$.

(b) Without solving $2x^2 - 5x + 1 = 0$, determine the values of m and n. [6]

7 – (MathAA/21_HL_Summer_2022_Q8) – *Quadratics*

[Maximum mark: 4]

Consider the equation $kx^2 - (k+3)x + 2k - 9 = 0$, where $k \in \mathbb{R}$.

(a) Write down an expression for the product of the roots, in terms of k. [1]

(b) Hence or otherwise, determine the values of k such that the equation has one positive and one negative real root. [3]

1 – (MAT/20_HL_Winter_2013_Q3) – *Graphs, Exponentials - Logarithms*

Consider $f(x) = \ln x - e^{\cos x}$, $0 < x \le 10$.

(a) Sketch the graph of $y = f(x)$, stating the coordinates of any maximum and minimum points and points of intersection with the x-axis. *[5]*

(b) Solve the inequality $\ln x \le e^{\cos x}$, $0 < x \le 10$. *[2]*

2 – (MAT/20_HL_Winter_2013_Q13) – *Functions - Roots, Exponentials - Logarithms, Integration*

A function f is defined by $f(x)=\frac{1}{2}\left(e^{x}+e^{-x}\right)$, $x \in \mathbb{R}$.

(a) (i) Explain why the inverse function f^{-1} does not exist.

(ii) Show that the equation of the normal to the curve at the point P where $x=\ln 3$ is given by $9x+12y-9\ln 3-20=0$.

(iii) Find the x-coordinates of the points Q and R on the curve such that the tangents at Q and R pass through $(0, 0)$. *[14]*

(b) The domain of f is now restricted to $x \geq 0$.

(i) Find an expression for $f^{-1}(x)$.

(ii) Find the volume generated when the region bounded by the curve $y=f(x)$ and the lines $x=0$ and $y=5$ is rotated through an angle of 2π radians about the y-axis. *[8]*

3 – (MAT/22_HL_Summer_2016_Q3) – *Exponentials - Logarithms*

[Maximum mark: 6]

Solve the simultaneous equations

$$\ln \frac{y}{x} = 2$$

$$\ln x^2 + \ln y^3 = 7 .$$

4 – (MAT/21_HL_Summer_2016_Q7) – *Exponentials - Logarithms*

It has been suggested that in rowing competitions the time, T seconds taken to complete a $2000\,\mathrm{m}$ race can be modelled by an equation of the form $T = aN^b$, where N is the number of rowers in the boat and a and b are constants for rowers of a similar standard.

To test this model the times for the finalists in all the $2000\,\mathrm{m}$ men's races at a recent Olympic games were recorded and the mean calculated.

The results are shown in the following table for $N = 1$ and $N = 2$.

N	T (seconds)
1	420.65
2	390.94

(a) Use these results to find estimates for the value of a and the value of b. Give your answers to five significant figures. [4]

(b) Use this model to estimate the mean time for the finalists in an Olympic race for boats with 8 rowers. Give your answer correct to two decimal places. [1]

It is now given that the mean time in the final for boats with 8 rowers was 342.08 seconds.

(c) Calculate the error in your estimate as a percentage of the actual value. [1]

(d) Comment on the likely validity of the model as N increases beyond 8. [2]

5 – (MAT/22_HL_Summer_2017_Q6) – *Exponentials - Logarithms*

[Maximum mark: 5]

Given that $\log_{10}\left(\frac{1}{2\sqrt{2}}(p+2q)\right)=\frac{1}{2}\left(\log_{10}p+\log_{10}q\right)$, $p>0$, $q>0$, find p in terms of q.

6 – (MAT/21_HL_Summer_2019_Q11) – *Exponentials - Logarithms*

Consider the equation $x^5 - 3x^4 + mx^3 + nx^2 + px + q = 0$, where $m, n, p, q \in \mathbb{R}$.
The equation has three distinct real roots which can be written as $\log_2 a$, $\log_2 b$ and $\log_2 c$.
The equation also has two imaginary roots, one of which is $d\mathrm{i}$ where $d \in \mathbb{R}$.

(a) Show that $abc = 8$. [5]

The values a, b, and c are consecutive terms in a geometric sequence.

(b) Show that one of the real roots is equal to 1. [3]

(c) Given that $q = 8d^2$, find the other two real roots. [9]

7 – (MathAA/22_HL_Summer_2021_Q5) – *Exponentials - Logarithms*

[Maximum mark: 7]

All living plants contain an isotope of carbon called carbon-14. When a plant dies, the isotope decays so that the amount of carbon-14 present in the remains of the plant decreases. The time since the death of a plant can be determined by measuring the amount of carbon-14 still present in the remains.

The amount, A, of carbon-14 present in a plant t years after its death can be modelled by $A = A_0 e^{-kt}$ where $t \geq 0$ and A_0, k are positive constants.

At the time of death, a plant is defined to have 100 units of carbon-14.

(a) Show that $A_0 = 100$. [1]

The time taken for half the original amount of carbon-14 to decay is known to be 5730 years.

(b) Show that $k = \frac{\ln 2}{5730}$. [3]

(c) Find, correct to the nearest 10 years, the time taken after the plant's death for 25% of the carbon-14 to decay. [3]

8 – (MathAA/21_HL_Summer_2021_Q10) – *Exponentials - Logarithms, Differentiation*

[Maximum mark: 15]

Consider the function f defined by $f(x) = 90\,\mathrm{e}^{-0.5x}$ for $x \in \mathbb{R}^+$.

The graph of f and the line $y = x$ intersect at point P.

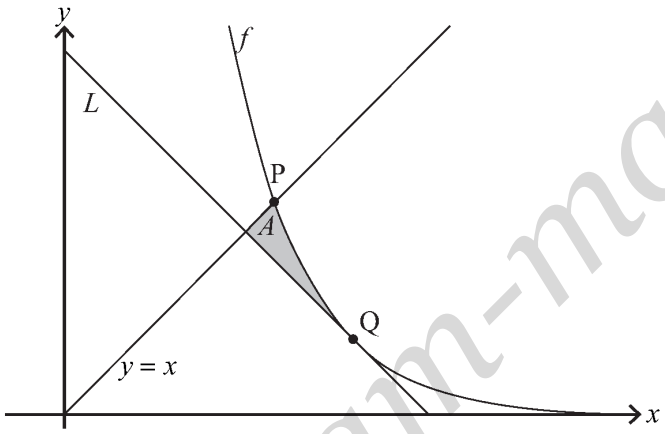
(a) Find the x-coordinate of P. [2]

The line L has a gradient of -1 and is a tangent to the graph of f at the point Q.

(b) Find the exact coordinates of Q. [4]

(c) Show that the equation of L is $y = -x + 2\ln 45 + 2$. [2]

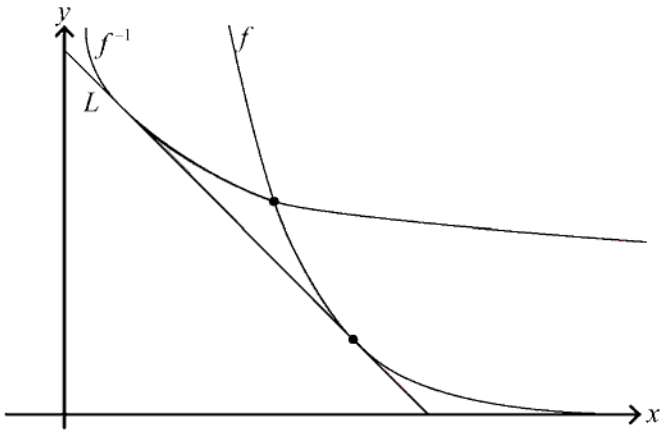
The shaded region A is enclosed by the graph of f and the lines $y = x$ and L.



(d) (i) Find the x-coordinate of the point where L intersects the line $y = x$.

(ii) Hence, find the area of A. [5]

The line L is tangent to the graphs of both f and the inverse function f^{-1}.



(e) Find the shaded area enclosed by the graphs of f and f^{-1} and the line L. [2]

9 – (MathAA/20_HL_Winter_2022_Q4) – *Exponentials - Logarithms*

[Maximum mark: 7]

The population of a town t years after 1 January 2014 can be modelled by the function

$$P(t) = 15\,000e^{kt}, \text{ where } k < 0 \text{ and } t \geq 0.$$

It is known that between 1 January 2014 and 1 January 2022 the population decreased by 11%.

Use this model to estimate the population of this town on 1 January 2041.

1 – (MAT/21_HL_Summer_2016_Q5) – *Graphs*

The function f is given by $f(x) = \dfrac{3x^2+10}{x^2-4}$, $x \in \mathbb{R}$, $x \neq 2$, $x \neq -2$.

(a) Prove that f is an even function. [2]

(b) (i) Sketch the graph $y = f(x)$.

(ii) Write down the range of f. [5]

2 – (MAT/21_HL_Summer_2017_Q6) – *Graphs*

Consider the graph of $y = \tan\left(x + \frac{\pi}{4}\right)$, where $-\pi \le x \le 2\pi$.

(a) Write down the equations of the vertical asymptotes of the graph. [2]

The graph is reflected in the y-axis, then stretched parallel to the y-axis by a factor $\frac{1}{2}$,

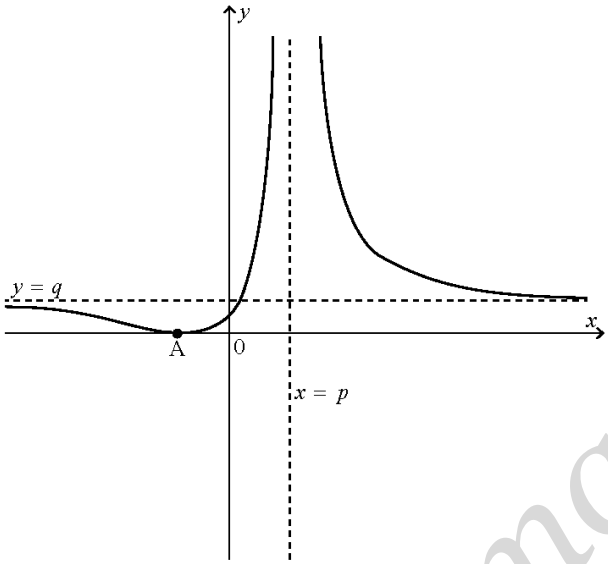
then translated by $\begin{pmatrix} \frac{\pi}{4} \\ -3 \end{pmatrix}$.

(b) Give the equation of the transformed graph. [4]

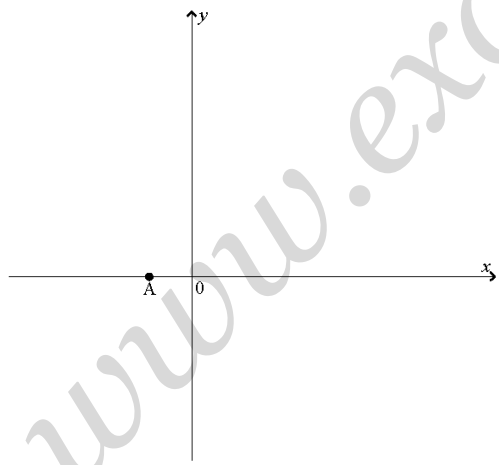
3 – (MAT/20_HL_Winter_2018_Q8) – *Graphs*

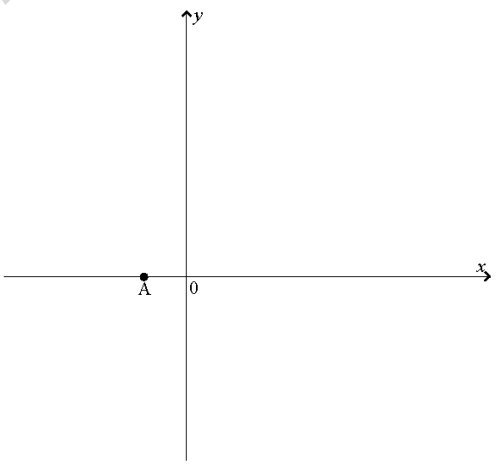
Consider the function $f(x) = \dfrac{ax+1}{bx+c}$, $x \neq -\dfrac{c}{b}$, where $a, b, c \in \mathbb{Z}$.

The following graph shows the curve $y = (f(x))^2$. It has asymptotes at $x = p$ and $y = q$ and meets the x-axis at A.



(a) On the following axes, sketch the two possible graphs of $y = f(x)$ giving the equations of any asymptotes in terms of p and q. [4]





(b) Given that $p = \dfrac{4}{3}$, $q = \dfrac{4}{9}$ and A has coordinates $\left(-\dfrac{1}{2}, 0\right)$, determine the possible sets of values for a, b and c. [4]

4 – (MAT/22_HL_Summer_2019_Q9) – *Graphs*

Consider the polynomial $P(z) \equiv z^4 - 6z^3 - 2z^2 + 58z - 51$. $z \in \mathbb{C}$.

(a) Express $P(z)$ in the form $(z^2 + az + b)(z^2 + cz + d)$ where $a, b, c, d \in \mathbb{R}$. [7]

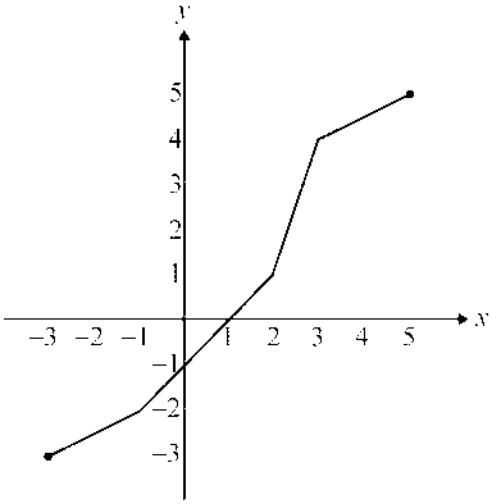
(b) Sketch the graph of $y = x^4 - 6x^3 - 2x^2 + 58x - 51$, stating clearly the coordinates of any maximum and minimum points and intersections with axes. [6]

(c) Hence, or otherwise, state the condition on $k \in \mathbb{R}$ such that all roots of the equation $P(z) = k$ are real. [2]

5 – (MAT/20_HL_Winter_2019_Q3) – *Graphs*

[Maximum mark: 6]

The following diagram shows the graph of $y = f(x)$, $-3 \le x \le 5$.



(a) Find the value of $(f \circ f)(1)$. [2]

(b) Given that $f^{-1}(a) = 3$, determine the value of a. [2]

(c) Given that $g(x) = 2f(x-1)$, find the domain and range of g. [2]

6 – (MathAA/21_HL_Summer_2021_Q11) – *Algebra-functions - Roots, Graphs*

[Maximum mark: 20]

The function f is defined by $f(x) = \dfrac{3x+2}{4x^2-1}$, for $x \in \mathbb{R}$, $x \neq p$, $x \neq q$.

(a) Find the value of p and the value of q. [2]

(b) Find an expression for $f'(x)$. [3]

The graph of $y = f(x)$ has exactly one point of inflexion.

(c) Find the x-coordinate of the point of inflexion. [2]

(d) Sketch the graph of $y = f(x)$ for $-3 \leq x \leq 3$, showing the values of any axes intercepts, the coordinates of any local maxima and local minima, and giving the equations of any asymptotes. [5]

The function g is defined by $g(x) = \dfrac{4x^2-1}{3x+2}$, for $x \in \mathbb{R}$, $x \neq -\dfrac{2}{3}$.

(e) Find the equations of all the asymptotes on the graph of $y = g(x)$. [4]

(f) By considering the graph of $y = g(x) - f(x)$, or otherwise, solve $f(x) < g(x)$ for $x \in \mathbb{R}$. [4]

7 – (MathAA/20_HL_Winter_2021_Q10) – *Algebra-functions - Roots, Graphs, Integration*

[Maximum mark: 18]

Consider the function $f(x) = \dfrac{x^2 - x - 12}{2x - 15}$, $x \in \mathbb{R}$, $x \neq \dfrac{15}{2}$.

(a) Find the coordinates where the graph of f crosses the

(i) x-axis;

(ii) y-axis. [3]

(b) Write down the equation of the vertical asymptote of the graph of f. [1]

(c) The oblique asymptote of the graph of f can be written as $y = ax + b$ where $a, b \in \mathbb{Z}$.

Find the value of a and the value of b. [4]

(d) Sketch the graph of f for $-30 \leq x \leq 30$, clearly indicating the points of intersection with each axis and any asymptotes. [3]

(e) (i) Express $\dfrac{1}{f(x)}$ in partial fractions.

(ii) Hence find the exact value of $\displaystyle\int_0^3 \frac{1}{f(x)}\,\mathrm{d}x$, expressing your answer as a single logarithm. [7]

1 – (MAT/22_HL_Summer_2012_Q1) – *Sequences - Series*

The sum of the first 16 terms of an arithmetic sequence is 212 and the fifth term is 8.

(a) Find the first term and the common difference. *[4 marks]*

(b) Find the smallest value of n such that the sum of the first n terms is greater than 600. *[3 marks]*

2 – (MAT/22_HL_Summer_2012_Q8) – *Sequences - Series*

Each time a ball bounces, it reaches 95 % of the height reached on the previous bounce. Initially, it is dropped from a height of 4 metres.

(a) What height does the ball reach after its fourth bounce? *[2 marks]*

(b) How many times does the ball bounce before it no longer reaches a height of 1 metre? *[3 marks]*

(c) What is the total distance travelled by the ball? *[3 marks]*

3 – (MAT/20_HL_Winter_2012_Q1) – *Sequences - Series*

[Maximum mark: 4]

Find the sum of all the multiples of 3 between 100 and 500.

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4 – (MAT/20_HL_Winter_2012_Q5) – *Sequences - Series*

[Maximum mark: 6]

A metal rod 1 metre long is cut into 10 pieces, the lengths of which form a geometric sequence. The length of the longest piece is 8 times the length of the shortest piece. Find, to the nearest millimetre, the length of the shortest piece.

5 – (MAT/22_HL_Summer_2013_Q5) – *Sequences - Series*

The arithmetic sequence $\{u_n : n \in \mathbb{Z}^+\}$ has first term $u_1 = 1.6$ and common difference $d = 1.5$. The geometric sequence $\{v_n : n \in \mathbb{Z}^+\}$ has first term $v_1 = 3$ and common ratio $r = 1.2$.

(a) Find an expression for $u_n - v_n$ in terms of n. *[2 marks]*

(b) Determine the set of values of n for which $u_n > v_n$. *[3 marks]*

(c) Determine the greatest value of $u_n - v_n$. Give your answer correct to four significant figures. *[1 mark]*

6 – (MAT/22_HL_Summer_2013_Q11) – *Sequences - Series, Probability*

(a) (i) Express the sum of the first n positive odd integers using sigma notation.

(ii) Show that the sum stated above is n^2.

(iii) Deduce the value of the difference between the sum of the first 47 positive odd integers and the sum of the first 14 positive odd integers. *[4 marks]*

(b) A number of distinct points are marked on the circumference of a circle, forming a polygon. Diagonals are drawn by joining all pairs of non-adjacent points.

(i) Show on a diagram all diagonals if there are 5 points.

(ii) Show that the number of diagonals is $\frac{n(n-3)}{2}$ if there are n points, where $n > 2$.

(iii) Given that there are more than one million diagonals, determine the least number of points for which this is possible. *[7 marks]*

(c) The random variable $X \sim B(n, p)$ has mean 4 and variance 3.

(i) Determine n and p.

(ii) Find the probability that in a single experiment the outcome is 1 or 3. *[8 marks]*

7 – (MAT/20_HL_Winter_2013_Q2) – *Sequences - Series*

The fourth term in an arithmetic sequence is 34 and the tenth term is 76.

(a) Find the first term and the common difference. *[3]*

(b) The sum of the first n terms exceeds 5000. Find the least possible value of n. *[4]*

8 – (MAT/22_HL_Summer_2014_Q1) – *Sequences - Series*

(a) (i) Find the sum of all integers, between 10 and 200, which are divisible by 7.

(ii) Express the above sum using sigma notation. *[4]*

An arithmetic sequence has first term 1000 and common difference of -6. The sum of the first n terms of this sequence is negative.

(b) Find the least value of n. *[2]*

9 – (MAT/20_HL_Winter_2014_Q7) – *Sequences - Series*

The seventh, third and first terms of an arithmetic sequence form the first three terms of a geometric sequence.

The arithmetic sequence has first term a and non-zero common difference d.

(a) Show that $d = \frac{a}{2}$. *[3]*

The seventh term of the arithmetic sequence is 3. The sum of the first n terms in the arithmetic sequence exceeds the sum of the first n terms in the geometric sequence by at least 200.

(b) Find the least value of n for which this occurs. *[6]*

10 – (MAT/22_HL_Summer_2016_Q4) – *Sequences - Series*

[Maximum mark: 6]

The sum of the second and third terms of a geometric sequence is 96.

The sum to infinity of this sequence is 500.

Find the possible values for the common ratio, r.

11 – (MAT/20_HL_Winter_2016_Q12) – *Sequences - Series*

On the day of her birth, 1st January 1998, Mary's grandparents invested $\$x$ in a savings account. They continued to deposit $\$x$ on the first day of each month thereafter.
The account paid a fixed rate of 0.4% interest per month. The interest was calculated on the last day of each month and added to the account.
Let $\$A_n$ be the amount in Mary's account on the last day of the nth month, immediately after the interest had been added.

(a) Find an expression for A_1 and show that $A_2 = 1.004^2x + 1.004x$. [2]

(b) (i) Write down a similar expression for A_3 and A_4.

(ii) Hence show that the amount in Mary's account the day before she turned 10 years old is given by $251(1.004^{120} - 1)x$. [6]

(c) Write down an expression for A_n in terms of x on the day before Mary turned 18 years old showing clearly the value of n. [1]

(d) Mary's grandparents wished for the amount in her account to be at least $\$20000$ the day before she was 18. Determine the minimum value of the monthly deposit $\$x$ required to achieve this. Give your answer correct to the nearest dollar. [4]

(e) As soon as Mary was 18 she decided to invest $\$15000$ of this money in an account of the same type earning 0.4% interest per month. She withdraws $\$1000$ every year on her birthday to buy herself a present. Determine how long it will take until there is no money in the account. [5]

12 – (MAT/20_HL_Winter_2017_Q12) – *Sequences - Series*

[Maximum mark: 15]

Phil takes out a bank loan of $\$150\,000$ to buy a house, at an annual interest rate of 3.5%. The interest is calculated at the end of each year and added to the amount outstanding.

(a) Find the amount Phil would owe the bank after 20 years. Give your answer to the nearest dollar. [3]

To pay off the loan, Phil makes annual deposits of $\$P$ at the end of every year in a savings account, paying an annual interest rate of 2%. He makes his first deposit at the end of the first year after taking out the loan.

(b) Show that the total value of Phil's savings after 20 years is $\dfrac{\left(1.02^{20}-1\right)P}{\left(1.02-1\right)}$. [3]

(c) Given that Phil's aim is to own the house after 20 years, find the value for P to the nearest dollar. [3]

David visits a different bank and makes a single deposit of $\$Q$, the annual interest rate being 2.8%.

(d) (i) David wishes to withdraw $\$5000$ at the end of each year for a period of n years. Show that an expression for the minimum value of Q is $\dfrac{5000}{1.028}+\dfrac{5000}{1.028^2}+\ldots+\dfrac{5000}{1.028^n}$.

(ii) Hence or otherwise, find the minimum value of Q that would permit David to withdraw annual amounts of $\$5000$ indefinitely. Give your answer to the nearest dollar. [6]

13 – (MAT/21_HL_Summer_2018_Q1) – *Sequences - Series*

The 3rd term of an arithmetic sequence is 1407 and the 10th term is 1183.

(a) Find the first term and the common difference of the sequence. [4]

(b) Calculate the number of positive terms in the sequence. [3]

14 – (MAT/21_HL_Summer_2018_Q7) – *Sequences - Series*

It is known that the number of fish in a given lake will decrease by 7% each year unless some new fish are added. At the end of each year, 250 new fish are added to the lake. At the start of 2018, there are 2500 fish in the lake.

(a) Show that there will be approximately 2645 fish in the lake at the start of 2020. [3]

(b) Find the approximate number of fish in the lake at the start of 2042. [5]

15 – (MAT/20_HL_Winter_2018_Q1) – *Sequences - Series*

Consider a geometric sequence with a first term of 4 and a fourth term of -2.916.

(a) Find the common ratio of this sequence. [3]

(b) Find the sum to infinity of this sequence. [2]

16 – (MAT/22_HL_Summer_2019_Q7) – *Sequences - Series*

[Maximum mark: 7]

Suppose that u_1 is the first term of a geometric series with common ratio r.
Prove, by mathematical induction, that the sum of the first n terms, S_n is given by
$S_n = \frac{u_1\left(1-r^n\right)}{1-r}$, where $n \in \mathbb{Z}^+$.

17 – (MAT/20_HL_Winter_2019_Q1) – *Sequences - Series*

[Maximum mark: 5]

A geometric sequence has $u_4 = -70$ and $u_7 = 8.75$. Find the second term of the sequence.

18 – (MAT/20_HL_Winter_2020_Q11) – *Kinematics, Sequences - Series*

A particle P moves in a straight line such that after time t seconds, its velocity, v in $\mathrm{m\,s^{-1}}$, is given by $v = \mathrm{e}^{-3t}\sin 6t$, where $0 < t < \frac{\pi}{2}$.

(a) Find the times when P comes to instantaneous rest. [2]

At time t, P has displacement $s(t)$; at time $t = 0$, $s(0) = 0$.

(b) Find an expression for s in terms of t. [7]

(c) Find the maximum displacement of P, in metres, from its initial position. [2]

(d) Find the total distance travelled by P in the first 1.5 seconds of its motion. [2]

At successive times when the acceleration of P is $0\,\mathrm{m\,s^{-2}}$, the velocities of P form a geometric sequence. The acceleration of P is zero at times t_1, t_2, t_3 where $t_1 < t_2 < t_3$ and the respective velocities are v_1, v_2, v_3.

(e) (i) Show that, at these times, $\tan 6t = 2$.

(ii) Hence show that $\frac{v_2}{v_1} = \frac{v_3}{v_2} = -\mathrm{e}^{\frac{\pi}{2}}$. [7]

19 – (MathAA/22_HL_Summer_2021_Q2) – *Sequences - Series*

[Maximum mark: 5]

An arithmetic sequence has first term 60 and common difference -2.5.

(a) Given that the kth term of the sequence is zero, find the value of k. [2]

Let S_n denote the sum of the first n terms of the sequence.

(b) Find the maximum value of S_n. [3]

20 – (MathAA/22_HL_Summer_2021_Q9) – *Sequences - Series*

[Maximum mark: 8]

(a) Write down the first three terms of the binomial expansion of $(1+t)^{-1}$ in ascending powers of t. [1]

(b) By using the Maclaurin series for $\cos x$ and the result from part (a), show that the Maclaurin series for $\sec x$ up to and including the term in x^4 is $1+\frac{x^2}{2}+\frac{5x^4}{24}$. [4]

(c) By using the Maclaurin series for $\arctan x$ and the result from part (b), find $\lim_{x\to 0}\left(\frac{x\arctan 2x}{\sec x-1}\right)$. [3]

21 – (MathAA/20_HL_Winter_2021_Q5) – *Sequences - Series*

[Maximum mark: 9]

The sum of the first n terms of a geometric sequence is given by $S_n = \sum_{r=1}^{n} \frac{2}{3}\left(\frac{7}{8}\right)^r$.

(a) Find the first term of the sequence, u_1. [2]

(b) Find S_∞. [3]

(c) Find the least value of n such that $S_\infty - S_n < 0.001$. [4]

22 – (MathAA/20_HL_Winter_2022_Q3) – *Sequences - Series*

[Maximum mark: 5]

A geometric sequence has a first term of 50 and a fourth term of 86.4.

The sum of the first n terms of the sequence is S_n.

Find the smallest value of n such that $S_n > 33\,500$.

1 – (MAT/20_HL_Winter_2012_Q10) – *Complex Numbers*

[Maximum mark: 7]

Let $\omega = \cos\theta + \mathrm{i}\sin\theta$. Find, in terms of θ, the modulus and argument of $(1-\omega^2)^*$.

2 - (MAT/21_HL_Summer_2013_Q6) - *Complex Numbers*

6. *[Maximum mark: 7]*

A polynomial $p(x)$ with real coefficients is of degree five. The equation $p(x)=0$ has a complex root $2+\mathrm{i}$. The graph of $y=p(x)$ has the x-axis as a tangent at $(2,\ 0)$ and intersects the coordinate axes at $(-1,\ 0)$ and $(0,\ 4)$.

Find $p(x)$ in factorised form with real coefficients.

3 – (MAT/20_HL_Winter_2013_Q6) – *Complex Numbers*

A complex number z is given by $z = \dfrac{a+\mathrm{i}}{a-\mathrm{i}}$, $a \in \mathbb{R}$.

(a) Determine the set of values of a such that

(i) z is real;

(ii) z is purely imaginary. *[4]*

(b) Show that $|z|$ is constant for all values of a. *[2]*

4 – (MAT/21_HL_Summer_2014_Q1) – *Complex Numbers*

[Maximum mark: 4]

One root of the equation $x^2 + ax + b = 0$ is $2 + 3\mathrm{i}$ where $a,\ b \in \mathbb{R}$. Find the value of a and the value of b.

5 - (MAT/22_HL_Summer_2014_Q13) - *Complex Numbers*

Consider $z = r(\cos\theta + \mathrm{i}\sin\theta)$, $z \in \mathbb{C}$.

(a) Use mathematical induction to prove that $z^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta)$, $n \in \mathbb{Z}^+$. *[7]*

Given $u = 1 + \sqrt{3}\mathrm{i}$ and $v = 1 - \mathrm{i}$,

(b) (i) express u and v in modulus-argument form;

(ii) hence find u^3v^4. *[4]*

The complex numbers u and v are represented by point A and point B respectively on an Argand diagram.

(c) Plot point A and point B on the Argand diagram. *[1]*

Point A is rotated through $\frac{\pi}{2}$ in the anticlockwise direction about the origin O to become point A'. Point B is rotated through $\frac{\pi}{2}$ in the clockwise direction about O to become point B'.

(d) Find the area of triangle $\mathrm{OA'B'}$. *[3]*

Given that u and v are roots of the equation $z^4 + bz^3 + cz^2 + dz + e = 0$, where $b, c, d, e \in \mathbb{R}$,

(e) find the values of b, c, d and e. *[5]*

6 – (MAT/21_HL_Summer_2015_Q12) – *Complex Numbers*

(a) (i) Use the binomial theorem to expand $(\cos\theta + \mathrm{i}\sin\theta)^5$.

(ii) Hence use De Moivre's theorem to prove

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta.$$

(iii) State a similar expression for $\cos 5\theta$ in terms of $\cos\theta$ and $\sin\theta$. [6]

Let $z = r(\cos\alpha + \mathrm{i}\sin\alpha)$, where α is measured in degrees, be the solution of $z^5 - 1 = 0$ which has the smallest positive argument.

(b) Find the value of r and the value of α. [4]

(c) Using (a) (ii) and your answer from (b) show that $16\sin^4\alpha - 20\sin^2\alpha + 5 = 0$. [4]

(d) Hence express $\sin 72^\circ$ in the form $\dfrac{\sqrt{a + b\sqrt{c}}}{d}$ where $a, b, c, d \in \mathbb{Z}$. [5]

7 – (MAT/21_HL_Summer_2016_Q9) – *Complex Numbers*

Two distinct roots for the equation $z^4 - 10z^3 + az^2 + bz + 50 = 0$ are $c + \mathrm{i}$ and $2 + \mathrm{i}d$ where $a, b, c, d \in \mathbb{R}, d > 0$.

(a) Write down the other two roots in terms of c and d. [1]

(b) Find the value of c and the value of d. [6]

8 – (MAT/22_HL_Summer_2018_Q1) – *Complex Numbers*

Consider the complex number $z = \dfrac{2 + 7\mathrm{i}}{6 + 2\mathrm{i}}$.

(a) Express z in the form $a + \mathrm{i}b$, where $a, b \in \mathbb{Q}$. [2]

(b) Find the exact value of the modulus of z. [2]

(c) Find the argument of z, giving your answer to 4 decimal places. [2]

9 – (MAT/21_HL_Summer_2019_Q2) – *Complex Numbers*

Solve $z^2 = 4e^{\frac{\pi}{2}i}$, giving your answers in the form

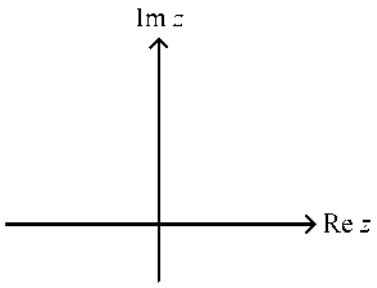
(a) $re^{i\theta}$ where $r, \theta \in \mathbb{R}, r > 0$; [3]

(b) $a + ib$ where $a, b \in \mathbb{R}$. [2]

10 – (MAT/21_HL_Summer_2019_Q6) – *Complex Numbers*

Let $z = a + b\mathrm{i}$, $a, b \in \mathbb{R}^+$ and let $\arg z = \theta$.

(a) Show the points represented by z and $z - 2a$ on the following Argand diagram. [1]



(b) Find an expression in terms of θ for

(i) $\arg(z - 2a)$;

(ii) $\arg\left(\dfrac{z}{z - 2a}\right)$. [3]

(c) Hence or otherwise find the value of θ for which $\mathrm{Re}\left(\dfrac{z}{z - 2a}\right) = 0$. [3]

11 – (MAT/22_HL_Summer_2019_Q8) – *Complex Numbers*

(a) Find the roots of the equation $w^3 = 8\mathrm{i}$, $w \in \mathbb{C}$. Give your answers in Cartesian form. [4]

One of the roots w_1 satisfies the condition $\mathrm{Re}(w_1) = 0$.

(b) Given that $w_1 = \dfrac{z}{z - \mathrm{i}}$, express z in the form $a + b\mathrm{i}$ where $a, b \in \mathbb{Q}$. [3]

12 – (MAT/20_HL_Winter_2019_Q6) – *Complex Numbers*

[Maximum mark: 6]

Let $P(z) = az^3 - 37z^2 + 66z - 10$, where $z \in \mathbb{C}$ and $a \in \mathbb{Z}$.

One of the roots of $P(z) = 0$ is $3 + \mathrm{i}$. Find the value of a.

13 – (MathAA/21_HL_Summer_2021_Q8) – *Complex Numbers*

[Maximum mark: 7]

Consider the complex numbers $z = 2\left(\cos\frac{\pi}{5} + \mathrm{i}\sin\frac{\pi}{5}\right)$ and $w = 8\left(\cos\frac{2k\pi}{5} - \mathrm{i}\sin\frac{2k\pi}{5}\right)$, where $k \in \mathbb{Z}^+$.

(a) Find the modulus of zw. [1]

(b) Find the argument of zw in terms of k. [2]

Suppose that $zw \in \mathbb{Z}$.

(c) (i) Find the minimum value of k.

(ii) For the value of k found in part (i), find the value of zw. [4]

14 – (MathAA/22_HL_Summer_2021_Q8) – *Complex Numbers*

[Maximum mark: 5]

Consider $z = \cos\theta + \mathrm{i}\sin\theta$ where $z \in \mathbb{C}$, $z \neq 1$.

Show that $\mathrm{Re}\left(\frac{1+z}{1-z}\right) = 0$.

15 – (MathAA/20_HL_Winter_2022_Q7) – *Complex Numbers*

[Maximum mark: 6]

Consider the vectors $\boldsymbol{u} = \boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{v} = \left(\cos\frac{1}{n}\right)\boldsymbol{i} + \left(\sin\frac{1}{n}\right)\boldsymbol{j}$, where $n \in \mathbb{Z}$.

Let θ be the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

(a) Find an expression for $\cos\theta$ in terms of n. [3]

(b) Find the exact value of the limit approached by θ as $n \to \infty$. [3]

1 – (MAT/21_HL_Summer_2012_Q3) – *Permutation - Combination*

A team of 6 players is to be selected from 10 volleyball players, of whom 8 are boys and 2 are girls.

(a) In how many ways can the team be selected? *[2 marks]*

(b) In how many of these selections is exactly one girl in the team? *[3 marks]*

(c) If the selection of the team is made at random, find the probability that exactly one girl is in the team. *[2 marks]*

2 – (MAT/22_HL_Summer_2012_Q4) – *Permutation - Combination*

Fifteen boys and ten girls sit in a single line.

(a) In how many ways can they be seated in a single line so that the boys and girls are in two separate groups? *[3 marks]*

(b) Two boys and three girls are selected to go the theatre. In how many ways can this selection be made? *[3 marks]*

3 – (MAT/21_HL_Summer_2013_Q8) – *Permutation - Combination*

Three boys and three girls are to sit on a bench for a photograph.

(a) Find the number of ways this can be done if the three girls must sit together. *[3 marks]*

(b) Find the number of ways this can be done if the three girls must all sit apart. *[4 marks]*

4 – (MAT/21_HL_Summer_2014_Q3) – *Permutation - Combination*

[Maximum mark: 4]

Find the number of ways in which seven different toys can be given to three children, if the youngest is to receive three toys and the others receive two toys each.

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5 – (MAT/22_HL_Summer_2015_Q2) – *Permutation - Combination*

From a group of five males and six females, four people are chosen.

(a) Determine how many possible groups can be chosen. [2]

(b) Determine how many groups can be formed consisting of two males and two females. [2]

(c) Determine how many groups can be formed consisting of at least one female. [2]

6 – (MAT/22_HL_Summer_2017_Q8) – *Permutation - Combination*

[Maximum mark: 6]

In a trial examination session a candidate at a school has to take 18 examination papers including the physics paper, the chemistry paper and the biology paper. No two of these three papers may be taken consecutively. There is no restriction on the order in which the other examination papers may be taken.

Find the number of different orders in which these 18 examination papers may be taken.

7 – (MAT/20_HL_Winter_2017_Q9) – *Permutation - Combination*

[Maximum mark: 6]

Twelve students are to take an exam in advanced combinatorics.
The exam room is set out in three rows of four desks, with the invigilator at the front of the room, as shown in the following diagram.

INVIGILATOR

Desk 1	Desk 2	Desk 3	Desk 4
Desk 5	Desk 6	Desk 7	Desk 8
Desk 9	Desk 10	Desk 11	Desk 12

(a) Find the number of ways the twelve students may be arranged in the exam hall. [1]

Two of the students, Helen and Nicky, are suspected of cheating in a previous exam.

(b) Find the number of ways the students may be arranged if Helen and Nicky must sit so that one is directly behind the other (with no desk in between). For example Desk 5 and Desk 9. [2]

(c) Find the number of ways the students may be arranged if Helen and Nicky must not sit next to each other in the same row. [3]

8 – (MAT/20_HL_Winter_2019_Q8) – *Permutation - Combination*

[Maximum mark: 8]

Eight boys and two girls sit on a bench. Determine the number of possible arrangements, given that

(a) the girls do not sit together; [3]

(b) the girls do not sit on either end; [2]

(c) the girls do not sit on either end and do not sit together. [3]

9 – (MAT/20_HL_Winter_2020_Q7) – *Permutation - Combination*

At a gathering of 12 teachers, seven are male and five are female. A group of five of these teachers go out for a meal together. Determine the possible number of groups in each of the following situations:

(a) There are more males than females in the group. [4]

(b) Two of the teachers, Gary and Gerwyn, refuse to go out for a meal together. [3]

10 – (MathAA/22_HL_Summer_2021_Q7) – *Permutation - Combination*

[Maximum mark: 5]

Eight runners compete in a race where there are no tied finishes. Andrea and Jack are two of the eight competitors in this race.

Find the total number of possible ways in which the eight runners can finish if Jack finishes

(a) in the position immediately after Andrea; [2]

(b) in any position after Andrea. [3]

11 – (MathAA/21_HL_Summer_2022_Q9) – *Permutation - Combination*

[Maximum mark: 7]

Mary, three female friends, and her brother, Peter, attend the theatre. In the theatre there is a row of 10 empty seats. For the first half of the show, they decide to sit next to each other in this row.

(a) Find the number of ways these five people can be seated in this row. [3]

For the second half of the show, they return to the same row of 10 empty seats. The four girls decide to sit at least one seat apart from Peter. The four girls do not have to sit next to each other.

(b) Find the number of ways these five people can now be seated in this row. [4]

12 – (MathAA/22_HL_Summer_2022_Q9) – *Permutation - Combination*

[Maximum mark: 4]

Consider the set of six-digit positive integers that can be formed from the digits $0, 1, 2, 3, 4, 5, 6, 7, 8$ and 9.

Find the total number of six-digit positive integers that can be formed such that

(a) the digits are distinct; [2]

(b) the digits are distinct and are in increasing order. [2]

1 – (MAT/21_HL_Summer_2012_Q9) – *Binomial Theorem*

[Maximum mark: 7]

Find the constant term in the expansion of $\left(x-\frac{2}{x}\right)^4\left(x^2+\frac{2}{x}\right)^3$.

2 – (MAT/22_HL_Summer_2014_Q5) – *Binomial Theorem*

[Maximum mark: 6]

Find the coefficient of x^{-2} in the expansion of $(x-1)^3\left(\frac{1}{x}+2x\right)^6$.

3 – (MAT/21_HL_Summer_2014_Q8) – *Binomial Theorem, Probability*

(a) Find the term in x^5 in the expansion of $(3x+A)(2x+B)^6$. *[4]*

Mina and Norbert each have a fair cubical die with faces labelled 1, 2, 3, 4, 5 and 6; they throw it to decide if they are going to eat a cookie.

Mina throws her die just once and she eats a cookie if she throws a four, a five or a six. Norbert throws his die six times and each time eats a cookie if he throws a five or a six.

(b) Calculate the probability that five cookies are eaten. *[4]*

4 – (MAT/20_HL_Winter_2016_Q4) – *Binomial Theorem*

[Maximum mark: 5]

Find the constant term in the expansion of $\left(4x^2 - \frac{3}{2x}\right)^{12}$.

5 – (MAT/21_HL_Summer_2017_Q3) – *Binomial Theorem*

[Maximum mark: 6]

The coefficient of x^2 in the expansion of $\left(\frac{1}{x}+5x\right)^8$ is equal to the coefficient of x^4 in the expansion of $(a+5x)^7$, $a \in \mathbb{R}$. Find the value of a.

6 – (MAT/22_HL_Summer_2018_Q5) – *Binomial Theorem*

(a) Express the binomial coefficient $\binom{3n+1}{3n-2}$ as a polynomial in n. [3]

(b) Hence find the least value of n for which $\binom{3n+1}{3n-2} > 10^6$. [3]

7 – (MAT/20_HL_Winter_2018_Q4) – *Binomial Theorem*

[Maximum mark: 5]

Find the value of the constant term in the expansion of $x^4\left(x+\frac{3}{x^2}\right)^5$.

8 – (MAT/20_HL_Winter_2019_Q5) – *Binomial Theorem*

[Maximum mark: 6]

Consider the expansion of $(2+x)^n$, where $n \geq 3$ and $n \in \mathbb{Z}$.

The coefficient of x^3 is four times the coefficient of x^2. Find the value of n.

9 – (MAT/20_HL_Winter_2020_Q4) – *Binomial Theorem*

[Maximum mark: 6]

Find the term independent of x in the expansion of $\frac{1}{x^3}\left(\frac{1}{3x^2}-\frac{x}{2}\right)^9$.

10 – (MathAA/21_HL_Summer_2021_Q5) – *Binomial Theorem*

[Maximum mark: 5]

Consider the expansion of $(3+x^2)^{n+1}$, where $n \in \mathbb{Z}^+$.

Given that the coefficient of x^4 is 20412, find the value of n.

11 – (MathAA/20_HL_Winter_2022_Q5) – *Binomial Theorem*

[Maximum mark: 6]

Consider the expansion of $\frac{(ax+1)^9}{21x^2}$, where $a \neq 0$. The coefficient of the term in x^2 is $\frac{8}{7}a^5$.

Find the value of a.

1 – (MAT/20_HL_Winter_2014_Q6) – *Remainder & Factor Theorem*

[Maximum mark: 6]

Consider $p(x) = 3x^3 + ax + 5a$, $a \in \mathbb{R}$.

The polynomial $p(x)$ leaves a remainder of -7 when divided by $(x - a)$.

Show that only one value of a satisfies the above condition and state its value.

2 - (MAT/21_HL_Summer_2016_Q8) - *Remainder & Factor Theorem*

[Maximum mark: 5]

When $x^2 + 4x - b$ is divided by $x - a$ the remainder is 2.
Given that $a, b \in \mathbb{R}$, find the smallest possible value for b.

3 - (MAT/22_HL_Summer_2018_Q2) - *Remainder & Factor Theorem*

[Maximum mark: 5]

The polynomial $x^4 + px^3 + qx^2 + rx + 6$ is exactly divisible by each of $(x-1)$, $(x-2)$ and $(x-3)$.

Find the values of p, q and r.

4 - (MAT/20_HL_Winter_2018_Q6) - *Remainder & Factor Theorem*

Let $P(x) = 2x^4 - 15x^3 + ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$.

(a) Given that $(x-5)$ is a factor of $P(x)$, find a relationship between a, b and c. [2]

(b) Given that $(x-5)^2$ is a factor of $P(x)$, write down the value of $P'(5)$. [1]

(c) Given that $(x-5)^2$ is a factor of $P(x)$, and that $a = 2$, find the values of b and c. [3]

1 – (MAT/22_HL_Summer_2013_Q8) – *Mathematical Induction*

[Maximum mark: 7]

Use the method of mathematical induction to prove that $5^{2n} - 24n - 1$ is divisible by 576 for $n \in \mathbb{Z}^+$.

2 – (MAT/21_HL_Summer_2014_Q7) – *Mathematical Induction*

[Maximum mark: 8]

Prove, by mathematical induction, that $7^{8n+3}+2$, $n \in \mathbb{N}$, is divisible by 5.

3 - (MAT/22_HL_Summer_2018_Q6) - *Mathematical Induction*

[Maximum mark: 7]

Use mathematical induction to prove that $(1-a)^n > 1 - na$ for $\{n : n \in \mathbb{Z}^+, n \geq 2\}$ where $0 < a < 1$.

4 – (MAT/21_HL_Summer_2019_Q8) – *Mathematical Induction*

(a) Solve the inequality $x^2 > 2x + 1$. [2]

(b) Use mathematical induction to prove that $2^{n+1} > n^2$ for $n \in \mathbb{Z}$, $n \geq 3$. [7]

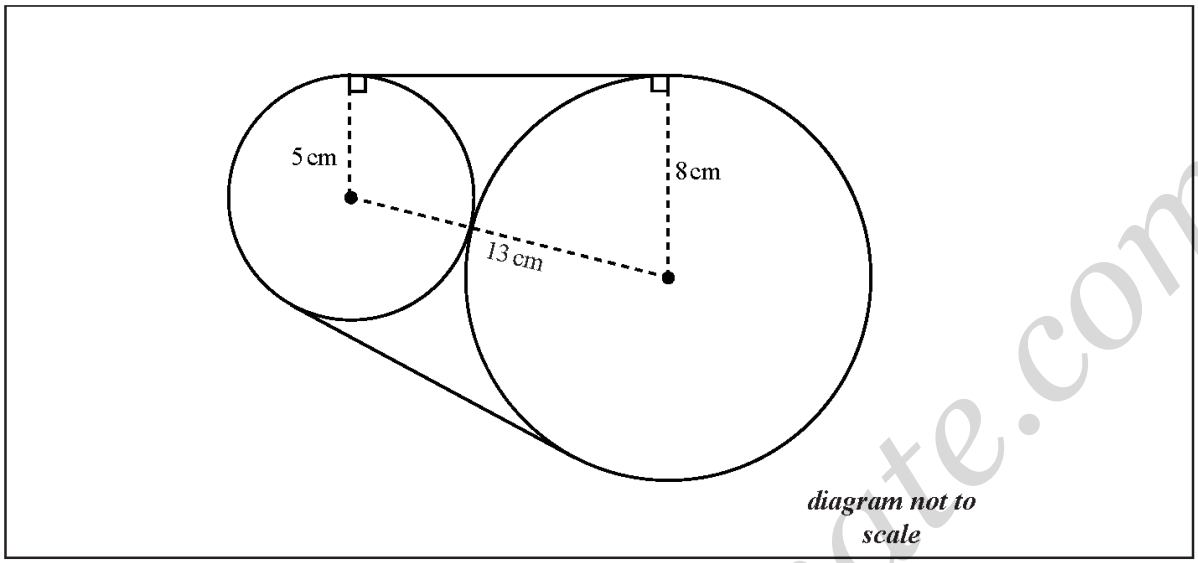
5 – (MAT/20_HL_Winter_2020_Q6) – *Mathematical Induction*

[Maximum mark: 7]

Use mathematical induction to prove that $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x\mathrm{e}^{px}\right) = p^{n-1}\left(px+n\right)\mathrm{e}^{px}$ for $n \in \mathbb{Z}^+$, $p \in \mathbb{Q}$.

1 – (MAT/22_HL_Summer_2012_Q9) – *Radian*

Two discs, one of radius 8 cm and one of radius 5 cm, are placed such that they touch each other. A piece of string is wrapped around the discs. This is shown in the diagram below.



Calculate the length of string needed to go around the discs.

2 – (MAT/21_HL_Summer_2013_Q5) – *Radian*

[Maximum mark: 6]

A rectangle is drawn around a sector of a circle as shown. If the angle of the sector is 1 radian and the area of the sector is $7\,\text{cm}^2$, find the dimensions of the rectangle, giving your answers to the nearest millimetre.

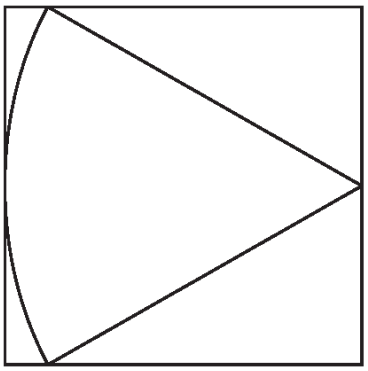
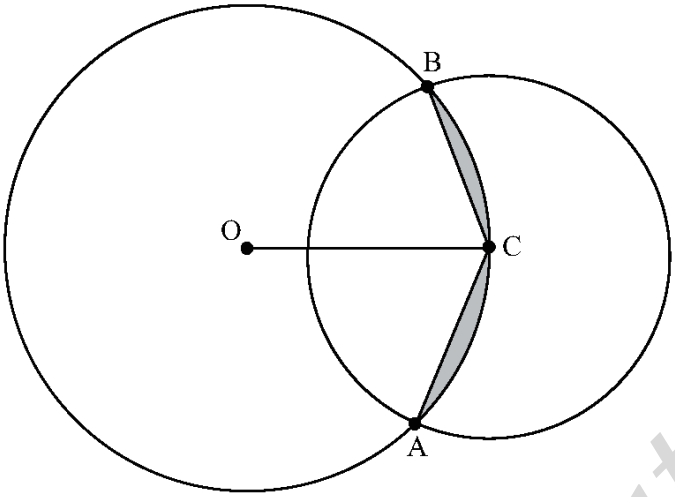


diagram not to scale

3 – (MAT/22_HL_Summer_2014_Q4) – *Trigonometry, Radian*

The following diagram shows two intersecting circles of radii 4 cm and 3 cm. The centre C of the smaller circle lies on the circumference of the bigger circle. O is the centre of the bigger circle and the two circles intersect at points A and B.



Find:

(a) $B\hat{O}C$; *[2]*

(b) the area of the shaded region. *[4]*

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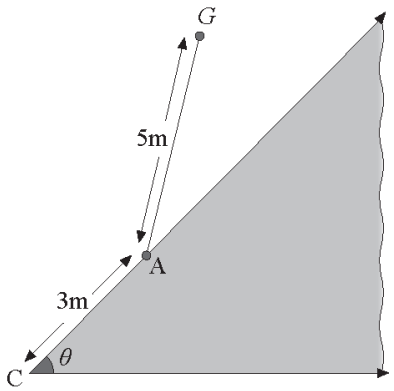
4 – (MAT/21_HL_Summer_2016_Q4) – *Radian*

[Maximum mark: 6]

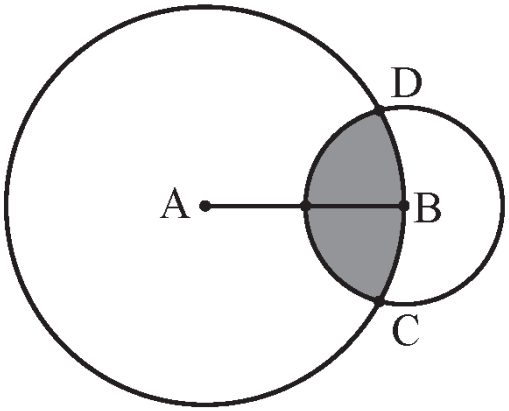
The diagram below shows a fenced triangular enclosure in the middle of a large grassy field. The points A and C are $3\,\mathrm{m}$ apart. A goat G is tied by a $5\,\mathrm{m}$ length of rope at point A on the outside edge of the enclosure.

Given that the corner of the enclosure at C forms an angle of θ radians and the area of field that can be reached by the goat is $44\,\mathrm{m}^2$, find the value of θ.



5 – (MAT/20_HL_Winter_2016_Q9) – *Radian, Trigonometry*

The diagram shows two circles with centres at the points A and B and radii $2r$ and r, respectively. The point B lies on the circle with centre A. The circles intersect at the points C and D.



Let α be the measure of the angle CAD and θ be the measure of the angle CBD in radians.

(a) Find an expression for the shaded area in terms of α, θ and r. [3]

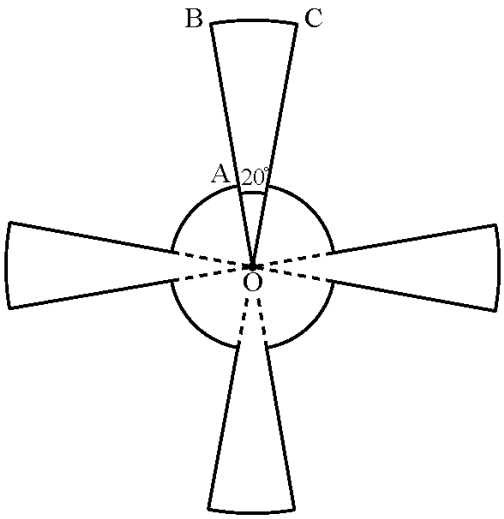
(b) Show that $\alpha = 4\arcsin\frac{1}{4}$. [2]

(c) Hence find the value of r given that the shaded area is equal to 4. [3]

6 – (MAT/20_HL_Winter_2017_Q3) – *Radian*

[Maximum mark: 4]

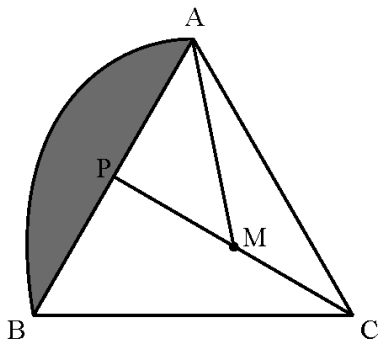
This diagram shows a metallic pendant made out of four equal sectors of a larger circle of radius $\mathrm{OB} = 9\,\mathrm{cm}$ and four equal sectors of a smaller circle of radius $\mathrm{OA} = 3\,\mathrm{cm}$.
The angle $\mathrm{BOC} = 20^\circ$.



Find the area of the pendant.

7 – (MAT/22_HL_Summer_2018_Q4) – *Radian*

Consider the following diagram.



The sides of the equilateral triangle ABC have lengths $1\,\mathrm{m}$. The midpoint of $[\mathrm{AB}]$ is denoted by P. The circular arc AB has centre, M, the midpoint of $[\mathrm{CP}]$.

(a) (i) Find AM.

(ii) Find $\mathrm{A\hat{M}P}$ in radians. [5]

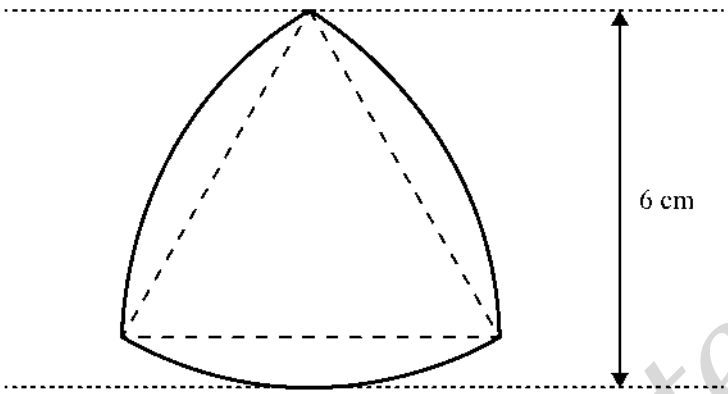
(b) Find the area of the shaded region. [3]

8 – (MAT/20_HL_Winter_2019_Q4) – *Radian*

[Maximum mark: 7]

The following shape consists of three arcs of a circle, each with centre at the opposite vertex of an equilateral triangle as shown in the diagram.

diagram not to scale



For this shape, calculate

(a) the perimeter; [2]

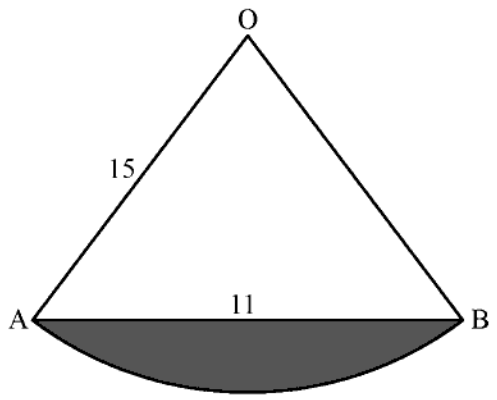
(b) the area. [5]

9 – (MAT/20_HL_Winter_2020_Q1) – *Radian, Trigonometry*

[Maximum mark: 5]

The following diagram shows a sector OAB of radius $15\,\text{cm}$. The length of [AB] is $11\,\text{cm}$.

diagram not to scale

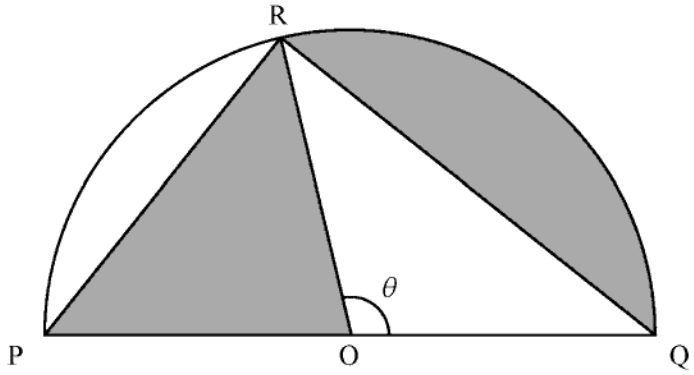


Find the area of the shaded region.

10 – (MathAA/20_HL_Winter_2021_Q4) – *Radian*

[Maximum mark: 6]

The following diagram shows a semicircle with centre O and radius r. Points P, Q and R lie on the circumference of the circle, such that $\mathrm{PQ} = 2r$ and $\mathrm{R\hat{O}Q} = \theta$, where $0 < \theta < \pi$.



(a) Given that the areas of the two shaded regions are equal, show that $\theta = 2\sin\theta$. [5]

(b) Hence determine the value of θ. [1]

1 – (MAT/22_HL_Summer_2012_Q3) – *Trigonometry*

Consider a triangle ABC with $B\hat{A}C = 45.7^\circ$, $AB = 9.63$ cm and $BC = 7.5$ cm.

(a) By drawing a diagram, show why there are two triangles consistent with this information. *[2 marks]*

(b) Find the possible values of AC. *[6 marks]*

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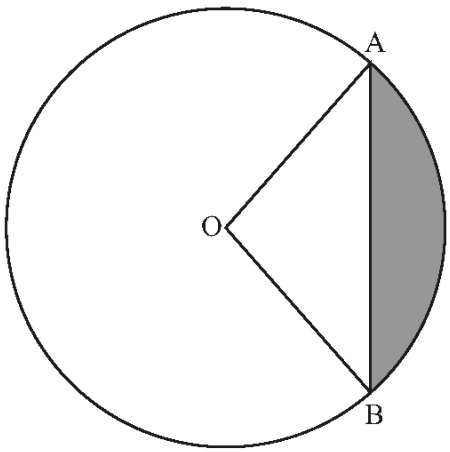
2 – (MAT/22_HL_Summer_2013_Q1) – *Trigonometry*

A circle of radius 4 cm, centre O, is cut by a chord $[\text{AB}]$ of length 6 cm.

diagram not to scale



(a) Find $\text{A}\hat{\text{O}}\text{B}$, expressing your answer in radians correct to four significant figures. *[2 marks]*

(b) Determine the area of the shaded region. *[3 marks]*

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3 – (MAT/22_HL_Summer_2013_Q6) – *Trigonometry*

(a) Solve the equation $3\cos^2 x - 8\cos x + 4 = 0$, where $0 \le x \le 180°$, expressing your answer(s) to the nearest degree. *[3 marks]*

(b) Find the exact values of $\sec x$ satisfying the equation $3\sec^4 x - 8\sec^2 x + 4 = 0$. *[3 marks]*

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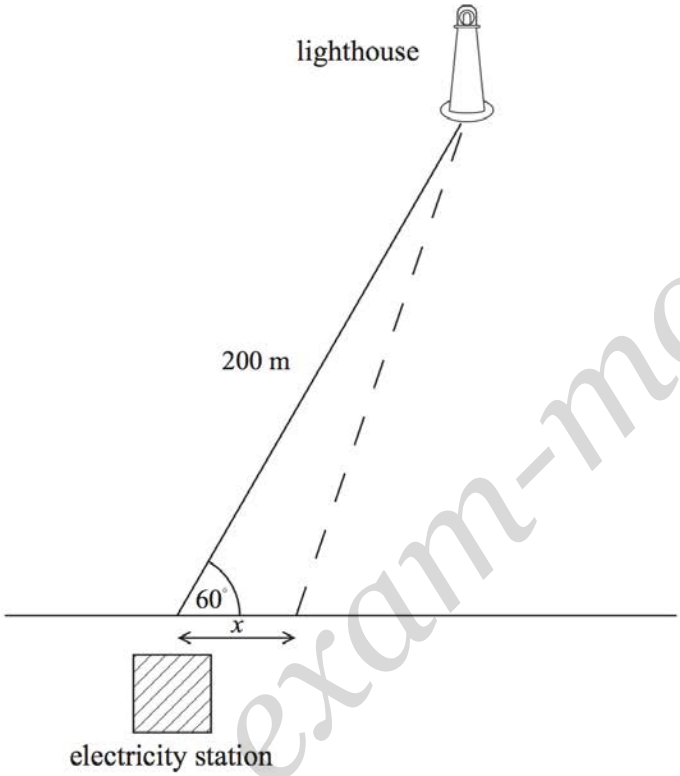
4 – (MAT/21_HL_Summer_2013_Q7) – *Trigonometry*

[Maximum mark: 6]

An electricity station is on the edge of a straight coastline. A lighthouse is located in the sea 200 m from the electricity station. The angle between the coastline and the line joining the lighthouse with the electricity station is 60°. A cable needs to be laid connecting the lighthouse to the electricity station. It is decided to lay the cable in a straight line to the coast and then along the coast to the electricity station. The length of cable laid along the coastline is x metres. This information is illustrated in the diagram below.



The cost of laying the cable along the sea bed is US$80 per metre, and the cost of laying it on land is US$20 per metre.

(a) Find, in terms of x, an expression for the cost of laying the cable. *[4 marks]*

(b) Find the value of x, to the nearest metre, such that this cost is minimized. *[2 marks]*

5 – (MAT/20_HL_Winter_2013_Q7) – *Vectors - Lines - Planes, Trigonometry*

The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are such that $\boldsymbol{a} = (3\cos\theta + 6)\boldsymbol{i} + 7\boldsymbol{j}$ and $\boldsymbol{b} = (\cos\theta - 2)\boldsymbol{i} + (1 + \sin\theta)\boldsymbol{j}$.

Given that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular,

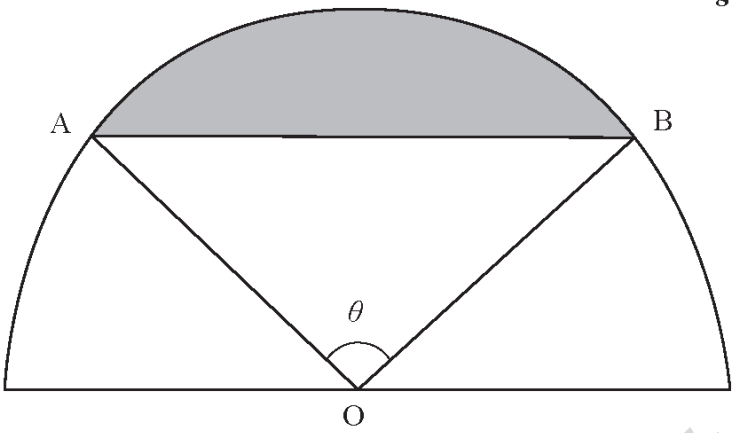
(a) show that $3\sin^2\theta - 7\sin\theta + 2 = 0$; *[3]*

(b) find the smallest possible positive value of θ. *[3]*

6 – (MAT/20_HL_Winter_2013_Q8) – *Trigonometry*

The diagram below shows a semi-circle of diameter 20 cm, centre O and two points A and B such that $A\hat{O}B = \theta$, where θ is in radians.

diagram not to scale

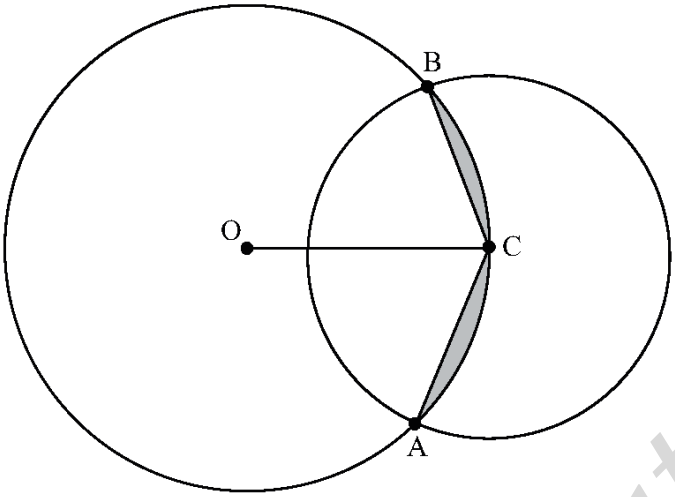


(a) Show that the shaded area can be expressed as $50\theta - 50\sin\theta$. *[2]*

(b) Find the value of θ for which the shaded area is equal to half that of the unshaded area, giving your answer correct to four significant figures. *[3]*

7 – (MAT/22_HL_Summer_2014_Q4) – *Trigonometry, Radian*

The following diagram shows two intersecting circles of radii 4 cm and 3 cm. The centre C of the smaller circle lies on the circumference of the bigger circle. O is the centre of the bigger circle and the two circles intersect at points A and B.



Find:

(a) $B\hat{O}C$; *[2]*

(b) the area of the shaded region. *[4]*

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8 – (MAT/20_HL_Winter_2014_Q9) – *Trigonometry*

Compactness is a measure of how compact an enclosed region is.

The compactness, C, of an enclosed region can be defined by $C = \frac{4A}{\pi d^2}$, where A is the area of the region and d is the maximum distance between any two points in the region.

For a circular region, $C = 1$.

Consider a regular polygon of n sides constructed such that its vertices lie on the circumference of a circle of diameter x units.

(a) If $n > 2$ and even, show that $C = \frac{n}{2\pi}\sin\frac{2\pi}{n}$. *[3]*

If $n > 1$ and odd, it can be shown that $C = \frac{n\sin\frac{2\pi}{n}}{\pi\left(1+\cos\frac{\pi}{n}\right)}$.

(b) Find the regular polygon with the least number of sides for which the compactness is more than 0.99. *[4]*

(c) Comment briefly on whether C is a good measure of compactness. *[1]*

9 – (MAT/20_HL_Winter_2014_Q14) – *Trigonometry*

In triangle ABC,

$$3\sin B + 4\cos C = 6 \text{ and}$$
$$4\sin C + 3\cos B = 1.$$

(a) Show that $\sin(B+C) = \frac{1}{2}$. *[6]*

Robert conjectures that $\mathrm{C\hat{A}B}$ can have two possible values.

(b) Show that Robert's conjecture is incorrect by proving that $\mathrm{C\hat{A}B}$ has only one possible value. *[5]*

10 – (MAT/22_HL_Summer_2015_Q1) – *Trigonometry*

In triangle ABC, $\mathrm{AB} = 5\,\mathrm{cm}$, $\mathrm{BC} = 12\,\mathrm{cm}$ and $\mathrm{A\hat{B}C} = 100^\circ$.

(a) Find the area of the triangle. [2]

(b) Find AC. [2]

11 – (MAT/21_HL_Summer_2015_Q4) – *Trigonometry*

[Maximum mark: 6]

A triangle ABC has $\hat{A} = 50°$, $\mathrm{AB} = 7\,\mathrm{cm}$ and $\mathrm{BC} = 6\,\mathrm{cm}$. Find the area of the triangle given that it is smaller than $10\,\mathrm{cm}^2$.

12 – (MAT/22_HL_Summer_2015_Q8) – *Trigonometry*

Farmer Bill owns a rectangular field, $10\,\text{m}$ by $4\,\text{m}$. Bill attaches a rope to a wooden post at one corner of his field, and attaches the other end to his goat Gruff.

(a) Given that the rope is $5\,\text{m}$ long, calculate the percentage of Bill's field that Gruff is able to graze. Give your answer correct to the nearest integer. [4]

(b) Bill replaces Gruff's rope with another, this time of length a, $4 < a < 10$, so that Gruff can now graze exactly one half of Bill's field.

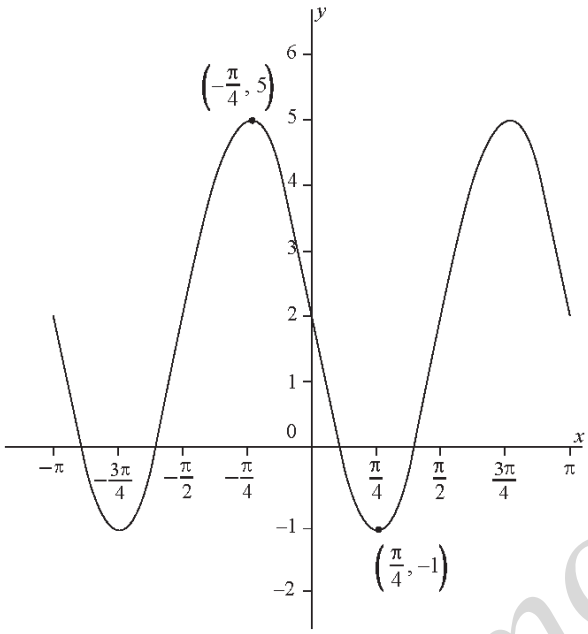
Show that a satisfies the equation

$$a^2 \arcsin\left(\frac{4}{a}\right) + 4\sqrt{a^2 - 16} = 40\,.$$ [4]

(c) Find the value of a. [2]

13 – (MAT/20_HL_Winter_2015_Q4) – *Trigonometry*

A function is defined by $f(x) = A\sin(Bx) + C$, $-\pi \le x \le \pi$, where $A, B, C \in \mathbb{Z}$. The following diagram represents the graph of $y = f(x)$.



(a) Find the value of

(i) A;

(ii) B;

(iii) C. [4]

(b) Solve $f(x) = 3$ for $0 \le x \le \pi$. [2]

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14 – (MAT/20_HL_Winter_2015_Q7) – *Trigonometry*

[Maximum mark: 6]

Triangle ABC has area $21\,\mathrm{cm}^2$. The sides AB and AC have lengths $6\,\mathrm{cm}$ and $11\,\mathrm{cm}$ respectively. Find the two possible lengths of the side BC.

15 – (MAT/22_HL_Summer_2016_Q1) – *Trigonometry*

[Maximum mark: 5]

$ABCD$ is a quadrilateral where $AB = 6.5$, $BC = 9.1$, $CD = 10.4$, $DA = 7.8$ and $C\hat{D}A = 90°$.
Find $A\hat{B}C$, giving your answer correct to the nearest degree.

16 – (MAT/20_HL_Winter_2016_Q7) – *Trigonometry*

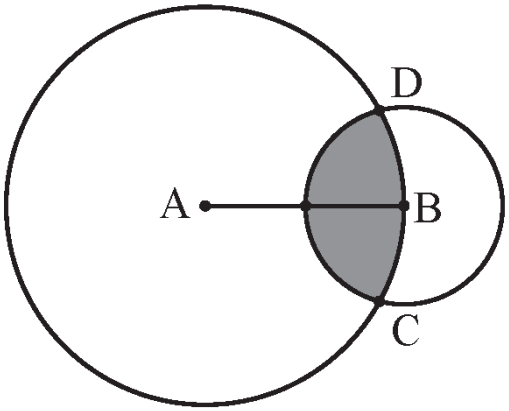
In a triangle ABC, $\mathrm{AB} = 4\,\mathrm{cm}$, $\mathrm{BC} = 3\,\mathrm{cm}$ and $\mathrm{B\hat{A}C} = \frac{\pi}{9}$.

(a) Use the cosine rule to find the two possible values for AC. [5]

(b) Find the difference between the areas of the two possible triangles ABC. [3]

17 – (MAT/20_HL_Winter_2016_Q9) – *Radian, Trigonometry*

The diagram shows two circles with centres at the points A and B and radii $2r$ and r, respectively. The point B lies on the circle with centre A. The circles intersect at the points C and D.



Let α be the measure of the angle CAD and θ be the measure of the angle CBD in radians.

(a) Find an expression for the shaded area in terms of α, θ and r. [3]

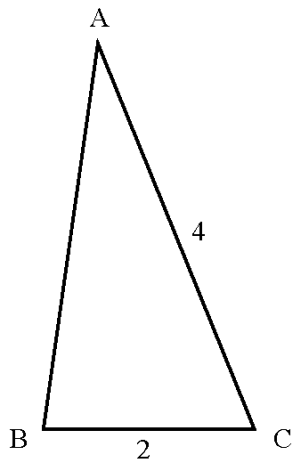
(b) Show that $\alpha = 4\arcsin\frac{1}{4}$. [2]

(c) Hence find the value of r given that the shaded area is equal to 4. [3]

18 – (MAT/22_HL_Summer_2017_Q4) – *Quadratics, Trigonometry*

(a) Find the set of values of k that satisfy the inequality $k^2 - k - 12 < 0$. [2]

(b) The triangle ABC is shown in the following diagram. Given that $\cos B < \frac{1}{4}$, find the range of possible values for AB. [4]



19 – (MAT/21_HL_Summer_2017_Q10) – *Trigonometry*

In triangle PQR, $\mathrm{PR} = 12\,\mathrm{cm}$, $\mathrm{QR} = p\,\mathrm{cm}$, $\mathrm{PQ} = r\,\mathrm{cm}$ and $\mathrm{Q\hat{P}R} = 30°$.

(a) Use the cosine rule to show that $r^2 - 12\sqrt{3}\,r + 144 - p^2 = 0$. [2]

Consider the possible triangles with $\mathrm{QR} = 8\,\mathrm{cm}$.

(b) Calculate the two corresponding values of PQ. [3]

(c) Hence, find the area of the smaller triangle. [3]

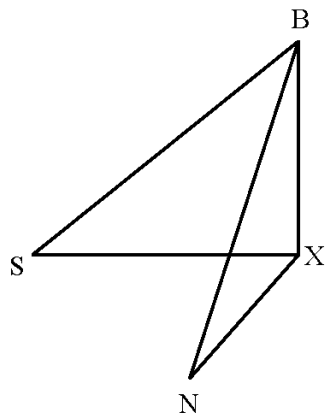
Consider the case where p, the length of QR is not fixed at $8\,\mathrm{cm}$.

(d) Determine the range of values of p for which it is possible to form two triangles. [7]

20 – (MAT/20_HL_Winter_2017_Q5) – *Trigonometry*

[Maximum mark: 6]

Barry is at the top of a cliff, standing $80\,\mathrm{m}$ above sea level, and observes two yachts in the sea.
"*Seaview*" (S) is at an angle of depression of 25°.
"*Nauti Buoy*" (N) is at an angle of depression of 35°.
The following three dimensional diagram shows Barry and the two yachts at S and N.
X lies at the foot of the cliff and angle $\mathrm{SXN} = 70^\circ$.



Find, to 3 significant figures, the distance between the two yachts.

21 – (MAT/20_HL_Winter_2017_Q11) – *Trigonometry, Differentiation*

[Maximum mark: 18]

Consider the function $f(x) = 2\sin^2 x + 7\sin 2x + \tan x - 9,\ 0 \le x < \frac{\pi}{2}$.

(a) (i) Determine an expression for $f'(x)$ in terms of x.

(ii) Sketch a graph of $y = f'(x)$ for $0 \le x < \frac{\pi}{2}$.

(iii) Find the x-coordinate(s) of the point(s) of inflexion of the graph of $y = f(x)$, labelling these clearly on the graph of $y = f'(x)$. [8]

(b) Let $u = \tan x$.

(i) Express $\sin x$ in terms of u.

(ii) Express $\sin 2x$ in terms of u.

(iii) Hence show that $f(x) = 0$ can be expressed as $u^3 - 7u^2 + 15u - 9 = 0$. [7]

(c) Solve the equation $f(x) = 0$, giving your answers in the form $\arctan k$ where $k \in \mathbb{Z}$. [3]

22 – (MAT/21_HL_Summer_2018_Q3) – *Trigonometry*

[Maximum mark: 5]

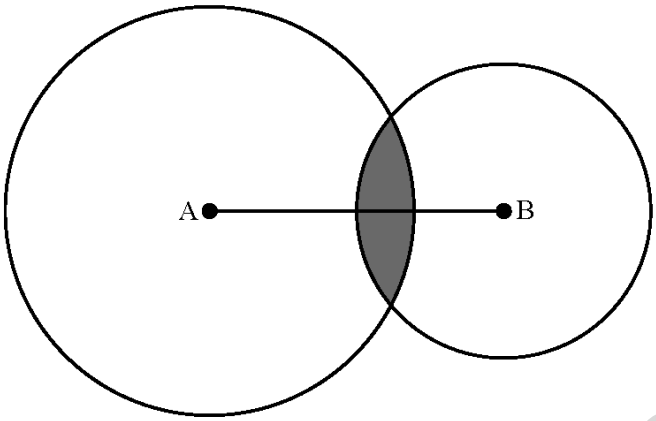
Let $f(x) = \tan(x + \pi)\cos\left(x - \frac{\pi}{2}\right)$ where $0 < x < \frac{\pi}{2}$.

Express $f(x)$ in terms of $\sin x$ and $\cos x$.

23 – (MAT/20_HL_Winter_2018_Q7) – *Trigonometry*

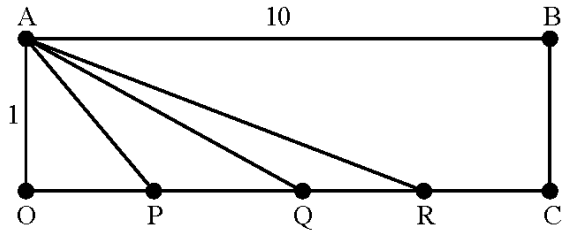
[Maximum mark: 6]

Boat A is situated 10km away from boat B, and each boat has a marine radio transmitter on board. The range of the transmitter on boat A is 7km, and the range of the transmitter on boat B is 5km. The region in which both transmitters can be detected is represented by the shaded region in the following diagram. Find the area of this region.



24 – (MAT/20_HL_Winter_2018_Q11) – *Trigonometry*

Consider the rectangle OABC such that $\mathrm{AB} = \mathrm{OC} = 10$ and $\mathrm{BC} = \mathrm{OA} = 1$, with the points P, Q and R placed on the line OC such that $\mathrm{OP} = p$, $\mathrm{OQ} = q$ and $\mathrm{OR} = r$, such that $0 < p < q < r < 10$.



Let θ_p be the angle APO, θ_q be the angle AQO and θ_r be the angle ARO.

(a) Find an expression for θ_p in terms of p. [3]

Consider the case when $\theta_p = \theta_q + \theta_r$ and $\mathrm{QR} = 1$.

(b) Show that $p = \dfrac{q^2 + q - 1}{2q + 1}$. [6]

(c) By sketching the graph of p as a function of q, determine the range of values of p for which there are possible values of q. [4]

25 - (MAT/22_HL_Summer_2019_Q1) - *Trigonometry*

[Maximum mark: 5]

In triangle ABC, $AB = 5$, $BC = 14$ and $AC = 11$.
Find all the interior angles of the triangle. Give your answers in degrees to one decimal place.

26 – (MAT/21_HL_Summer_2019_Q4) – *Trigonometry*

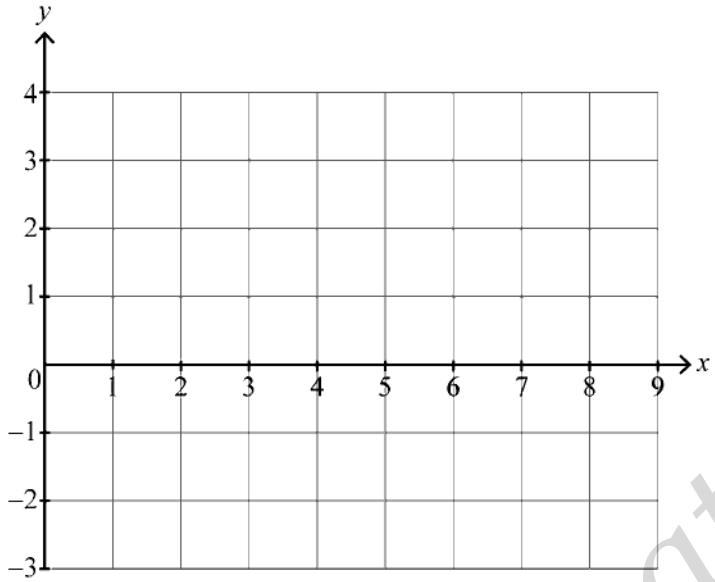
The function f is defined by $f(x) = \sec x + 2,\ 0 \le x < \frac{\pi}{2}$.

(a) Write down the range of f. [1]

(b) Find $f^{-1}(x)$, stating its domain. [4]

27 – (MAT/22_HL_Summer_2019_Q4) – *Trigonometry*

(a) Sketch the graphs of $y = \sin^3 x + \ln x$ and $y = 1 + \cos x$ on the following axes for $0 < x \leq 9$. [2]



(b) Hence solve $\sin^3 x + \ln x - \cos x - 1 < 0$ in the range $0 < x \leq 9$. [4]

28 – (MAT/22_HL_Summer_2019_Q5) – *Trigonometry*

(a) Prove the identity $\frac{1+\sin 2x}{\cos 2x} \equiv \frac{1+\tan x}{1-\tan x}$. [4]

(b) Solve the equation $\frac{1+\sin 2x}{\cos 2x} = \sqrt{3}$ for $0 \le x < 2\pi$. [2]

29 – (MAT/21_HL_Summer_2019_Q10) – *Trigonometry*

The voltage v in a circuit is given by the equation

$$v(t) = 3\sin(100\pi t),\ t \geq 0 \text{ where } t \text{ is measured in seconds.}$$

(a) Write down the maximum and minimum value of v. [2]

The current i in this circuit is given by the equation

$$i(t) = 2\sin\left(100\pi(t + 0.003)\right).$$

(b) Write down two transformations that will transform the graph of $y = v(t)$ onto the graph of $y = i(t)$. [2]

The power p in this circuit is given by $p(t) = v(t) \times i(t)$.

(c) Sketch the graph of $y = p(t)$ for $0 \leq t \leq 0.02$, showing clearly the coordinates of the first maximum and the first minimum. [3]

(d) Find the total time in the interval $0 \leq t \leq 0.02$ for which $p(t) \geq 3$. [3]

The average power p_{av} in this circuit from $t = 0$ to $t = T$ is given by the equation

$$p_{av}(T) = \frac{1}{T}\int_0^T p(t)\mathrm{d}t \text{, where } T > 0.$$

(e) Find $p_{av}(0.007)$. [2]

(f) With reference to your graph of $y = p(t)$ explain why $p_{av}(T) > 0$ for all $T > 0$. [2]

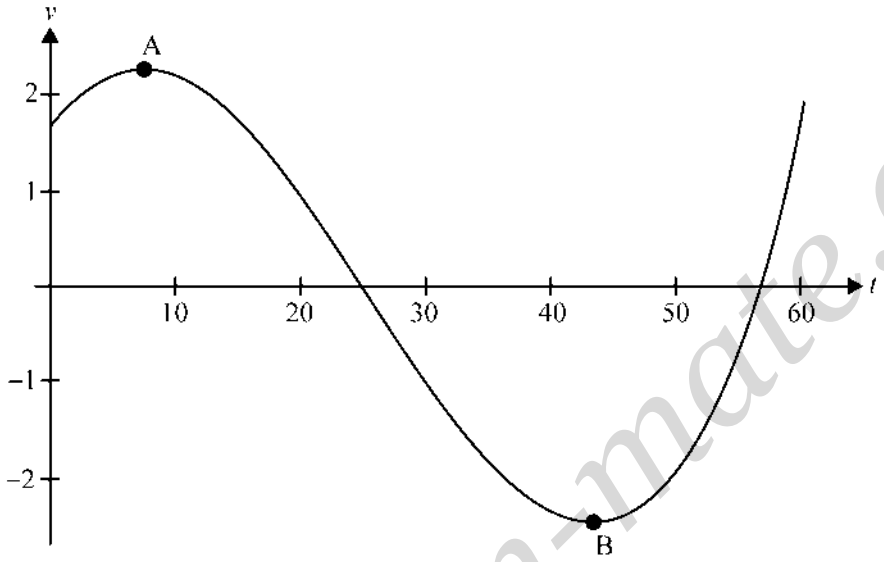
(g) Given that $p(t)$ can be written as $p(t) = a\sin(b(t - c)) + d$ where $a, b, c, d > 0$, use your graph to find the values of a, b, c and d. [6]

30 – (MAT/20_HL_Winter_2019_Q9) – *Trigonometry*

[Maximum mark: 14]

A body moves in a straight line such that its velocity, v m s^{-1}, after t seconds is given by $v = 2\sin\left(\frac{t}{10} + \frac{\pi}{5}\right)\csc\left(\frac{t}{30} + \frac{\pi}{4}\right)$ for $0 \le t \le 60$.

The following diagram shows the graph of v against t. Point A is a local maximum and point B is a local minimum.



(a) (i) Determine the coordinates of point A and the coordinates of point B.

(ii) Hence, write down the maximum speed of the body. [5]

(b) The body first comes to rest at time $t = t_1$. Find

(i) the value of t_1;

(ii) the distance travelled between $t = 0$ and $t = t_1$;

(iii) the acceleration when $t = t_1$. [6]

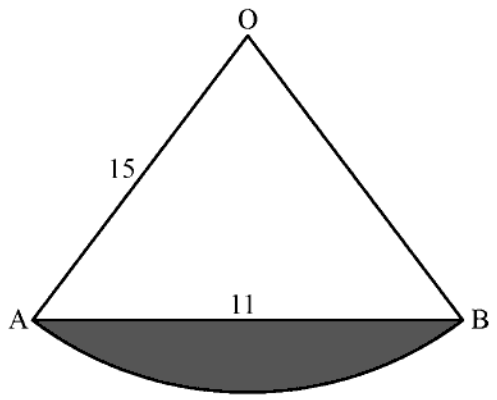
(c) Find the distance travelled in the first 30 seconds. [3]

31 – (MAT/20_HL_Winter_2020_Q1) – *Radian, Trigonometry*

[Maximum mark: 5]

The following diagram shows a sector OAB of radius $15\,\text{cm}$. The length of [AB] is $11\,\text{cm}$.

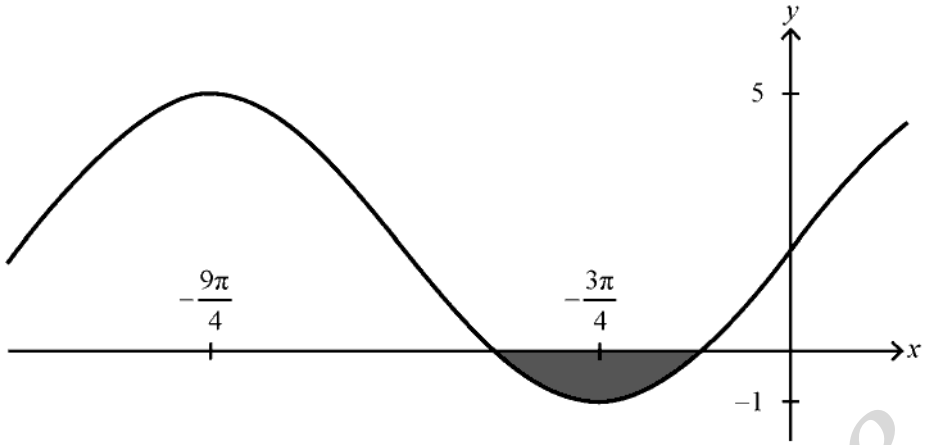
diagram not to scale



Find the area of the shaded region.

32 – (MAT/20_HL_Winter_2020_Q3) – *Trigonometry, Integration*

The following diagram shows part of the graph of $y = p + q\sin(rx)$. The graph has a local maximum point at $\left(-\frac{9\pi}{4}, 5\right)$ and a local minimum point at $\left(-\frac{3\pi}{4}, -1\right)$.



(a) Determine the values of p, q and r. [4]

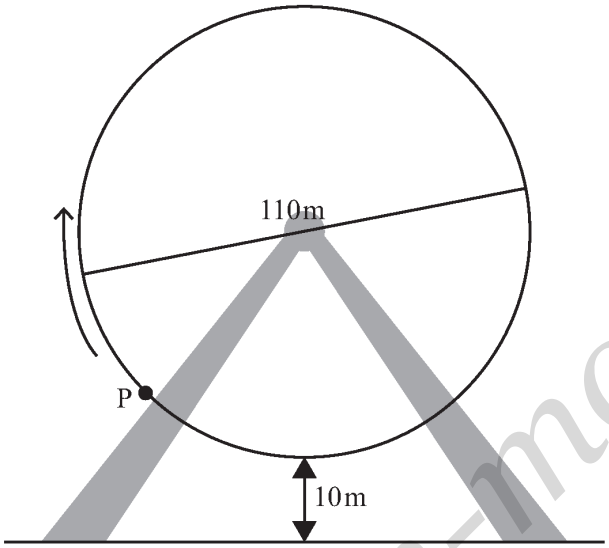
(b) Hence find the area of the shaded region. [4]

33 – (MathAA/21_HL_Summer_2021_Q3) – *Trigonometry*

[Maximum mark: 5]

A Ferris wheel with diameter 110 metres rotates at a constant speed. The lowest point on the wheel is 10 metres above the ground, as shown on the following diagram. P is a point on the wheel. The wheel starts moving with P at the lowest point and completes one revolution in 20 minutes.

diagram not to scale



The height, h metres, of P above the ground after t minutes is given by $h(t) = a\cos(bt) + c$, where a, b, $c \in \mathbb{R}$.

Find the values of a, b and c.

34 - (MathAA/20_HL_Winter_2021_Q2) - *Trigonometry*

[Maximum mark: 5]

Consider a triangle ABC, where $AC = 12$, $CB = 7$ and $B\hat{A}C = 25°$.

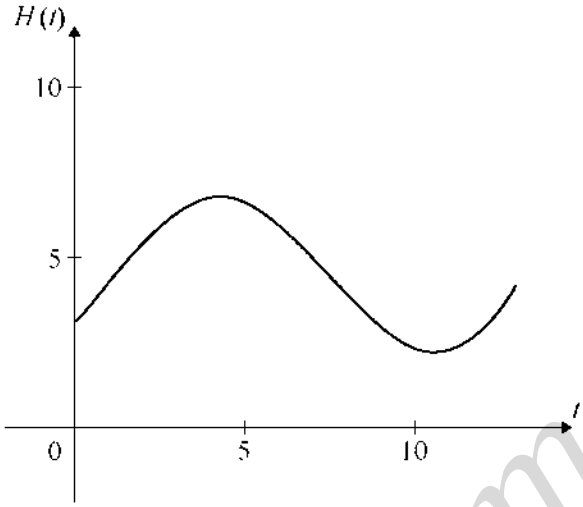
Find the smallest possible perimeter of triangle ABC.

35 – (MathAA/20_HL_Winter_2021_Q9) – *Trigonometry*

[Maximum mark: 15]

The height of water, in metres, in Dungeness harbour is modelled by the function $H(t) = a\sin(b(t-c)) + d$, where t is the number of hours after midnight, and a, b, c and d are constants, where $a > 0$, $b > 0$ and $c > 0$.

The following graph shows the height of the water for 13 hours, starting at midnight.



The first high tide occurs at 04:30 and the next high tide occurs 12 hours later. Throughout the day, the height of the water fluctuates between $2.2\,\text{m}$ and $6.8\,\text{m}$.

All heights are given correct to one decimal place.

(a) Show that $b = \frac{\pi}{6}$. [1]

(b) Find the value of a. [2]

(c) Find the value of d. [2]

(d) Find the smallest possible value of c. [3]

(e) Find the height of the water at 12:00. [2]

(f) Determine the number of hours, over a 24-hour period, for which the tide is higher than 5 metres. [3]

A fisherman notes that the water height at nearby Folkestone harbour follows the same sinusoidal pattern as that of Dungeness harbour, with the exception that high tides (and low tides) occur 50 minutes earlier than at Dungeness.

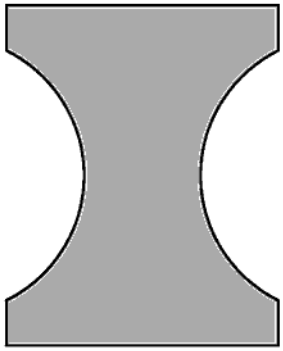
(g) Find a suitable equation that may be used to model the tidal height of water at Folkestone harbour. [2]

36 – (MathAA/21_HL_Summer_2022_Q2) – *Trigonometry*

[Maximum mark: 6]

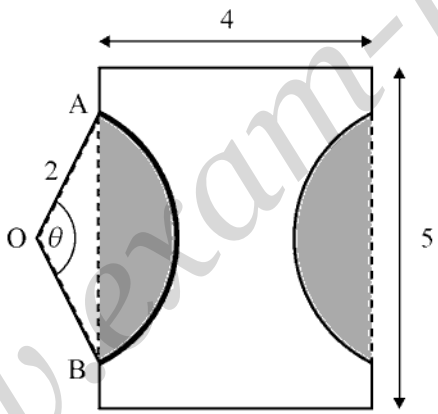
A company is designing a new logo. The logo is created by removing two equal segments from a rectangle, as shown in the following diagram.

diagram not to scale



The rectangle measures $5\,\text{cm}$ by $4\,\text{cm}$. The points A and B lie on a circle, with centre O and radius $2\,\text{cm}$, such that $\text{A}\hat{\text{O}}\text{B} = \theta$, where $0 < \theta < \pi$. This information is shown in the following diagram.

diagram not to scale



(a) Find the area of one of the shaded segments in terms of θ. [3]

(b) Given that the area of the logo is $13.4\,\text{cm}^2$, find the value of θ. [3]

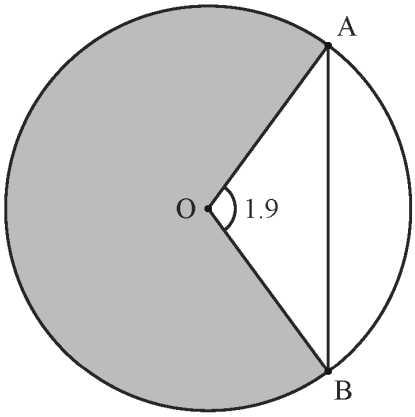
37 – (MathAA/22_HL_Summer_2022_Q1) – *Trigonometry*

[Maximum mark: 6]

The following diagram shows a circle with centre O and radius 5 metres.

Points A and B lie on the circle and $\mathrm{A\hat{O}B} = 1.9$ radians.

diagram not to scale



(a) Find the length of the chord $[\mathrm{AB}]$. [3]

(b) Find the area of the shaded sector. [3]

38 – (MathAA/22_HL_Summer_2022_Q10) – *Trigonometry, Differentiation*

[Maximum mark: 15]

A scientist conducted a nine-week experiment on two plants, A and B, of the same species. He wanted to determine the effect of using a new plant fertilizer. Plant A was given fertilizer regularly, while Plant B was not.

The scientist found that the height of Plant A, h_A cm, at time t weeks can be modelled by the function $h_A(t) = \sin(2t + 6) + 9t + 27$, where $0 \le t \le 9$.

The scientist found that the height of Plant B, h_B cm, at time t weeks can be modelled by the function $h_B(t) = 8t + 32$, where $0 < t < 9$.

(a) Use the scientist's models to find the initial height of

(i) Plant B;

(ii) Plant A correct to three significant figures. [3]

(b) Find the values of t when $h_A(t) = h_B(t)$. [3]

(c) For $t > 6$, prove that Plant A was always taller than Plant B. [3]

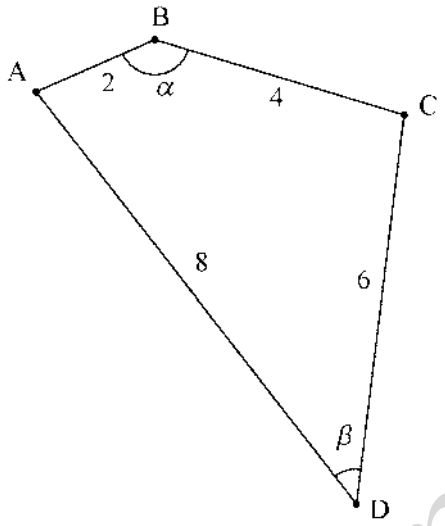
(d) For $0 \le t \le 9$, find the total amount of time when the rate of growth of Plant B was greater than the rate of growth of Plant A. [6]

39 – (MathAA/20_HL_Winter_2022_Q9) – *Trigonometry*

[Maximum mark: 8]

Consider a quadrilateral $ABCD$ such that $AB = 2$, $BC = 4$, $CD = 6$ and $DA = 8$, as shown in the following diagram. Let $\alpha = A\hat{B}C$ and $\beta = A\hat{D}C$.

diagram not to scale



(a) (i) Find AC in terms of α.

(ii) Find AC in terms of β.

(iii) Hence or otherwise, find an expression for α in terms of β. [4]

(b) Find the maximum area of the quadrilateral $ABCD$. [4]

1 – (MAT/20_HL_Winter_2012_Q3) – *Matrices*

[Maximum mark: 5]

Consider the matrix $A = \begin{pmatrix} \ln x & \ln(5-x) \\ 2 & 3 \end{pmatrix}$, where $0 < x < 5$. Find the value of x for which A is singular.

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2 – (MAT/21_HL_Summer_2013_Q2) – *Matrices*

[Maximum mark: 5]

Find the value of k such that the following system of equations does not have a unique solution.

$$\begin{aligned} kx + y + 2z &= 4 \\ -y + 4z &= 5 \\ 3x + 4y + 2z &= 1 \end{aligned}$$

3 – (MAT/22_HL_Summer_2013_Q2) – *Matrices*

Consider the system of equations

$$\begin{aligned} 0.1x - 1.7y + 0.9z &= -4.4 \\ -2.4x + 0.3y + 3.2z &= 1.2 \\ 2.5x + 0.6y - 3.7z &= 0.8\,. \end{aligned}$$

(a) Express the system of equations in matrix form. *[2 marks]*

(b) Find the solution to the system of equations. *[3 marks]*

4 – (MAT/20_HL_Winter_2013_Q1) – *Matrices*

[Maximum mark: 4]

Consider the matrices $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$. Find the matrix X such that $\boldsymbol{AX} = \boldsymbol{B}$.

5 – (MAT/21_HL_Summer_2014_Q4) – *Matrices*

A system of equations is given below.

$$x + 2y - z = 2$$
$$2x + y + z = 1$$
$$-x + 4y + az = 4$$

(a) Find the value of a so that the system does not have a unique solution. *[4]*

(b) Show that the system has a solution for any value of a. *[2]*

6 – (MAT/22_HL_Summer_2015_Q7) – *Matrices*

Consider the following system of equations

$$\begin{aligned} 2x + y + 6z &= 0 \\ 4x + 3y + 14z &= 4 \\ 2x - 2y + (\alpha - 2)z &= \beta - 12 \ . \end{aligned}$$

(a) Find conditions on α and β for which

(i) the system has no solutions;

(ii) the system has only one solution;

(iii) the system has an infinite number of solutions. [6]

(b) In the case where the number of solutions is infinite, find the general solution of the system of equations in Cartesian form. [3]

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1 – (MAT/21_HL_Summer_2012_Q4) – *Vectors - Lines - Planes*

[Maximum mark: 5]

The planes $2x+3y-z=5$ and $x-y+2z=k$ intersect in the line $5x+1=9-5y=-5z$.
Find the value of k.

2 – (MAT/22_HL_Summer_2012_Q11) – *Vectors - Lines - Planes*

(a) Find the values of k for which the following system of equations has no solutions and the value of k for the system to have an infinite number of solutions.

$$x - 3y + z = 3$$
$$x + 5y - 2z = 1$$
$$16y - 6z = k$$

[5 marks]

(b) Given that the system of equations can be solved, find the solutions in the form of a vector equation of a line, $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$, where the components of $\boldsymbol{b}$ are integers. *[7 marks]*

(c) The plane ÷ is parallel to both the line in part (b) and the line $\frac{x-4}{3} = \frac{y-6}{-2} = \frac{z-2}{0}$.

Given that ÷ contains the point $(1, 2, 0)$, show that the Cartesian equation of ÷ is $16x + 24y - 11z = 64$. *[5 marks]*

(d) The z-axis meets the plane ÷ at the point P. Find the coordinates of P. *[2 marks]*

(e) Find the angle between the line $\frac{x-2}{3} = \frac{y+5}{4} = \frac{z}{2}$ and the plane ÷ . *[5 marks]*

3 – (MAT/21_HL_Summer_2012_Q13) – *Vectors - Lines - Planes*

The coordinates of points A, B and C are given as $(5, -2, 5)$, $(5, 4, -1)$ and $(-1, -2, -1)$ respectively.

(a) Show that $AB = AC$ and that $B\hat{A}C = 60^\circ$. *[4 marks]*

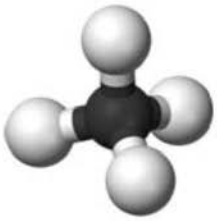
(b) Find the Cartesian equation of Π, the plane passing through A, B, and C. *[4 marks]*

(c) (i) Find the Cartesian equation of Π_1, the plane perpendicular to (AB) passing through the midpoint of [AB] .

(ii) Find the Cartesian equation of Π_2, the plane perpendicular to (AC) passing through the midpoint of [AC]. *[4 marks]*

(d) Find the vector equation of L, the line of intersection of Π_1 and Π_2, and show that it is perpendicular to Π. *[3 marks]*

A methane molecule consists of a carbon atom with four hydrogen atoms symmetrically placed around it in three dimensions.



The positions of the centres of three of the hydrogen atoms are A, B and C as given. The position of the centre of the fourth hydrogen atom is D.

(e) Using the fact that $AB = AD$, show that the coordinates of one of the possible positions of the fourth hydrogen atom is $(-1, 4, 5)$. *[3 marks]*

(f) Letting D be $(-1, 4, 5)$, show that the coordinates of G, the position of the centre of the carbon atom, are $(2, 1, 2)$. Hence calculate $D\hat{G}A$, the bonding angle of carbon. *[6 marks]*

4 – (MAT/20_HL_Winter_2012_Q13) – *Vectors - Lines - Planes*

Consider the planes $\pi_1 : x - 2y - 3z = 2$ and $\pi_2 : 2x - y - z = k$.

(a) Find the angle between the planes π_1 and π_2. *[4 marks]*

(b) The planes π_1 and π_2 intersect in the line L_1. Show that the vector equation of L_1 is $\boldsymbol{r} = \begin{pmatrix} 0 \\ 2-3k \\ 2k-2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$. *[5 marks]*

(c) The line L_2 has Cartesian equation $5 - x = y + 3 = 2 - 2z$. The lines L_1 and L_2 intersect at a point X. Find the coordinates of X. *[5 marks]*

(d) Determine a Cartesian equation of the plane π_3 containing both lines L_1 and L_2. *[5 marks]*

(e) Let Y be a point on L_1 and Z be a point on L_2 such that XY is perpendicular to YZ and the area of the triangle XYZ is 3. Find the perimeter of the triangle XYZ. *[5 marks]*

5 – (MAT/21_HL_Summer_2013_Q11) – *Vectors - Lines - Planes*

Consider the points $\mathrm{P}(-3, -1, 2)$ and $\mathrm{Q}(5, 5, 6)$.

(a) Find a vector equation for the line, L_1, which passes through the points P and Q. *[3 marks]*

The line L_2 has equation

$$\boldsymbol{r} = \begin{pmatrix} -4 \\ 0 \\ 4 \end{pmatrix} + s\begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}.$$

(b) Show that L_1 and L_2 intersect at the point $\mathrm{R}(1, 2, 4)$. *[4 marks]*

(c) Find the acute angle between L_1 and L_2. *[3 marks]*

Let S be a point on L_2 such that $|\overrightarrow{\mathrm{RP}}| = |\overrightarrow{\mathrm{RS}}|$.

(d) Show that one of the possible positions for S is $\mathrm{S}_1(-4, 0, 4)$ and find the coordinates of the other possible position, S_2. *[6 marks]*

(e) Find a vector equation of the line which passes through R and bisects $\overline{\mathrm{PRS}_1}$. *[4 marks]*

6 – (MAT/20_HL_Winter_2013_Q7) – *Vectors - Lines - Planes, Trigonometry*

The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are such that $\boldsymbol{a} = (3\cos\theta + 6)\boldsymbol{i} + 7\boldsymbol{j}$ and $\boldsymbol{b} = (\cos\theta - 2)\boldsymbol{i} + (1 + \sin\theta)\boldsymbol{j}$.

Given that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular,

(a) show that $3\sin^2\theta - 7\sin\theta + 2 = 0$; *[3]*

(b) find the smallest possible positive value of θ. *[3]*

7 – (MAT/20_HL_Winter_2013_Q9) – *Vectors - Lines - Planes*

A line L_1 has equation $\boldsymbol{r} = \begin{pmatrix} -5 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$.

A line L_2 passing through the origin intersects L_1 and is perpendicular to L_1.

(a) Find a vector equation of L_2. *[5]*

(b) Determine the shortest distance from the origin to L_1. *[2]*

8 – (MAT/20_HL_Winter_2014_Q1) – *Vectors - Lines - Planes*

[Maximum mark: 6]

Consider the two planes

$$\pi_1 : 4x + 2y - z = 8$$
$$\pi_2 : x + 3y + 3z = 3 .$$

Find the angle between π_1 and π_2, giving your answer correct to the nearest degree.

9 – (MAT/20_HL_Winter_2014_Q5) – *Vectors - Lines - Planes*

The lines l_1 and l_2 are defined as

$$l_1 : \frac{x-1}{3} = \frac{y-5}{2} = \frac{z-12}{-2}$$

$$l_2 : \frac{x-1}{8} = \frac{y-5}{11} = \frac{z-12}{6}.$$

The plane π contains both l_1 and l_2.

(a) Find the Cartesian equation of π. *[4]*

The line l_3 passing through the point $(4, 0, 8)$ is perpendicular to π.

(b) Find the coordinates of the point where l_3 meets π. *[4]*

10 – (MAT/22_HL_Summer_2015_Q5) – *Vectors - Lines - Planes*

Consider the vectors given by $\boldsymbol{u} = \boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j}$, where a and b are constants.

It is given that $\boldsymbol{u} \times \boldsymbol{v} = 4\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$, where c is a constant.

(a) Find the value of each of the constants a, b and c. [5]

(b) Hence find the Cartesian equation of the plane containing the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ and passing through the point $(0, 0, 0)$. [2]

11 – (MAT/21_HL_Summer_2015_Q8) – *Vectors - Lines - Planes*

Let $\boldsymbol{v} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} 4 \\ \lambda \\ 10 \end{pmatrix}$.

(a) Find the value of λ for $\boldsymbol{v}$ and $\boldsymbol{w}$ to be parallel. [2]

(b) Find the value of λ for $\boldsymbol{v}$ and $\boldsymbol{w}$ to be perpendicular. [2]

(c) Find the two values of λ if the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ is 10°. [4]

12 – (MAT/22_HL_Summer_2015_Q13) – *Vectors - Lines - Planes*

The equations of the lines L_1 and L_2 are

$$L_1 : \boldsymbol{r}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$L_2 : \boldsymbol{r}_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}.$$

(a) Show that the lines L_1 and L_2 are skew. [4]

(b) Find the acute angle between the lines L_1 and L_2. [4]

(c) (i) Find a vector perpendicular to both lines.

(ii) Hence determine an equation of the line L_3 that is perpendicular to both L_1 and L_2 and intersects both lines. [10]

13 – (MAT/20_HL_Winter_2015_Q2) – *Vectors - Lines - Planes*

[Maximum mark: 4]

The three planes having Cartesian equations $2x + 3y - z = 11$, $x + 2y + z = 3$ and $5x - y - z = 10$ meet at a point P. Find the coordinates of P.

14 – (MAT/20_HL_Winter_2015_Q10) – *Vectors - Lines - Planes*

Ed walks in a straight line from point $\mathrm{P}(-1, 4)$ to point $\mathrm{Q}(4, 16)$ with constant speed. Ed starts from point P at time $t = 0$ and arrives at point Q at time $t = 3$, where t is measured in hours.
Given that, at time t, Ed's position vector, relative to the origin, can be given in the form, $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$,

(a) find the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. [3]

Roderick is at a point $\mathrm{C}(11, 9)$. During Ed's walk from P to Q Roderick wishes to signal to Ed. He decides to signal when Ed is at the closest point to C.

(b) Find the time when Roderick signals to Ed. [5]

15 – (MAT/21_HL_Summer_2016_Q1) – *Vectors - Lines - Planes*

The points A and B have position vectors $\overrightarrow{\mathrm{OA}} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and $\overrightarrow{\mathrm{OB}} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$.

(a) Find $\overrightarrow{\mathrm{OA}} \times \overrightarrow{\mathrm{OB}}$. [2]

(b) Hence find the area of the triangle OAB. [2]

16 – (MAT/22_HL_Summer_2016_Q9) – *Vectors - Lines - Planes*

OACB is a parallelogram with $\overrightarrow{\mathrm{OA}} = \boldsymbol{a}$ and $\overrightarrow{\mathrm{OB}} = \boldsymbol{b}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors.

(a) Show that

(i) $|\overrightarrow{\mathrm{OC}}|^2 = |\boldsymbol{a}|^2 + 2\boldsymbol{a}\cdot\boldsymbol{b} + |\boldsymbol{b}|^2$;

(ii) $|\overrightarrow{\mathrm{AB}}|^2 = |\boldsymbol{a}|^2 - 2\boldsymbol{a}\cdot\boldsymbol{b} + |\boldsymbol{b}|^2$. [4]

(b) Given that $|\overrightarrow{\mathrm{OC}}| = |\overrightarrow{\mathrm{AB}}|$, prove that OACB is a rectangle. [4]

17 – (MAT/20_HL_Winter_2016_Q2) – *Vectors - Lines - Planes*

[Maximum mark: 5]

Find the acute angle between the planes with equations $x + y + z = 3$ and $2x - z = 2$.

18 - (MAT/21_HL_Summer_2017_Q7) - *Vectors - Lines - Planes*

[Maximum mark: 6]

Find the Cartesian equation of plane Π containing the points $A(6, 2, 1)$ and $B(3, -1, 1)$ and perpendicular to the plane $x + 2y - z - 6 = 0$.

19 – (MAT/22_HL_Summer_2017_Q7) – *Vectors - Lines - Planes*

[Maximum mark: 4]

Given that $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c} \neq \boldsymbol{0}$ prove that $\boldsymbol{a} + \boldsymbol{c} = s\boldsymbol{b}$ where s is a scalar.

20 – (MAT/22_HL_Summer_2017_Q9) – *Vectors - Lines - Planes*

The points A, B and C have the following position vectors with respect to an origin O.

$$\overrightarrow{\mathrm{OA}} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$$

$$\overrightarrow{\mathrm{OB}} = 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$$

$$\overrightarrow{\mathrm{OC}} = \boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}$$

(a) Find the vector equation of the line (BC). [3]

(b) Determine whether or not the lines (OA) and (BC) intersect. [6]

(c) Find the Cartesian equation of the plane Π_1, which passes through C and is perpendicular to $\overrightarrow{\mathrm{OA}}$. [3]

(d) Show that the line (BC) lies in the plane Π_1. [2]

The plane Π_2 contains the points O, A and B and the plane Π_3 contains the points O, A and C.

(e) Verify that $2\boldsymbol{j} + \boldsymbol{k}$ is perpendicular to the plane Π_2. [3]

(f) Find a vector perpendicular to the plane Π_3. [1]

(g) Find the acute angle between the planes Π_2 and Π_3. [4]

21 – (MAT/21_HL_Summer_2018_Q11) – *Vectors - Lines - Planes*

Two submarines A and B have their routes planned so that their positions at time t hours, $0 \le t < 20$, would be defined by the position vectors $\boldsymbol{r}_A = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ -0.15 \end{pmatrix}$ and $\boldsymbol{r}_B = \begin{pmatrix} 0 \\ 3.2 \\ -2 \end{pmatrix} + t\begin{pmatrix} -0.5 \\ 1.2 \\ 0.1 \end{pmatrix}$ relative to a fixed point on the surface of the ocean (all lengths are in kilometres).

(a) Show that the two submarines would collide at a point P and write down the coordinates of P. [4]

To avoid the collision submarine B adjusts its velocity so that its position vector is now given by

$$\boldsymbol{r}_B = \begin{pmatrix} 0 \\ 3.2 \\ -2 \end{pmatrix} + t\begin{pmatrix} -0.45 \\ 1.08 \\ 0.09 \end{pmatrix}.$$

(b) (i) Show that submarine B travels in the same direction as originally planned.

(ii) Find the value of t when submarine B passes through P. [3]

(c) (i) Find an expression for the distance between the two submarines in terms of t.

(ii) Find the value of t when the two submarines are closest together.

(iii) Find the distance between the two submarines at this time. [8]

22 – (MAT/22_HL_Summer_2019_Q11) – *Vectors - Lines - Planes*

The plane Π_1 contains the points $\mathrm{P}(1, 6, -7)$, $\mathrm{Q}(0, 1, 1)$ and $\mathrm{R}(2, 0, -4)$.

(a) Find the Cartesian equation of the plane containing P, Q and R. [6]

The Cartesian equation of the plane Π_2 is given by $x - 3y - z = 3$.

(b) Given that Π_1 and Π_2 meet in a line L, verify that the vector equation of L can be given by $\boldsymbol{r} = \begin{pmatrix} \frac{5}{4} \\ 0 \\ -\frac{7}{4} \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{2} \\ 1 \\ -\frac{5}{2} \end{pmatrix}$. [3]

The Cartesian equation of the plane Π_3 is given by $ax + by + cz = 1$.

(c) Given that Π_3 is parallel to the line L, show that $a + 2b - 5c = 0$. [1]

Consider the case that Π_3 contains L.

(d) (i) Show that $5a - 7c = 4$.

(ii) Given that Π_3 is equally inclined to both Π_1 and Π_2, determine two distinct possible Cartesian equations for Π_3. [9]

23 – (MAT/20_HL_Winter_2020_Q10) – *Vectors - Lines - Planes*

The plane Π_1 has equation $3x - y + z = -13$ and the line L has vector equation

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -1 \\ 4 \end{pmatrix}, \lambda \in \mathbb{R}.$$

(a) Given that L meets Π_1 at the point P, find the coordinates of P. [4]

(b) Find the shortest distance from the point $\mathrm{O}(0, 0, 0)$ to Π_1. [4]

The plane Π_2 contains the point O and the line L.

(c) Find the equation of Π_2, giving your answer in the form $\boldsymbol{r}.\boldsymbol{n} = d$. [3]

(d) Determine the acute angle between Π_1 and Π_2. [5]

24 – (MathAA/21_HL_Summer_2021_Q6) – *Vectors - Lines - Planes*

[Maximum mark: 5]

Consider the planes Π_1 and Π_2 with the following equations.

$\Pi_1 : 3x + 2y + z = 6$

$\Pi_2 : x - 2y + z = 4$

(a) Find a Cartesian equation of the plane Π_3 which is perpendicular to Π_1 and Π_2 and passes through the origin $(0, 0, 0)$. [3]

(b) Find the coordinates of the point where Π_1, Π_2 and Π_3 intersect. [2]

25 – (MathAA/20_HL_Winter_2021_Q11) – *Vectors - Lines - Planes*

[Maximum mark: 21]

Three points $A(3, 0, 0)$, $B(0, -2, 0)$ and $C(1, 1, -7)$ lie on the plane Π_1.

(a) (i) Find the vector $\overrightarrow{AB}$ and the vector $\overrightarrow{AC}$.

(ii) Hence find the equation of Π_1, expressing your answer in the form $ax + by + cz = d$, where $a, b, c, d \in \mathbb{Z}$. [7]

Plane Π_2 has equation $3x - y + 2z = 2$.

(b) The line L is the intersection of Π_1 and Π_2. Verify that the vector equation of L can be written as $\boldsymbol{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. [2]

(c) The plane Π_3 is given by $2x - 2z = 3$. The line L and the plane Π_3 intersect at the point P.

(i) Show that at the point P, $\lambda = \frac{3}{4}$.

(ii) Hence find the coordinates of P. [3]

(d) The point $B(0, -2, 0)$ lies on L.

(i) Find the reflection of the point B in the plane Π_3.

(ii) Hence find the vector equation of the line formed when L is reflected in the plane Π_3. [9]

26 – (MathAA/21_HL_Summer_2022_Q7) – *Vectors - Lines - Planes*

[Maximum mark: 9]

Consider the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\boldsymbol{a} = \begin{pmatrix} 12 \\ -5 \end{pmatrix}$ and $|\boldsymbol{b}| = 15$.

(a) Find the possible range of values for $|\boldsymbol{a} + \boldsymbol{b}|$. [2]

Consider the vector $\boldsymbol{p}$ such that $\boldsymbol{p} = \boldsymbol{a} + \boldsymbol{b}$.

(b) Given that $|\boldsymbol{a} + \boldsymbol{b}|$ is a minimum, find $\boldsymbol{p}$. [2]

Consider the vector $\boldsymbol{q}$ such that $\boldsymbol{q} = \begin{pmatrix} x \\ y \end{pmatrix}$, where $x, y \in \mathbb{R}^+$.

(c) Find $\boldsymbol{q}$ such that $|\boldsymbol{q}| = |\boldsymbol{b}|$ and $\boldsymbol{q}$ is perpendicular to $\boldsymbol{a}$. [5]

27 – (MathAA/22_HL_Summer_2022_Q11) – *Vectors - Lines - Planes*

[Maximum mark: 20]

Two airplanes, A and B, have position vectors with respect to an origin O given respectively by

$$\boldsymbol{r}_A = \begin{pmatrix} 19 \\ -1 \\ 1 \end{pmatrix} + t \begin{pmatrix} -6 \\ 2 \\ 4 \end{pmatrix}$$

$$\boldsymbol{r}_B = \begin{pmatrix} 1 \\ 0 \\ 12 \end{pmatrix} + t \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$$

where t represents the time in minutes and $0 \le t \le 2.5$.

Entries in each column vector give the displacement east of O, the displacement north of O and the distance above sea level, all measured in kilometres.

(a) Find the three-figure bearing on which airplane B is travelling. [2]

(b) Show that airplane A travels at a greater speed than airplane B. [2]

(c) Find the acute angle between the two airplanes' lines of flight. Give your answer in degrees. [4]

The two airplanes' lines of flight cross at point P.

(d) (i) Find the coordinates of P.

(ii) Determine the length of time between the first airplane arriving at P and the second airplane arriving at P. [7]

Let $D(t)$ represent the distance between airplane A and airplane B for $0 \le t \le 2.5$.

(e) Find the minimum value of $D(t)$. [5]

28 – (MathAA/20_HL_Winter_2022_Q12) – *Vectors - Lines - Planes*

[Maximum mark: 22]

Consider the points $\mathrm{A}(1, 2, 3)$, $\mathrm{B}(k, -2, 1)$ and $\mathrm{C}(5, 0, 2)$, where $k \in \mathbb{R}$.

(a) Write down $\overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{AC}}$. [2]

(b) Given that the points A, B and C lie on a straight line, show that $k = 9$. [1]

(c) For $k = 9$, let L_1 be the line passing through A, B and C.

(i) Find a vector equation of the line L_1.

(ii) Line L_2 has the equation $\frac{x-1}{2} = \frac{y}{3} = 1 - z$. Show that the lines L_1 and L_2 are skew. [10]

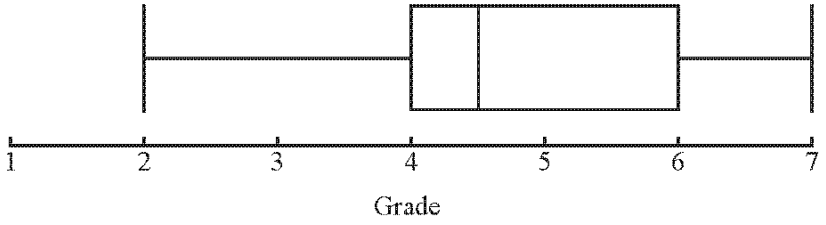
(d) For $k \neq 9$, let Π be the plane containing A, B and C.

(i) Find the Cartesian equation of the plane Π.

(ii) Find the coordinates of the point on the plane Π which is closest to the origin $(0, 0, 0)$. [9]

1 – (MAT/21_HL_Summer_2012_Q5) – *Statistics*

The box and whisker plot below illustrates the IB grades obtained by 100 students.



IB grades can only take integer values.

(a) How many students obtained a grade of more than 4? *[1 mark]*

(b) State, with reasons, the maximum possible number and minimum possible number of students who obtained a 4 in the exam. *[4 marks]*

2 – (MAT/22_HL_Summer_2012_Q10) – *Probability, Statistics*

A market stall sells apples, pears and plums.

(a) The weights of the apples are normally distributed with a mean of 200 grams and a standard deviation of 25 grams.

(i) Given that there are 450 apples on the stall, what is the expected number of apples with a weight of more than 225 grams?

(ii) Given that 70 % of the apples weigh less than m grams, find the value of m. *[5 marks]*

(b) The weights of the pears are normally distributed with a mean of $\propto$ grams and a standard deviation of σ grams. Given that 8 % of these pears have a weight of more than 270 grams and 15 % have a weight less than 250 grams, find $\propto$ and σ. *[6 marks]*

(c) The weights of the plums are normally distributed with a mean of 80 grams and a standard deviation of 4 grams. 5 plums are chosen at random. What is the probability that exactly 3 of them weigh more than 82 grams? *[3 marks]*

3 – (MAT/21_HL_Summer_2012_Q12) – *Probability, Statistics*

A ski resort finds that the mean number of accidents on any given weekday (Monday to Friday) is 2.2. The number of accidents can be modelled by a Poisson distribution.

(a) Find the probability that in a certain week (Monday to Friday only)

(i) there are fewer than 12 accidents;

(ii) there are more than 8 accidents, given that there are fewer than 12 accidents. *[6 marks]*

Due to the increased usage, it is found that the probability of more than 3 accidents in a day at the weekend (Saturday and Sunday) is 0.24.

(b) Assuming a Poisson model,

(i) calculate the mean number of accidents per day at the weekend (Saturday and Sunday);

(ii) calculate the probability that, in the four weekends in February, there will be more than 5 accidents during at least two of the weekends. *[10 marks]*

It is found that 20 % of skiers having accidents are at least 25 years of age and 40 % are under 18 years of age.

(c) Assuming that the ages of skiers having accidents are normally distributed, find the mean age of skiers having accidents. *[6 marks]*

4 – (MAT/20_HL_Winter_2012_Q4) – *Statistics*

[Maximum mark: 6]

A set of 15 observations has mean 11.5 and variance 9.3. One observation of 22.1 is considered unreliable and is removed. Find the mean and variance of the remaining 14 observations.

5 – (MAT/20_HL_Winter_2012_Q11) – *Probability, Statistics*

The number of visitors that arrive at a museum every minute can be modelled by a Poisson distribution with mean 2.2.

(a) If the museum is open 6 hours daily, find the expected number of visitors in 1 day. *[2 marks]*

(b) Find the probability that the number of visitors arriving during an hour exceeds 100. *[3 marks]*

(c) Find the probability that the number of visitors in each of the 6 hours the museum is open exceeds 100. *[2 marks]*

The ages of the visitors to the museum can be modelled by a normal distribution with mean μ and variance σ^2. The records show that 29 % of the visitors are under 35 years of age and 23 % are at least 55 years of age.

(d) Find the values of μ and σ. *[6 marks]*

(e) One day, 100 visitors under 35 years of age come to the museum. Estimate the number of visitors under 50 years of age that were at the museum on that day. *[5 marks]*

6 – (MAT/21_HL_Summer_2013_Q1) – *Statistics*

[Maximum mark: 4]

The marks obtained by a group of students in a class test are shown below.

Marks	Frequency
5	6
6	k
7	3
8	1
9	2
10	1

Given the mean of the marks is 6.5, find the value of k.

7 – (MAT/20_HL_Winter_2013_Q11) – *Probability, Statistics*

(a) The number of cats visiting Helena's garden each week follows a Poisson distribution with mean $\lambda = 0.6$.

Find the probability that

(i) in a particular week no cats will visit Helena's garden;

(ii) in a particular week at least three cats will visit Helena's garden;

(iii) over a four-week period no more than five cats in total will visit Helena's garden;

(iv) over a twelve-week period there will be exactly four weeks in which at least one cat will visit Helena's garden. *[9]*

(b) A continuous random variable X has probability distribution function f given by

$$f(x) = k \ln x \quad 1 \le x \le 3$$
$$f(x) = 0 \quad \text{otherwise}$$

(i) Find the value of k to six decimal places.

(ii) Find the value of $\mathrm{E}(X)$.

(iii) State the mode of X.

(iv) Find the median of X. *[9]*

8 – (MAT/21_HL_Summer_2014_Q2) – *Statistics*

[Maximum mark: 5]

A student sits a national test and is told that the marks follow a normal distribution with mean 100. The student receives a mark of 124 and is told that he is at the 68th percentile. Calculate the variance of the distribution.

9 – (MAT/22_HL_Summer_2014_Q2) – *Statistics, Probability*

The weights, in kg, of one-year-old bear cubs are modelled by a normal distribution with mean μ and standard deviation σ.

(a) Given that the upper quartile weight is 21.3 kg and the lower quartile weight is 17.1 kg, calculate the value of μ and the value of σ. *[4]*

A random sample of 100 of these bear cubs is selected.

(b) Find the expected number of bear cubs weighing more than 22 kg. *[1]*

10 – (MAT/20_HL_Winter_2014_Q3) – *Statistics*

[Maximum mark: 6]

Consider the data set $\{2, x, y, 10, 17\}$, $x, y \in \mathbb{Z}^+$ and $x < y$.

The mean of the data set is 8 and its variance is 27.6.

Find the value of x and the value of y.

11 – (MAT/20_HL_Winter_2015_Q3) – *Statistics*

The data of the goals scored by players in a football club during a season are given in the following table.

Goals	Frequency
0	4
1	k
2	3
3	2
4	3
8	1

(a) Given that the mean number of goals scored per player is 1.95, find the value of k. [3]

It is discovered that there is a mistake in the data and that the top scorer, who scored 22 goals, has not been included in the table.

(b) (i) Find the correct mean number of goals scored per player.

(ii) Find the correct standard deviation of the number of goals scored per player. [3]

12 – (MAT/21_HL_Summer_2016_Q13) – *Statistics, Probability*

Six balls numbered $1, 2, 2, 3, 3, 3$ are placed in a bag. Balls are taken one at a time from the bag at random and the number noted. Throughout the question a ball is always replaced before the next ball is taken.

(a) A single ball is taken from the bag. Let X denote the value shown on the ball. Find $\mathrm{E}(X)$. [2]

(b) Three balls are taken from the bag. Find the probability that

(i) the total of the three numbers is 5;

(ii) the median of the three numbers is 1. [6]

(c) Ten balls are taken from the bag. Find the probability that less than four of the balls are numbered 2. [3]

(d) Find the least number of balls that must be taken from the bag for the probability of taking out at least one ball numbered 2 to be greater than 0.95. [3]

(e) Another bag also contains balls numbered $1, 2$ or 3.
Eight balls are to be taken from this bag at random. It is calculated that the expected number of balls numbered 1 is 4.8, and the variance of the number of balls numbered 2 is 1.5.

Find the least possible number of balls numbered 3 in this bag. [8]

13 – (MAT/22_HL_Summer_2017_Q1) – *Probability, Statistics*

There are 75 players in a golf club who take part in a golf tournament. The scores obtained on the 18th hole are as shown in the following table.

Score	2	3	4	5	6	7
Frequency	3	15	28	17	9	3

(a) One of the players is chosen at random. Find the probability that this player's score was 5 or more. [2]

(b) Calculate the mean score. [2]

14 – (MAT/22_HL_Summer_2017_Q3) – *Statistics, Probability*

Packets of biscuits are produced by a machine. The weights X, in grams, of packets of biscuits can be modelled by a normal distribution where $X \sim \mathrm{N}(\mu, \sigma^2)$. A packet of biscuits is considered to be underweight if it weighs less than 250 grams.

(a) Given that $\mu = 253$ and $\sigma = 1.5$ find the probability that a randomly chosen packet of biscuits is underweight. [2]

The manufacturer makes the decision that the probability that a packet is underweight should be 0.002. To do this μ is increased and σ remains unchanged.

(b) Calculate the new value of μ giving your answer correct to two decimal places. [3]

The manufacturer is happy with the decision that the probability that a packet is underweight should be 0.002, but is unhappy with the way in which this was achieved. The machine is now adjusted to reduce σ and return μ to 253.

(c) Calculate the new value of σ. [2]

15 – (MAT/21_HL_Summer_2017_Q9) – *Probability, Statistics*

The times taken for male runners to complete a marathon can be modelled by a normal distribution with a mean 196 minutes and a standard deviation 24 minutes.

(a) Find the probability that a runner selected at random will complete the marathon in less than 3 hours. [2]

It is found that 5% of the male runners complete the marathon in less than T_1 minutes.

(b) Calculate T_1. [2]

The times taken for female runners to complete the marathon can be modelled by a normal distribution with a mean 210 minutes. It is found that 58% of female runners complete the marathon between 185 and 235 minutes.

(c) Find the standard deviation of the times taken by female runners. [4]

16 – (MAT/20_HL_Winter_2017_Q4) – *Statistics*

[Maximum mark: 6]

It is given that one in five cups of coffee contain more than $120\,\mathrm{mg}$ of caffeine.
It is also known that three in five cups contain more than $110\,\mathrm{mg}$ of caffeine.

Assume that the caffeine content of coffee is modelled by a normal distribution.
Find the mean and standard deviation of the caffeine content of coffee.

17 – (MAT/22_HL_Summer_2017_Q10) – *Probability, Statistics*

A continuous random variable X has probability density function f given by

$$f(x) = \begin{cases} \dfrac{x^2}{a} + b, & 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$ where a and b are positive constants.

It is given that $\mathrm{P}(X \ge 2) = 0.75$.

(a) Show that $a = 32$ and $b = \dfrac{1}{12}$. [5]

(b) Find $\mathrm{E}(X)$. [2]

(c) Find $\mathrm{Var}(X)$. [2]

(d) Find the median of X. [3]

Eight independent observations of X are now taken and the random variable Y is the number of observations such that $X \ge 2$.

(e) Find $\mathrm{E}(Y)$. [2]

(f) Find $\mathrm{P}(Y \ge 3)$. [1]

18 – (MAT/21_HL_Summer_2018_Q8) – *Statistics*

[Maximum mark: 7]

Each of the 25 students in a class are asked how many pets they own. Two students own three pets and no students own more than three pets. The mean and standard deviation of the number of pets owned by students in the class are $\frac{18}{25}$ and $\frac{24}{25}$ respectively.

Find the number of students in the class who do not own a pet.

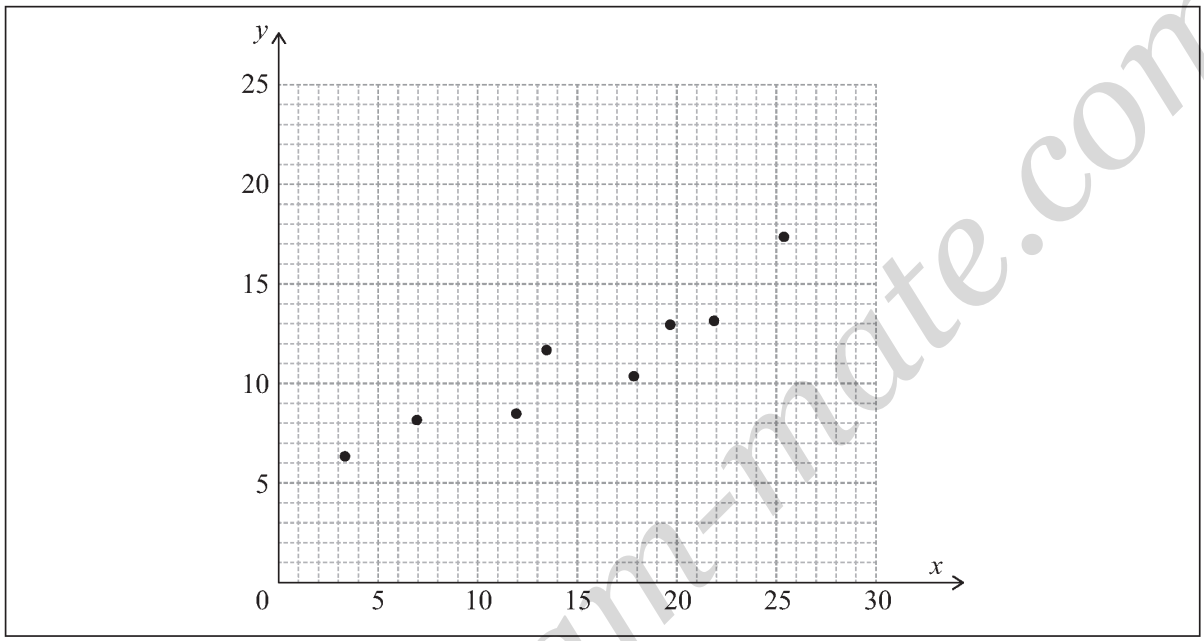
19 – (MathAA/21_HL_Summer_2021_Q1) – *Statistics*

[Maximum mark: 7]

The following table shows the data collected from an experiment.

x	3.3	6.9	11.9	13.4	17.8	19.6	21.8	25.3
y	6.3	8.1	8.4	11.6	10.3	12.9	13.1	17.3

The data is also represented on the following scatter diagram.



The relationship between x and y can be modelled by the regression line of y on x with equation $y = ax + b$, where $a, b \in \mathbb{R}$.

(a) Write down the value of a and the value of b. [2]

(b) Use this model to predict the value of y when $x = 18$. [2]

(c) Write down the value of $\overline{x}$ and the value of $\overline{y}$. [1]

(d) Draw the line of best fit on the scatter diagram. [2]

20 – (MathAA/22_HL_Summer_2021_Q1) – *Statistics*

[Maximum mark: 6]

At a café, the waiting time between ordering and receiving a cup of coffee is dependent upon the number of customers who have already ordered their coffee and are waiting to receive it.

Sarah, a regular customer, visited the café on five consecutive days. The following table shows the number of customers, x, ahead of Sarah who have already ordered and are waiting to receive their coffee and Sarah's waiting time, y minutes.

Number of customers (x)	3	9	11	10	5
Sarah's waiting time (y)	6	10	12	11	6

The relationship between x and y can be modelled by the regression line of y on x with equation $y = ax + b$.

(a) (i) Find the value of a and the value of b.

(ii) Write down the value of Pearson's product-moment correlation coefficient, r. [3]

(b) Interpret, in context, the value of a found in part (a)(i). [1]

On another day, Sarah visits the café to order a coffee. Seven customers have already ordered their coffee and are waiting to receive it.

(c) Use the result from part (a)(i) to estimate Sarah's waiting time to receive her coffee. [2]

21 – (MathAA/20_HL_Winter_2021_Q1) – *Statistics*

[Maximum mark: 7]

In Lucy's music academy, eight students took their piano diploma examination and achieved scores out of 150. For her records, Lucy decided to record the average number of hours per week each student reported practising in the weeks prior to their examination. These results are summarized in the table below.

Average weekly practice time (h)	28	13	45	33	17	29	39	36
Diploma score (D)	115	82	120	116	79	101	110	121

(a) Find Pearson's product-moment correlation coefficient, r, for these data. [2]

(b) The relationship between the variables can be modelled by the regression equation $D = ah + b$. Write down the value of a and the value of b. [1]

(c) One of these eight students was disappointed with her result and wished she had practised more. Based on the given data, determine how her score could have been expected to alter had she practised an extra five hours per week. [2]

(d) Lucy asserts that the number of hours a student practises has a direct effect on their final diploma result. Comment on the validity of Lucy's assertion. [1]

Lucy suspected that each student had not been practising as much as they reported. In order to compensate for this, Lucy deducted a fixed number of hours per week from each of the students' recorded hours.

(e) State how, if at all, the value of r would be affected. [1]

22 – (MathAA/21_HL_Summer_2022_Q1) – *Statistics*

[Maximum mark: 4]

The number of hours spent exercising each week by a group of students is shown in the following table.

Exercising time (in hours)	Number of students
2	5
3	1
4	4
5	3
6	x

The median is 4.5 hours.

(a) Find the value of x. [2]

(b) Find the standard deviation. [2]

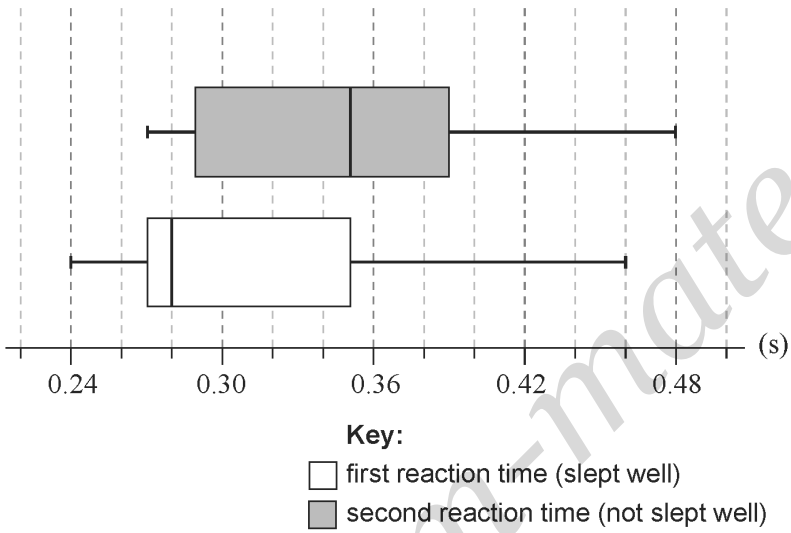
23 – (MathAA/22_HL_Summer_2022_Q4) – *Statistics*

[Maximum mark: 6]

A random sample of nine adults were selected to see whether sleeping well affected their reaction times to a visual stimulus. Each adult's reaction time was measured twice.

The first measurement for reaction time was taken on a morning after the adult had slept well. The second measurement was taken on a morning after the same adult had not slept well.

The box and whisker diagrams for the reaction times, measured in seconds, are shown below.



Consider the box and whisker diagram representing the reaction times after sleeping well.

(a) State the median reaction time after sleeping well. [1]

(b) Verify that the measurement of 0.46 seconds is not an outlier. [3]

(c) State why it appears that the mean reaction time is greater than the median reaction time. [1]

Now consider the two box and whisker diagrams.

(d) Comment on whether these box and whisker diagrams provide any evidence that might suggest that not sleeping well causes an increase in reaction time. [1]

24 – (MathAA/20_HL_Winter_2022_Q1) – *Statistics*

[Maximum mark: 5]

The following table shows the Mathematics test scores (x) and the Science test scores (y) for a group of eight students.

Mathematics scores (x)	64	68	72	75	80	82	85	86
Science scores (y)	67	72	77	76	84	83	89	91

The regression line of y on x for this data can be written in the form $y = ax + b$.

(a) Find the value of a and the value of b. [2]

(b) Write down the value of the Pearson's product-moment correlation coefficient, r. [1]

(c) Use the equation of your regression line to predict the Science test score for a student who has a score of 78 on the Mathematics test. Express your answer to the nearest integer. [2]

25 – (MathAA/20_HL_Winter_2022_Q6) – *Statistics*

[Maximum mark: 8]

The continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} axe^{x}, & 0 \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

where $a, b \in \mathbb{R}$.

(a) Find an expression for a in terms of b. [5]

(b) In the case where $a = b = 1$, find the median of X. [3]

1 – (MAT/21_HL_Summer_2012_Q2) – *Probability*

[Maximum mark: 4]

The probability density function of the continuous random variable X is given by

$$f(x)=\begin{cases} k2^{\frac{1}{x}}, & 1\le x\le 2 \\ 0, & \text{otherwise} \end{cases}$$

where k is a constant. Find the expected value of X.

2 – (MAT/22_HL_Summer_2012_Q2) – *Probability*

The random variable X has the distribution $\mathrm{B}(30,\ p)$. Given that $\mathrm{E}(X) = 10$, find

(a) the value of p; *[1 mark]*

(b) $\mathrm{P}(X = 10)$; *[2 marks]*

(c) $\mathrm{P}(X \geq 15)$. *[2 marks]*

3 – (MAT/22_HL_Summer_2012_Q5) – *Probability*

The random variable X has the distribution $\mathrm{Po}(m)$.
Given that $\mathrm{P}(X=5)=\mathrm{P}(X=3)+\mathrm{P}(X=4)$, find

(a) the value of m; *[3 marks]*

(b) $\mathrm{P}(X>2)$. *[2 marks]*

4 – (MAT/21_HL_Summer_2012_Q7) – *Probability*

[Maximum mark: 5]

A fisherman notices that in any hour of fishing, he is equally likely to catch exactly two fish, as he is to catch less than two fish. Assuming the number of fish caught can be modelled by a Poisson distribution, calculate the expected value of the number of fish caught when he spends four hours fishing.

5 – (MAT/22_HL_Summer_2012_Q7) – *Probability*

The probability density function of a continuous random variable X is given by

$$f(x) = \frac{1}{1+x^4}, \quad 0 \le x \le a.$$

(a) Find the value of a. *[3 marks]*

(b) Find the mean of X. *[2 marks]*

6 – (MAT/22_HL_Summer_2012_Q10) – *Probability, Statistics*

A market stall sells apples, pears and plums.

(a) The weights of the apples are normally distributed with a mean of 200 grams and a standard deviation of 25 grams.

(i) Given that there are 450 apples on the stall, what is the expected number of apples with a weight of more than 225 grams?

(ii) Given that 70 % of the apples weigh less than m grams, find the value of m. *[5 marks]*

(b) The weights of the pears are normally distributed with a mean of $\propto$ grams and a standard deviation of σ grams. Given that 8 % of these pears have a weight of more than 270 grams and 15 % have a weight less than 250 grams, find $\propto$ and σ. *[6 marks]*

(c) The weights of the plums are normally distributed with a mean of 80 grams and a standard deviation of 4 grams. 5 plums are chosen at random. What is the probability that exactly 3 of them weigh more than 82 grams? *[3 marks]*

7 – (MAT/21_HL_Summer_2012_Q12) – *Probability, Statistics*

A ski resort finds that the mean number of accidents on any given weekday (Monday to Friday) is 2.2. The number of accidents can be modelled by a Poisson distribution.

(a) Find the probability that in a certain week (Monday to Friday only)

(i) there are fewer than 12 accidents;

(ii) there are more than 8 accidents, given that there are fewer than 12 accidents. *[6 marks]*

Due to the increased usage, it is found that the probability of more than 3 accidents in a day at the weekend (Saturday and Sunday) is 0.24.

(b) Assuming a Poisson model,

(i) calculate the mean number of accidents per day at the weekend (Saturday and Sunday);

(ii) calculate the probability that, in the four weekends in February, there will be more than 5 accidents during at least two of the weekends. *[10 marks]*

It is found that 20 % of skiers having accidents are at least 25 years of age and 40 % are under 18 years of age.

(c) Assuming that the ages of skiers having accidents are normally distributed, find the mean age of skiers having accidents. *[6 marks]*

8 – (MAT/20_HL_Winter_2012_Q7) – *Probability*

Kathy plays a computer game in which she has to find the path through a maze within a certain time. The first time she attempts the game, the probability of success is known to be 0.75. In subsequent attempts, if Kathy is successful, the difficulty increases and the probability of success is half the probability of success on the previous attempt. However, if she is unsuccessful, the probability of success remains the same. Kathy plays the game three times consecutively.

(a) Find the probability that she is successful in all three games. *[2 marks]*

(b) Assuming that she is successful in the first game, find the probability that she is successful in exactly two games. *[6 marks]*

9 – (MAT/20_HL_Winter_2012_Q11) – *Probability, Statistics*

The number of visitors that arrive at a museum every minute can be modelled by a Poisson distribution with mean 2.2.

(a) If the museum is open 6 hours daily, find the expected number of visitors in 1 day. *[2 marks]*

(b) Find the probability that the number of visitors arriving during an hour exceeds 100. *[3 marks]*

(c) Find the probability that the number of visitors in each of the 6 hours the museum is open exceeds 100. *[2 marks]*

The ages of the visitors to the museum can be modelled by a normal distribution with mean μ and variance σ^2. The records show that 29 % of the visitors are under 35 years of age and 23 % are at least 55 years of age.

(d) Find the values of μ and σ. *[6 marks]*

(e) One day, 100 visitors under 35 years of age come to the museum. Estimate the number of visitors under 50 years of age that were at the museum on that day. *[5 marks]*

10 – (MAT/21_HL_Summer_2013_Q3) – *Probability*

Emily walks to school every day. The length of time this takes can be modelled by a normal distribution with a mean of 11 minutes and a standard deviation of 3 minutes. She is late if her journey takes more than 15 minutes.

(a) Find the probability she is late next Monday. *[2 marks]*

(b) Find the probability she is late at least once during the next week (Monday to Friday). *[3 marks]*

11 – (MAT/22_HL_Summer_2013_Q3) – *Probability*

It is believed that the lifespans of Manx cats are normally distributed with a mean of 13.5 years and a variance of 9.5 years^2.

(a) Calculate the range of lifespans of Manx cats whose lifespans are within one standard deviation of the mean. *[2 marks]*

(b) Estimate the number of Manx cats in a population of 10 000 that will have a lifespan of less than 10 years. Give your answer to the nearest whole number. *[3 marks]*

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12 – (MAT/22_HL_Summer_2013_Q7) – *Probability*

The length, X metres, of a species of fish has the probability density function

$$f(x)=\begin{cases} ax^2, & \text{for } 0\le x\le 0.5 \\ 0.5a(1-x), & \text{for } 0.5\le x\le 1 \\ 0, & \text{otherwise}. \end{cases}$$

(a) Show that $a=9.6$. *[3 marks]*

(b) Sketch the graph of the distribution. *[2 marks]*

(c) Find $\mathrm{P}(X<0.6)$. *[2 marks]*

13 – (MAT/22_HL_Summer_2013_Q9) – *Probability*

A small car hire company has two cars. Each car can be hired for one whole day at a time. The rental charge is US$60 per car per day. The number of requests to hire a car for one whole day may be modelled by a Poisson distribution with mean 1.2.

(a) Find the probability that on a particular weekend, three requests are received on Saturday and none are received on Sunday. *[2 marks]*

Over a weekend of two days, it is given that a total of three requests are received.

(b) Find the expected total rental income for the weekend. *[5 marks]*

14 – (MAT/21_HL_Summer_2013_Q10) – *Probability*

A ferry carries cars across a river. There is a fixed time of T minutes between crossings. The arrival of cars at the crossing can be assumed to follow a Poisson distribution with a mean of one car every four minutes. Let X denote the number of cars that arrive in T minutes.

(a) Find T, to the nearest minute, if $P(X \leq 3) = 0.6$. *[3 marks]*

It is now decided that the time between crossings, T, will be 10 minutes. The ferry can carry a maximum of three cars on each trip.

(b) One day all the cars waiting at 13:00 get on the ferry. Find the probability that all the cars that arrive in the next 20 minutes will get on either the 13:10 or the 13:20 ferry. *[4 marks]*

15 – (MAT/22_HL_Summer_2013_Q11) – *Sequences - Series, Probability*

(a) (i) Express the sum of the first n positive odd integers using sigma notation.

(ii) Show that the sum stated above is n^2.

(iii) Deduce the value of the difference between the sum of the first 47 positive odd integers and the sum of the first 14 positive odd integers. *[4 marks]*

(b) A number of distinct points are marked on the circumference of a circle, forming a polygon. Diagonals are drawn by joining all pairs of non-adjacent points.

(i) Show on a diagram all diagonals if there are 5 points.

(ii) Show that the number of diagonals is $\frac{n(n-3)}{2}$ if there are n points, where $n > 2$.

(iii) Given that there are more than one million diagonals, determine the least number of points for which this is possible. *[7 marks]*

(c) The random variable $X \sim B(n, p)$ has mean 4 and variance 3.

(i) Determine n and p.

(ii) Find the probability that in a single experiment the outcome is 1 or 3. *[8 marks]*

16 – (MAT/20_HL_Winter_2013_Q4) – *Probability*

[Maximum mark: 6]

The duration of direct flights from London to Singapore in a particular year followed a normal distribution with mean μ and standard deviation σ.

92% of flights took under 13 hours, while only 12% of flights took under 12 hours 35 minutes.

Find μ and σ to the nearest minute.

17 – (MAT/20_HL_Winter_2013_Q5) – *Probability*

At the start of each week, Eric and Marina pick a night at random on which they will watch a movie.

If they choose a Saturday night, the probability that they watch a French movie is $\frac{7}{9}$ and if they choose any other night the probability that they watch a French movie is $\frac{4}{9}$.

(a) Find the probability that they watch a French movie. *[3]*

(b) Given that last week they watched a French movie, find the probability that it was on a Saturday night. *[2]*

18 – (MAT/20_HL_Winter_2013_Q11) – *Probability, Statistics*

(a) The number of cats visiting Helena's garden each week follows a Poisson distribution with mean $\lambda = 0.6$.

Find the probability that

(i) in a particular week no cats will visit Helena's garden;

(ii) in a particular week at least three cats will visit Helena's garden;

(iii) over a four-week period no more than five cats in total will visit Helena's garden;

(iv) over a twelve-week period there will be exactly four weeks in which at least one cat will visit Helena's garden. *[9]*

(b) A continuous random variable X has probability distribution function f given by

$$f(x) = k \ln x \quad 1 \le x \le 3$$
$$f(x) = 0 \quad \text{otherwise}$$

(i) Find the value of k to six decimal places.

(ii) Find the value of $\mathrm{E}(X)$.

(iii) State the mode of X.

(iv) Find the median of X. *[9]*

19 – (MAT/22_HL_Summer_2014_Q2) – *Statistics, Probability*

The weights, in kg, of one-year-old bear cubs are modelled by a normal distribution with mean μ and standard deviation σ.

(a) Given that the upper quartile weight is 21.3 kg and the lower quartile weight is 17.1 kg, calculate the value of μ and the value of σ. *[4]*

A random sample of 100 of these bear cubs is selected.

(b) Find the expected number of bear cubs weighing more than 22 kg. *[1]*

20 – (MAT/22_HL_Summer_2014_Q6) – *Probability*

Six customers wait in a queue in a supermarket. A customer can choose to pay with cash or a credit card. Assume that whether or not a customer pays with a credit card is independent of any other customers' methods of payment.

It is known that 60 % of customers choose to pay with a credit card.

(a) Find the probability that:

(i) the first three customers pay with a credit card and the next three pay with cash;

(ii) exactly three of the six customers pay with a credit card. *[4]*

There are n customers waiting in another queue in the same supermarket. The probability that at least one customer pays with cash is greater than 0.995.

(b) Find the minimum value of n. *[3]*

21 – (MAT/21_HL_Summer_2014_Q8) – *Binomial Theorem, Probability*

(a) Find the term in x^5 in the expansion of $(3x+A)(2x+B)^6$. *[4]*

Mina and Norbert each have a fair cubical die with faces labelled 1, 2, 3, 4, 5 and 6; they throw it to decide if they are going to eat a cookie.

Mina throws her die just once and she eats a cookie if she throws a four, a five or a six. Norbert throws his die six times and each time eats a cookie if he throws a five or a six.

(b) Calculate the probability that five cookies are eaten. *[4]*

22 – (MAT/22_HL_Summer_2014_Q8) – *Probability*

The random variable X has a Poisson distribution with mean μ.

Given that $\mathrm{P}(X=2)+\mathrm{P}(X=3)=\mathrm{P}(X=5)$,

(a) find the value of μ; *[2]*

(b) find the probability that X lies within one standard deviation of the mean. *[2]*

23 – (MAT/21_HL_Summer_2014_Q9) – *Probability*

The number of birds seen on a power line on any day can be modelled by a Poisson distribution with mean 5.84.

(a) Find the probability that during a certain seven-day week, more than 40 birds have been seen on the power line. *[2]*

(b) On Monday there were more than 10 birds seen on the power line. Show that the probability of there being more than 40 birds seen on the power line from that Monday to the following Sunday, inclusive, can be expressed as:

$$\frac{\mathrm{P}(X>40)+\sum_{r=11}^{40}\mathrm{P}(X=r)\mathrm{P}(Y>40-r)}{\mathrm{P}(X>10)} \text{ where } X\sim\mathrm{Po}(5.84) \text{ and } Y\sim\mathrm{Po}(35.04).$$ *[5]*

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24 – (MAT/21_HL_Summer_2014_Q11) – *Probability*

A random variable X has probability density function

$$f(x)=\begin{cases} ax+b, & 2\le x\le 3 \\ 0, & \text{otherwise} \end{cases},\ a,\ b\in\mathbb{R}.$$

(a) Show that $5a+2b=2$. *[4]*

Let $\mathrm{E}(X)=\mu$.

(b) (i) Show that $a=12\mu-30$.

(ii) Find a similar expression for b in terms of μ. *[7]*

Let the median of the distribution be 2.3.

(c) (i) Find the value of μ.

(ii) Find the value of the standard deviation of X. *[10]*

25 – (MAT/22_HL_Summer_2014_Q11) – *Probability*

The probability density function of a random variable X is defined as:

$$f(x) = \begin{cases} a\,x\cos x, & 0 \le x \le \dfrac{\pi}{2}, \text{ where } a \in \mathbb{R}. \\ 0, & \text{elsewhere} \end{cases}$$

(a) Show that $a = \dfrac{2}{\pi - 2}$. *[5]*

(b) Find $\mathrm{P}\left(X < \dfrac{\pi}{4}\right)$. *[2]*

(c) Find:

(i) the mode of X;

(ii) the median of X. *[4]*

(d) Find $\mathrm{P}\left(X < \dfrac{\pi}{8} \mid X < \dfrac{\pi}{4}\right)$. *[2]*

26 – (MAT/20_HL_Winter_2014_Q2) – *Probability*

The wingspans of a certain species of bird can be modelled by a normal distribution with mean 60.2 cm and standard deviation 2.4 cm.

According to this model, 99 % of wingspans are greater than x cm.

(a) Find the value of x. *[2]*

In a field experiment, a research team studies a large sample of these birds. The wingspans of each bird are measured correct to the nearest 0.1 cm.

(b) Find the probability that a randomly selected bird has a wingspan measured as 60.2 cm. *[3]*

27 – (MAT/20_HL_Winter_2014_Q11) – *Probability*

The number of complaints per day received by customer service at a department store follows a Poisson distribution with a mean of 0.6.

(a) On a randomly chosen day, find the probability that

(i) there are no complaints;

(ii) there are at least three complaints. *[3]*

(b) In a randomly chosen five-day week, find the probability that there are no complaints. *[2]*

(c) On a randomly chosen day, find the most likely number of complaints received. Justify your answer. *[3]*

The department store introduces a new policy to improve customer service. The number of complaints received per day now follows a Poisson distribution with mean λ.

On a randomly chosen day, the probability that there are no complaints is now 0.8.

(d) Find the value of λ. *[2]*

28 – (MAT/20_HL_Winter_2014_Q12) – *Probability*

Ava and Barry play a game with a bag containing one green marble and two red marbles. Each player in turn randomly selects a marble from the bag, notes its colour and replaces it. Ava wins the game if she selects a green marble. Barry wins the game if he selects a red marble. Ava starts the game.

Find the probability that

(a) Ava wins on her first turn; *[1]*

(b) Barry wins on his first turn; *[2]*

(c) Ava wins in one of her first three turns; *[4]*

(d) Ava eventually wins. *[4]*

29 – (MAT/21_HL_Summer_2015_Q2) – *Probability*

The finishing times in a marathon race follow a normal distribution with mean 210 minutes and standard deviation 22 minutes.

(a) Find the probability that a runner finishes the race in under three hours. [2]

The fastest 90 % of the finishers receive a certificate.

(b) Find the time, below which a competitor has to complete the race, in order to gain a certificate. [2]

30 – (MAT/21_HL_Summer_2015_Q3) – *Probability*

A mosaic is going to be created by randomly selecting 1000 small tiles, each of which is either black or white. The probability that a tile is white is 0.1. Let the random variable W be the number of white tiles.

(a) State the distribution of W, including the values of any parameters. [2]

(b) Write down the mean of W. [1]

(c) Find $\mathrm{P}(W > 89)$. [2]

31 – (MAT/22_HL_Summer_2015_Q4) – *Probability*

Emma acquires a new cell phone for her birthday and receives texts from her friends. It is assumed that the daily number of texts Emma receives follows a Poisson distribution with mean $m = 5$.

(a) (i) Find the probability that on a certain day Emma receives more than 7 texts.

(ii) Determine the expected number of days in a week on which Emma receives more than 7 texts. [4]

(b) Find the probability that Emma receives fewer than 30 texts during a week. [3]

32 – (MAT/21_HL_Summer_2015_Q7) – *Probability*

The random variable X follows a Poisson distribution with mean $m \neq 0$.

(a) Given that $2\mathrm{P}(X=4)=\mathrm{P}(X=5)$, show that $m=10$. [3]

(b) Given that $X \leq 11$, find the probability that $X=6$. [4]

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33 – (MAT/22_HL_Summer_2015_Q9) – *Probability*

Natasha lives in Chicago and has relatives in Nashville and St. Louis.
Each time she visits her relatives, she either flies or drives.

When travelling to Nashville, the probability that she drives is $\frac{4}{5}$, and when travelling to St. Louis, the probability that she flies is $\frac{1}{3}$.

Given that the probability that she drives when visiting her relatives is $\frac{13}{18}$, find the probability that for a particular trip,

(a) she travels to Nashville; [3]

(b) she is on her way to Nashville, given that she is flying. [3]

34 – (MAT/22_HL_Summer_2015_Q10) – *Probability*

Farmer Suzie grows turnips and the weights of her turnips are normally distributed with a mean of $122\,\text{g}$ and standard deviation of $14.7\,\text{g}$.

(a) (i) Calculate the percentage of Suzie's turnips that weigh between $110\,\text{g}$ and $130\,\text{g}$.

(ii) Suzie has 100 turnips to take to market. Find the expected number weighing more than $130\,\text{g}$.

(iii) Find the probability that at least 30 of the 100 turnips weigh more than $130\,\text{g}$. [6]

Farmer Ray also grows turnips and the weights of his turnips are normally distributed with a mean of $144\,\text{g}$. Ray only takes to market turnips that weigh more than $130\,\text{g}$. Over a period of time, Ray finds he has to reject 1 in 15 turnips due to their being underweight.

(b) (i) Find the standard deviation of the weights of Ray's turnips.

(ii) Ray has 200 turnips to take to market. Find the expected number weighing more than $150\,\text{g}$. [6]

35 – (MAT/21_HL_Summer_2015_Q11) – *Probability*

The probability density function of a continuous random variable X is given by

$$f(x)=\begin{cases} 0 \ , \ x<0 \\ \dfrac{\sin x}{4} \ , \ 0 \le x \le \pi \\ a(x-\pi), \ \pi < x \le 2\pi \\ 0 \ , \ 2\pi < x \end{cases} .$$

(a) Sketch the graph of $y=f(x)$. [2]

(b) Find $\mathrm{P}(X \le \pi)$. [2]

(c) Show that $a=\dfrac{1}{\pi^2}$. [3]

(d) Write down the median of X. [1]

(e) Calculate the mean of X. [3]

(f) Calculate the variance of X. [3]

(g) Find $\mathrm{P}\left(\dfrac{\pi}{2} \le X \le \dfrac{3\pi}{2}\right)$. [2]

(h) Given that $\dfrac{\pi}{2} \le X \le \dfrac{3\pi}{2}$ find the probability that $\pi \le X \le 2\pi$. [4]

36 – (MAT/20_HL_Winter_2015_Q1) – *Probability*

The events A and B are such that $\mathrm{P}(A) = 0.65$, $\mathrm{P}(B) = 0.48$ and $\mathrm{P}(A \cup B) = 0.818$.

(a) Find $\mathrm{P}(A \cap B)$. [2]

(b) Hence show that the events A and B are independent. [2]

37 – (MAT/20_HL_Winter_2015_Q6) – *Probability*

[Maximum mark: 6]

Josie has three ways of getting to school. 30% of the time she travels by car, 20% of the time she rides her bicycle and 50% of the time she walks.
When travelling by car, Josie is late 5% of the time. When riding her bicycle she is late 10% of the time. When walking she is late 25% of the time. Given that she was on time, find the probability that she rides her bicycle.

38 – (MAT/20_HL_Winter_2015_Q8) – *Probability*

The continuous random variable X has the probability distribution function
$f(x) = A\sin(\ln(x)),\ 1 \le x \le 5$.

(a) Find the value of A to three decimal places. [2]

(b) Find the mode of X. [2]

(c) Find the value $\mathrm{P}(X \le 3 \mid X \ge 2)$. [2]

39 – (MAT/20_HL_Winter_2015_Q11) – *Probability*

A survey is conducted in a large office building. It is found that 30% of the office workers weigh less than 62 kg and that 25% of the office workers weigh more than 98 kg.
The weights of the office workers may be modelled by a normal distribution with mean μ and standard deviation σ.

(a) (i) Determine two simultaneous linear equations satisfied by μ and σ.

(ii) Find the values of μ and σ. [6]

(b) Find the probability that an office worker weighs more than 100 kg. [1]

There are elevators in the office building that take the office workers to their offices.
Given that there are 10 workers in a particular elevator,

(c) find the probability that at least four of the workers weigh more than 100 kg. [2]

Given that there are 10 workers in an elevator and at least one weighs more than 100 kg,

(d) find the probability that there are fewer than four workers exceeding 100 kg. [3]

The arrival of the elevators at the ground floor between $08{:}00$ and $09{:}00$ can be modelled by a Poisson distribution. Elevators arrive on average every 36 seconds.

(e) Find the probability that in any half hour period between $08{:}00$ and $09{:}00$ more than 60 elevators arrive at the ground floor. [3]

An elevator can take a maximum of 10 workers. Given that 400 workers arrive in a half hour period independently of each other,

(f) find the probability that there are sufficient elevators to take them to their offices. [3]

40 – (MAT/22_HL_Summer_2016_Q2) – *Probability*

A random variable X is normally distributed with mean 3 and variance 2^2.

(a) Find $\mathrm{P}(0 \le X \le 2)$. [2]

(b) Find $\mathrm{P}(|X| > 1)$. [3]

(c) If $\mathrm{P}(X > c) = 0.44$, find the value of c. [2]

41 – (MAT/21_HL_Summer_2016_Q6) – *Probability*

[Maximum mark: 6]

The heights of students in a single year group in a large school can be modelled by a normal distribution.

It is given that 40% of the students are shorter than $1.62\,\text{m}$ and 25% are taller than $1.79\,\text{m}$.

Find the mean and standard deviation of the heights of the students.

42 – (MAT/22_HL_Summer_2016_Q6) – *Probability*

[Maximum mark: 8]

A company produces rectangular sheets of glass of area 5 square metres. During manufacturing these glass sheets flaws occur at the rate of 0.5 per 5 square metres. It is assumed that the number of flaws per glass sheet follows a Poisson distribution.

(a) Find the probability that a randomly chosen glass sheet contains at least one flaw. [3]

Glass sheets with no flaws earn a profit of $\$5$. Glass sheets with at least one flaw incur a loss of $\$3$.

(b) Find the expected profit, P dollars, per glass sheet. [3]

This company also produces larger glass sheets of area 20 square metres. The rate of occurrence of flaws remains at 0.5 per 5 square metres.
A larger glass sheet is chosen at random.

(c) Find the probability that it contains no flaws. [2]

43 – (MAT/21_HL_Summer_2016_Q10) – *Probability*

[Maximum mark: 8]

Students sign up at a desk for an activity during the course of an afternoon. The arrival of each student is independent of the arrival of any other student and the number of students arriving per hour can be modelled as a Poisson distribution with a mean of λ.

The desk is open for 4 hours. If exactly 5 people arrive to sign up for the activity during that time find the probability that exactly 3 of them arrived during the first hour.

44 – (MAT/22_HL_Summer_2016_Q10) – *Probability*

A continuous random variable T has probability density function f defined by

$$f(t) = \begin{cases} \dfrac{t|\sin 2t|}{\pi}, & 0 \le t \le \pi \\ 0, & \text{otherwise} \end{cases}$$

(a) Sketch the graph of $y = f(t)$. [2]

(b) Use your sketch to find the mode of T. [1]

(c) Find the mean of T. [2]

(d) Find the variance of T. [3]

(e) Find the probability that T lies between the mean and the mode. [2]

(f) (i) Find $\int_0^T f(t)\mathrm{d}t$ where $0 \le T \le \frac{\pi}{2}$.

(ii) Hence verify that the lower quartile of T is $\frac{\pi}{2}$. [5]

45 – (MAT/21_HL_Summer_2016_Q13) – *Statistics, Probability*

Six balls numbered $1, 2, 2, 3, 3, 3$ are placed in a bag. Balls are taken one at a time from the bag at random and the number noted. Throughout the question a ball is always replaced before the next ball is taken.

(a) A single ball is taken from the bag. Let X denote the value shown on the ball. Find $\mathrm{E}(X)$. [2]

(b) Three balls are taken from the bag. Find the probability that

(i) the total of the three numbers is 5;

(ii) the median of the three numbers is 1. [6]

(c) Ten balls are taken from the bag. Find the probability that less than four of the balls are numbered 2. [3]

(d) Find the least number of balls that must be taken from the bag for the probability of taking out at least one ball numbered 2 to be greater than 0.95. [3]

(e) Another bag also contains balls numbered $1, 2$ or 3.
Eight balls are to be taken from this bag at random. It is calculated that the expected number of balls numbered 1 is 4.8, and the variance of the number of balls numbered 2 is 1.5.

Find the least possible number of balls numbered 3 in this bag. [8]

46 – (MAT/20_HL_Winter_2016_Q1) – *Probability*

A random variable X has a probability distribution given in the following table.

x	0.5	1.5	2.5	3.5	4.5	5.5
$\mathrm{P}(X=x)$	0.12	0.18	0.20	0.28	0.14	0.08

(a) Determine the value of $\mathrm{E}(X^2)$. [2]

(b) Find the value of $\mathrm{Var}(X)$. [3]

47 – (MAT/20_HL_Winter_2016_Q3) – *Probability*

A discrete random variable X follows a Poisson distribution $\mathrm{Po}(\mu)$.

(a) Show that $\mathrm{P}(X = x + 1) = \dfrac{\mu}{x+1} \times \mathrm{P}(X = x), x \in \mathbb{N}$. [3]

(b) Given that $\mathrm{P}(X = 2) = 0.241667$ and $\mathrm{P}(X = 3) = 0.112777$, use part (a) to find the value of μ. [3]

48 – (MAT/20_HL_Winter_2016_Q8) – *Probability*

A random variable X is normally distributed with mean μ and standard deviation σ, such that $\mathrm{P}(X < 30.31) = 0.1180$ and $\mathrm{P}(X > 42.52) = 0.3060$.

(a) Find μ and σ. [6]

(b) Find $\mathrm{P}(|X - \mu| < 1.2\sigma)$. [2]

49 – (MAT/20_HL_Winter_2016_Q11) – *Probability*

A Chocolate Shop advertises free gifts to customers that collect three vouchers.
The vouchers are placed at random into 10% of all chocolate bars sold at this shop.
Kati buys some of these bars and she opens them one at a time to see if they contain a voucher. Let $\mathrm{P}(X = n)$ be the probability that Kati obtains her third voucher on the nth bar opened.

(It is assumed that the probability that a chocolate bar contains a voucher stays at 10% throughout the question.)

(a) Show that $\mathrm{P}(X = 3) = 0.001$ and $\mathrm{P}(X = 4) = 0.0027$. [3]

It is given that $\mathrm{P}(X = n) = \dfrac{n^2 + an + b}{2000} \times 0.9^{n-3}$ for $n \geq 3$, $n \in \mathbb{N}$.

(b) Find the values of the constants a and b. [5]

(c) Deduce that $\dfrac{\mathrm{P}(X = n)}{\mathrm{P}(X = n-1)} = \dfrac{0.9(n-1)}{n-3}$ for $n > 3$. [4]

(d) (i) Hence show that X has two modes m_1 and m_2.

(ii) State the values of m_1 and m_2. [5]

Kati's mother goes to the shop and buys x chocolate bars. She takes the bars home for Kati to open.

(e) Determine the minimum value of x such that the probability Kati receives at least one free gift is greater than 0.5. [3]

50 – (MAT/21_HL_Summer_2017_Q1) – *Probability*

Consider two events A and B such that $\mathrm{P}(A) = k$, $\mathrm{P}(B) = 3k$, $\mathrm{P}(A \cap B) = k^2$ and $\mathrm{P}(A \cup B) = 0.5$.

(a) Calculate k; [3]

(b) Find $\mathrm{P}(A' \cap B)$. [3]

51 – (MAT/22_HL_Summer_2017_Q1) – *Probability, Statistics*

There are 75 players in a golf club who take part in a golf tournament. The scores obtained on the 18th hole are as shown in the following table.

Score	2	3	4	5	6	7
Frequency	3	15	28	17	9	3

(a) One of the players is chosen at random. Find the probability that this player's score was 5 or more. [2]

(b) Calculate the mean score. [2]

52 – (MAT/22_HL_Summer_2017_Q3) – *Statistics, Probability*

Packets of biscuits are produced by a machine. The weights X, in grams, of packets of biscuits can be modelled by a normal distribution where $X \sim \mathrm{N}(\mu, \sigma^2)$. A packet of biscuits is considered to be underweight if it weighs less than 250 grams.

(a) Given that $\mu = 253$ and $\sigma = 1.5$ find the probability that a randomly chosen packet of biscuits is underweight. [2]

The manufacturer makes the decision that the probability that a packet is underweight should be 0.002. To do this μ is increased and σ remains unchanged.

(b) Calculate the new value of μ giving your answer correct to two decimal places. [3]

The manufacturer is happy with the decision that the probability that a packet is underweight should be 0.002, but is unhappy with the way in which this was achieved. The machine is now adjusted to reduce σ and return μ to 253.

(c) Calculate the new value of σ. [2]

53 – (MAT/21_HL_Summer_2017_Q5) – *Probability*

When carpet is manufactured, small faults occur at random. The number of faults in Premium carpets can be modelled by a Poisson distribution with mean 0.5 faults per $20\,\mathrm{m}^2$.

Mr Jones chooses Premium carpets to replace the carpets in his office building. The office building has 10 rooms, each with the area of $80\,\mathrm{m}^2$.

(a) Find the probability that the carpet laid in the first room has fewer than three faults. [3]

(b) Find the probability that exactly seven rooms will have fewer than three faults in the carpet. [3]

54 – (MAT/22_HL_Summer_2017_Q5) – *Probability*

John likes to go sailing every day in July. To help him make a decision on whether it is safe to go sailing he classifies each day in July as windy or calm. Given that a day in July is calm, the probability that the next day is calm is 0.9. Given that a day in July is windy, the probability that the next day is calm is 0.3. The weather forecast for the 1st July predicts that the probability that it will be calm is 0.8.

(a) Draw a tree diagram to represent this information for the first three days of July. [3]

(b) Find the probability that the 3rd July is calm. [2]

(c) Find the probability that the 1st July was calm given that the 3rd July is windy. [4]

55 - (MAT/21_HL_Summer_2017_Q9) - *Probability, Statistics*

The times taken for male runners to complete a marathon can be modelled by a normal distribution with a mean 196 minutes and a standard deviation 24 minutes.

(a) Find the probability that a runner selected at random will complete the marathon in less than 3 hours. [2]

It is found that 5% of the male runners complete the marathon in less than T_1 minutes.

(b) Calculate T_1. [2]

The times taken for female runners to complete the marathon can be modelled by a normal distribution with a mean 210 minutes. It is found that 58% of female runners complete the marathon between 185 and 235 minutes.

(c) Find the standard deviation of the times taken by female runners. [4]

56 – (MAT/22_HL_Summer_2017_Q10) – *Probability, Statistics*

A continuous random variable X has probability density function f given by

$$f(x) = \begin{cases} \dfrac{x^2}{a} + b, & 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$ where a and b are positive constants.

It is given that $\mathrm{P}(X \ge 2) = 0.75$.

(a) Show that $a = 32$ and $b = \dfrac{1}{12}$. [5]

(b) Find $\mathrm{E}(X)$. [2]

(c) Find $\mathrm{Var}(X)$. [2]

(d) Find the median of X. [3]

Eight independent observations of X are now taken and the random variable Y is the number of observations such that $X \ge 2$.

(e) Find $\mathrm{E}(Y)$. [2]

(f) Find $\mathrm{P}(Y \ge 3)$. [1]

57 – (MAT/20_HL_Winter_2017_Q2) – *Probability*

[Maximum mark: 6]

Events A and B are such that $\mathrm{P}(A \cup B) = 0.95$, $\mathrm{P}(A \cap B) = 0.6$ and $\mathrm{P}(A \mid B) = 0.75$.

(a) Find $\mathrm{P}(B)$. [2]

(b) Find $\mathrm{P}(A)$. [2]

(c) Hence show that events A' and B are independent. [2]

58 – (MAT/20_HL_Winter_2017_Q6) – *Probability*

[Maximum mark: 6]

The number of bananas that Lucca eats during any particular day follows a Poisson distribution with mean 0.2.

(a) Find the probability that Lucca eats at least one banana in a particular day. [2]

(b) Find the expected number of weeks in the year in which Lucca eats no bananas. [4]

59 - (MAT/22_HL_Summer_2018_Q3) - *Probability*

The random variable X has a normal distribution with mean $\mu = 50$ and variance $\sigma^2 = 16$.

(a) Sketch the probability density function for X, and shade the region representing $\mathrm{P}(\mu - 2\sigma < X < \mu + \sigma)$. [2]

(b) Find the value of $\mathrm{P}(\mu - 2\sigma < X < \mu + \sigma)$. [2]

(c) Find the value of k for which $\mathrm{P}(\mu - k\sigma < X < \mu + k\sigma) = 0.5$. [2]

60 – (MAT/21_HL_Summer_2018_Q4) – *Probability*

The age, L, in years, of a wolf can be modelled by the normal distribution $L \sim \mathrm{N}(8, 5)$.

(a) Find the probability that a wolf selected at random is at least 5 years old. [2]

Eight wolves are independently selected at random and their ages recorded.

(b) Find the probability that more than six of these wolves are at least 5 years old. [3]

61 – (MAT/21_HL_Summer_2018_Q6) – *Probability*

[Maximum mark: 5]

The mean number of squirrels in a certain area is known to be 3.2 squirrels per hectare of woodland. Within this area, there is a 56 hectare woodland nature reserve. It is known that there are currently at least 168 squirrels in this reserve.

Assuming the population of squirrels follow a Poisson distribution, calculate the probability that there are more than 190 squirrels in the reserve.

62 – (MAT/22_HL_Summer_2018_Q8) – *Probability*

The random variable X has a binomial distribution with parameters n and p.
It is given that $\mathrm{E}(X) = 3.5$.

(a) Find the least possible value of n. [2]

It is further given that $\mathrm{P}(X \leq 1) = 0.09478$ correct to 4 significant figures.

(b) Determine the value of n and the value of p. [5]

63 – (MAT/22_HL_Summer_2018_Q9) – *Probability*

The number of taxis arriving at Cardiff Central railway station can be modelled by a Poisson distribution. During busy periods of the day, taxis arrive at a mean rate of 5.3 taxis every 10 minutes. Let T represent a random 10 minute busy period.

(a) (i) Find the probability that exactly 4 taxis arrive during T.

(ii) Find the most likely number of taxis that would arrive during T.

(iii) Given that more than 5 taxis arrive during T, find the probability that exactly 7 taxis arrive during T. [7]

During quiet periods of the day, taxis arrive at a mean rate of 1.3 taxis every 10 minutes.

(b) Find the probability that during a period of 15 minutes, of which the first 10 minutes is busy and the next 5 minutes is quiet, that exactly 2 taxis arrive. [6]

64 – (MAT/21_HL_Summer_2018_Q10) – *Probability*

The continuous random variable X has probability density function f given by

$$f(x) = \begin{cases} 3ax & , \quad 0 \le x < 0.5 \\ a(2-x) & , \quad 0.5 \le x < 2 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(a) Show that $a = \frac{2}{3}$. [3]

(b) Find $\mathrm{P}(X < 1)$. [3]

(c) Given that $\mathrm{P}(s < X < 0.8) = 2 \times \mathrm{P}(2s < X < 0.8)$, and that $0.25 < s < 0.4$, find the value of s. [7]

65 – (MAT/20_HL_Winter_2018_Q3) – *Probability*

It is known that 56 % of Infiglow batteries have a life of less than 16 hours, and 94 % have a life less than 17 hours. It can be assumed that battery life is modelled by the normal distribution $\mathrm{N}(\mu, \sigma^2)$.

(a) Find the value of μ and the value of σ. [6]

(b) Find the probability that a randomly selected Infiglow battery will have a life of at least 15 hours. [2]

66 – (MAT/20_HL_Winter_2018_Q10) – *Probability*

Willow finds that she receives approximately 70 emails per working day.
She decides to model the number of emails received per working day using the random variable X, where X follows a Poisson distribution with mean 70.

(a) Using this distribution model, find

(i) $\mathrm{P}(X < 60)$

(ii) the standard deviation of X. [4]

(b) In order to test her model, Willow records the number of emails she receives per working day over a period of 6 months. The results are shown in the following table.

Number of emails received (x)	Number of days
$40 \le x \le 49$	2
$50 \le x \le 59$	15
$60 \le x \le 69$	40
$70 \le x \le 79$	53
$80 \le x \le 89$	0
$90 \le x \le 99$	1
$100 \le x \le 109$	3
$110 \le x \le 119$	6

From the table, calculate

(i) an estimate for the mean number of emails received per working day;

(ii) an estimate for the standard deviation of the number of emails received per working day. [5]

(c) Give one piece of evidence that suggests Willow's Poisson distribution model is not a good fit. [1]

Archie works for a different company and knows that he receives emails according to a Poisson distribution, with a mean of λ emails per day.

(d) Suppose that the probability of Archie receiving more than 10 emails in total on any one day is 0.99. Find the value of λ. [3]

(e) Now suppose that Archie received exactly 20 emails in total in a consecutive two day period. Show that the probability that he received exactly 10 of them on the first day is independent of λ. [5]

67 – (MAT/22_HL_Summer_2019_Q2) – *Probability*

Timmy owns a shop. His daily income from selling his goods can be modelled as a normal distribution, with a mean daily income of \$820, and a standard deviation of \$230. To make a profit, Timmy's daily income needs to be greater than \$1000.

(a) Calculate the probability that, on a randomly selected day, Timmy makes a profit. [2]

The shop is open for 24 days every month.

(b) Calculate the probability that, in a randomly selected month, Timmy makes a profit on between 5 and 10 days (inclusive). [3]

68 – (MAT/21_HL_Summer_2019_Q3) – *Probability*

The marks achieved by eight students in a class test are given in the following list.

8	4	7	6	10	9	7	3

(a) Find

(i) the mean;

(ii) the standard deviation. [2]

(b) The teacher increases all the marks by 2. Write down the new value for

(i) the mean;

(ii) the standard deviation. [2]

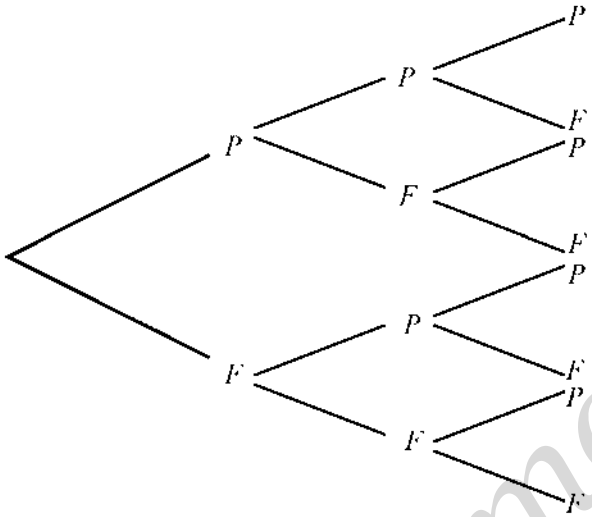
A ninth student also takes the test.

(c) Explain why the median is unchanged. [3]

69 – (MAT/22_HL_Summer_2019_Q3) – *Probability*

Iqbal attempts three practice papers in mathematics. The probability that he passes the first paper is 0.6. Whenever he gains a pass in a paper, his confidence increases so that the probability of him passing the next paper increases by 0.1. Whenever he fails a paper the probability of him passing the next paper is 0.6.

(a) Complete the given probability tree diagram for Iqbal's three attempts, labelling each branch with the correct probability. [3]



(b) Calculate the probability that Iqbal passes at least two of the papers he attempts. [2]

(c) Find the probability that Iqbal passes his third paper, given that he passed only one previous paper. [3]

70 – (MAT/21_HL_Summer_2019_Q9) – *Probability*

A café serves sandwiches and cakes. Each customer will choose one of the following three options; buy only a sandwich, buy only a cake or buy both a sandwich and a cake.
The probability that a customer buys a sandwich is 0.72 and the probability that a customer buys a cake is 0.45.

(a) Find the probability that a customer chosen at random will buy

(i) both a sandwich and a cake;

(ii) only a sandwich. [4]

On a typical day 200 customers come to the café.

(b) Find

(i) the expected number of cakes sold on a typical day;

(ii) the probability that more than 100 cakes will be sold on a typical day. [4]

It is known that 46 % of the customers who come to the café are male, and that 80 % of these buy a sandwich.

(c) (i) A customer is selected at random. Find the probability that the customer is male and buys a sandwich.

(ii) A female customer is selected at random. Find the probability that she buys a sandwich. [5]

71 – (MAT/22_HL_Summer_2019_Q10) – *Probability*

Steffi the stray cat often visits Will's house in search of food. Let X be the discrete random variable "the number of times per day that Steffi visits Will's house".
The random variable X can be modelled by a Poisson distribution with mean 2.1.

(a) Find the probability that on a randomly selected day, Steffi does not visit Will's house. [2]

Let Y be the discrete random variable "the number of times per day that Steffi is fed at Will's house". Steffi is only fed on the first four occasions that she visits each day.

(b) Copy and complete the probability distribution table for Y. [4]

y	0	1	2	3	4
$\mathrm{P}(Y=y)$					

(c) Hence find the expected number of times per day that Steffi is fed at Will's house. [3]

(d) In any given year of 365 days, the probability that Steffi does not visit Will for at most n days in total is 0.5 (to one decimal place). Find the value of n. [3]

(e) Show that the expected number of occasions per year on which Steffi visits Will's house and is not fed is at least 30. [4]

72 – (MAT/20_HL_Winter_2019_Q2) – *Probability*

[Maximum mark: 6]

The number of marathons that Audrey runs in any given year can be modelled by a Poisson distribution with mean 1.3.

(a) Calculate the probability that Audrey will run at least two marathons in a particular year. [2]

(b) Find the probability that she will run at least two marathons in exactly four out of the following five years. [4]

73 – (MAT/20_HL_Winter_2019_Q7) – *Probability*

[Maximum mark: 6]

Runners in an athletics club have season's best times for the $100\,\text{m}$, which can be modelled by a normal distribution with mean 11.6 seconds and standard deviation 0.8 seconds. To qualify for a particular competition a runner must have a season's best time of under 11 seconds. A runner from this club who has qualified for the competition is selected at random. Find the probability that he has a season's best time of under 10.7 seconds.

74 – (MAT/20_HL_Winter_2019_Q10) – *Probability*

[Maximum mark: 19]

A random variable X has probability density function

$$f(x)=\begin{cases} 3a & , \quad 0 \le x < 2 \\ a(x-5)(1-x) & , \quad 2 \le x \le b \\ 0 & , \quad \text{otherwise} \end{cases} \qquad a, b \in \mathbb{R}^+, 3 < b \le 5.$$

(a) Find, in terms of a, the probability that X lies between 1 and 3. [4]

Consider the case where $b = 5$.

(b) Sketch the graph of f. State the coordinates of the end points and any local maximum or minimum points, giving your answers in terms of a. [4]

(c) Find the value of

(i) a;

(ii) $\mathrm{E}(X)$;

(iii) the median of X. [11]

75 – (MAT/20_HL_Winter_2020_Q2) – *Probability*

[Maximum mark: 4]

Jenna is a keen book reader. The number of books she reads during one week can be modelled by a Poisson distribution with mean 2.6.

Determine the expected number of weeks in one year, of 52 weeks, during which Jenna reads at least four books.

76 – (MAT/20_HL_Winter_2020_Q5) – *Probability*

A survey of British holidaymakers found that 15% of those surveyed took a holiday in the Lake District in 2019.

(a) A random sample of 16 British holidaymakers was taken. The number of people in the sample who took a holiday in the Lake District in 2019 can be modelled by a binomial distribution.

(i) State two assumptions made in order for this model to be valid.

(ii) Find the probability that at least three people from the sample took a holiday in the Lake District in 2019. [4]

(b) From a random sample of n holidaymakers, the probability that at least one of them took a holiday in the Lake District in 2019 is greater than 0.999.

Determine the least possible value of n. [3]

77 – (MAT/20_HL_Winter_2020_Q9) – *Probability*

The weights, in grams, of individual packets of coffee can be modelled by a normal distribution, with mean $102\,\text{g}$ and standard deviation $8\,\text{g}$.

(a) Find the probability that a randomly selected packet has a weight less than $100\,\text{g}$. [2]

(b) The probability that a randomly selected packet has a weight greater than w grams is 0.444. Find the value of w. [2]

(c) A packet is randomly selected. Given that the packet has a weight greater than $105\,\text{g}$, find the probability that it has a weight greater than $110\,\text{g}$. [3]

(d) From a random sample of 500 packets, determine the number of packets that would be expected to have a weight lying within 1.5 standard deviations of the mean. [3]

(e) Packets are delivered to supermarkets in batches of 80. Determine the probability that at least 20 packets from a randomly selected batch have a weight less than $95\,\text{g}$. [4]

78 – (MathAA/21_HL_Summer_2021_Q2) – *Probability*

[Maximum mark: 6]

A company produces bags of sugar whose masses, in grams, can be modelled by a normal distribution with mean 1000 and standard deviation 3.5. A bag of sugar is rejected for sale if its mass is less than 995 grams.

(a) Find the probability that a bag selected at random is rejected. [2]

(b) Estimate the number of bags which will be rejected from a random sample of 100 bags. [1]

(c) Given that a bag is not rejected, find the probability that it has a mass greater than 1005 grams. [3]

79 – (MathAA/22_HL_Summer_2021_Q3) – *Probability*

[Maximum mark: 8]

At a school, $70\,\%$ of the students play a sport and $20\,\%$ of the students are involved in theatre. $18\,\%$ of the students do neither activity.

A student is selected at random.

(a) Find the probability that the student plays a sport and is involved in theatre. [2]

(b) Find the probability that the student is involved in theatre, but does not play a sport. [2]

At the school $48\,\%$ of the students are girls, and $25\,\%$ of the girls are involved in theatre.

A student is selected at random. Let G be the event "the student is a girl" and let T be the event "the student is involved in theatre".

(c) Find $\mathrm{P}(G \cap T)$. [2]

(d) Determine if the events G and T are independent. Justify your answer. [2]

80 – (MathAA/22_HL_Summer_2021_Q6) – *Probability*

[Maximum mark: 6]

A continuous random variable X has the probability density function f_n given by

$$f_n(x) = \begin{cases} (n+1)x^n, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

where $n \in \mathbb{R}$, $n \ge 0$.

(a) Show that $\mathrm{E}(X) = \dfrac{n+1}{n+2}$. [2]

(b) Show that $\mathrm{Var}(X) = \dfrac{n+1}{(n+2)^2(n+3)}$. [4]

81 – (MathAA/21_HL_Summer_2021_Q7) – *Probability*

[Maximum mark: 7]

A continuous random variable X has the probability density function f given by

$$f(x)=\begin{cases}\dfrac{x}{\sqrt{\left(x^2+k\right)^3}} & 0\le x\le 4\\ 0 & \text{otherwise}\end{cases}$$

where $k\in\mathbb{R}^+$.

(a) Show that $\sqrt{16+k}-\sqrt{k}=\sqrt{k}\sqrt{16+k}$. [5]

(b) Find the value of k. [2]

82 – (MathAA/22_HL_Summer_2021_Q10) – *Probability*

[Maximum mark: 15]

The flight times, T minutes, between two cities can be modelled by a normal distribution with a mean of 75 minutes and a standard deviation of σ minutes.

(a) Given that 2% of the flight times are longer than 82 minutes, find the value of σ. [3]

(b) Find the probability that a randomly selected flight will have a flight time of more than 80 minutes. [2]

(c) Given that a flight between the two cities takes longer than 80 minutes, find the probability that it takes less than 82 minutes. [4]

On a particular day, there are 64 flights scheduled between these two cities.

(d) Find the expected number of flights that will have a flight time of more than 80 minutes. [3]

(e) Find the probability that more than 6 of the flights on this particular day will have a flight time of more than 80 minutes. [3]

83 – (MathAA/20_HL_Winter_2021_Q3) – *Probability*

[Maximum mark: 7]

A factory manufactures lamps. It is known that the probability that a lamp is found to be defective is 0.05. A random sample of 30 lamps is tested.

(a) Find the probability that there is at least one defective lamp in the sample. [3]

(b) Given that there is at least one defective lamp in the sample, find the probability that there are at most two defective lamps. [4]

84 – (MathAA/20_HL_Winter_2021_Q7) – *Probability*

[Maximum mark: 6]

A continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} \arccos x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

The median of this distribution is m.

(a) Determine the value of m. [2]

(b) Given that $P(|X - m| \le a) = 0.3$, determine the value of a. [4]

85 – (MathAA/21_HL_Summer_2022_Q3) – *Probability*

[Maximum mark: 6]

A discrete random variable, X, has the following probability distribution:

x	0	1	2	3
$\mathrm{P}(X = x)$	0.41	$k - 0.28$	0.46	$0.29 - 2k^2$

(a) Show that $2k^2 - k + 0.12 = 0$. [1]

(b) Find the value of k, giving a reason for your answer. [3]

(c) Hence, find $\mathrm{E}(X)$. [2]

86 – (MathAA/21_HL_Summer_2022_Q5) – *Probability*

[Maximum mark: 6]

Let A and B be two independent events such that $\mathrm{P}(A \cap B') = 0.16$ and $\mathrm{P}(A' \cap B) = 0.36$.

(a) Given that $\mathrm{P}(A \cap B) = x$, find the value of x. [4]

(b) Find $\mathrm{P}(A' \mid B')$. [2]

87 – (MathAA/21_HL_Summer_2022_Q11) – *Probability*

[Maximum mark: 16]

A bakery makes two types of muffins: chocolate muffins and banana muffins.

The weights, C grams, of the chocolate muffins are normally distributed with a mean of $62\,\text{g}$ and standard deviation of $2.9\,\text{g}$.

(a) Find the probability that a randomly selected chocolate muffin weighs less than $61\,\text{g}$. [2]

(b) In a random selection of 12 chocolate muffins, find the probability that exactly 5 weigh less than $61\,\text{g}$. [2]

The weights, B grams, of the banana muffins are normally distributed with a mean of $68\,\text{g}$ and standard deviation of $3.4\,\text{g}$.

Each day $60\,\%$ of the muffins made are chocolate.

On a particular day, a muffin is randomly selected from all those made at the bakery.

(c) (i) Find the probability that the randomly selected muffin weighs less than $61\,\text{g}$.

(ii) Given that a randomly selected muffin weighs less than $61\,\text{g}$, find the probability that it is chocolate. [7]

The machine that makes the chocolate muffins is adjusted so that the mean weight of the chocolate muffins remains the same but their standard deviation changes to $\sigma\,\text{g}$. The machine that makes the banana muffins is not adjusted. The probability that the weight of a randomly selected muffin from these machines is less than $61\,\text{g}$ is now 0.157.

(d) Find the value of σ. [5]

88 – (MathAA/22_HL_Summer_2022_Q3) – *Probability*

[Maximum mark: 6]

Events A and B are independent and $\mathrm{P}(A) = 3\mathrm{P}(B)$.

Given that $\mathrm{P}(A \cup B) = 0.68$, find $\mathrm{P}(B)$.

89 – (MathAA/22_HL_Summer_2022_Q8) – *Probability*

[Maximum mark: 7]

Rachel and Sophia are competing in a javelin-throwing competition.

The distances, R metres, thrown by Rachel can be modelled by a normal distribution with mean 56.5 and standard deviation 3.

The distances, S metres, thrown by Sophia can be modelled by a normal distribution with mean 57.5 and standard deviation 1.8.

In the first round of competition, each competitor must have five throws. To qualify for the next round of competition, a competitor must record at least one throw of 60 metres or greater in the first round.

Find the probability that only one of Rachel or Sophia qualifies for the next round of competition.

90 – (MathAA/20_HL_Winter_2022_Q10) – *Probability*

[Maximum mark: 16]

The time worked, T, in hours per week by employees of a large company is normally distributed with a mean of 42 and standard deviation 10.7.

(a) Find the probability that an employee selected at random works more than 40 hours per week. [2]

(b) A group of four employees is selected at random. Each employee is asked in turn whether they work more than 40 hours per week. Find the probability that the fourth employee is the only one in the group who works more than 40 hours per week. [3]

(c) A large group of employees work more than 40 hours per week.

(i) An employee is selected at random from this large group.

Find the probability that this employee works less than 55 hours per week.

(ii) Ten employees are selected at random from this large group.

Find the probability that exactly five of them work less than 55 hours per week. [7]

It is known that $\mathrm{P}(a \le T \le b) = 0.904$ and that $\mathrm{P}(T > b) = 2\mathrm{P}(T < a)$, where a and b are numbers of hours worked per week. An employee who works fewer than a hours per week is considered to be a part-time employee.

(d) Find the maximum time, in hours per week, that an employee can work and still be considered part-time. [4]

1 – (MAT/21_HL_Summer_2012_Q1) – *Differentiation*

[Maximum mark: 5]

Given that the graph of $y = x^3 - 6x^2 + kx - 4$ has exactly one point at which the gradient is zero, find the value of k.

2 – (MAT/22_HL_Summer_2012_Q6) – *Differentiation*

(a) Sketch the curve $y = \frac{\cos x}{\sqrt{x^2+1}}$, $-4 \le x \le 4$ showing clearly the coordinates of the x-intercepts, any maximum points and any minimum points. *[4 marks]*

(b) Write down the gradient of the curve at $x = 1$. *[1 mark]*

(c) Find the equation of the normal to the curve at $x = 1$. *[3 marks]*

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3 – (MAT/21_HL_Summer_2012_Q10) – *Differentiation*

[Maximum mark: 8]

A triangle is formed by the three lines $y = 10 - 2x$, $y = mx$ and $y = -\frac{1}{m}x$, where $m > \frac{1}{2}$.
Find the value of m for which the area of the triangle is a minimum.

4 – (MAT/21_HL_Summer_2012_Q11) – *Differentiation*

The function $f(x) = 3\sin x + 4\cos x$ is defined for $0 < x < 2\pi$.

(a) Write down the coordinates of the minimum point on the graph of f. *[1 mark]*

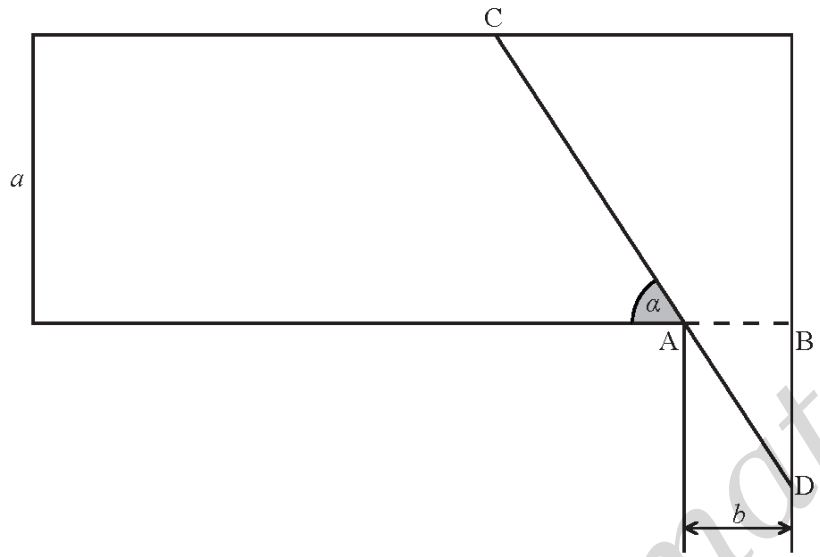
(b) The points $\mathrm{P}(p, 3)$ and $\mathrm{Q}(q, 3)$, $q > p$, lie on the graph of $y = f(x)$. Find p and q. *[2 marks]*

(c) Find the coordinates of the point, on $y = f(x)$, where the gradient of the graph is 3. *[4 marks]*

(d) Find the coordinates of the point of intersection of the normals to the graph at the points P and Q. *[7 marks]*

5 - (MAT/20_HL_Winter_2012_Q12) - *Differentiation*

The diagram shows the plan of an art gallery a metres wide. [AB] represents a doorway, leading to an exit corridor b metres wide. In order to remove a painting from the art gallery, CD (denoted by L) is measured for various values of α, as represented in the diagram.



(a) If α is the angle between [CD] and the wall, show that $L = \frac{a}{\sin\alpha} + \frac{b}{\cos\alpha}$, $0 < \alpha < \frac{\pi}{2}$. *[3 marks]*

(b) If $a = 5$ and $b = 1$, find the maximum length of a painting that can be removed through this doorway. *[4 marks]*

Let $a = 3k$ and $b = k$.

(c) Find $\frac{dL}{d\alpha}$. *[3 marks]*

(d) Find, in terms of k, the maximum length of a painting that can be removed from the gallery through this doorway. *[6 marks]*

(e) Find the minimum value of k if a painting 8 metres long is to be removed through this doorway. *[2 marks]*

6 – (MAT/22_HL_Summer_2013_Q12) – *Differentiation, Integration*

Consider the differential equation $y\frac{\mathrm{d}y}{\mathrm{d}x} = \cos 2x$.

(a) (i) Show that the function $y = \cos x + \sin x$ satisfies the differential equation.

(ii) Find the general solution of the differential equation. Express your solution in the form $y = f(x)$, involving a constant of integration.

(iii) For which value of the constant of integration does your solution coincide with the function given in part (i)? *[10 marks]*

(b) A different solution of the differential equation, satisfying $y = 2$ when $x = \frac{\pi}{4}$, defines a curve C.

(i) Determine the equation of C in the form $y = g(x)$, and state the range of the function g.

A region R in the xy plane is bounded by C, the x-axis and the vertical lines $x = 0$ and $x = \frac{\pi}{2}$.

(ii) Find the area of R.

(iii) Find the volume generated when that part of R above the line $y = 1$ is rotated about the x-axis through 2π radians. *[12 marks]*

7 – (MAT/21_HL_Summer_2013_Q13) – *Differentiation*

The function f has inverse f^{-1} and derivative $f'(x)$ for all $x \in \mathbb{R}$. For all functions with these properties you are given the result that for $a \in \mathbb{R}$ with $b = f(a)$ and $f'(a) \neq 0$

$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

(a) Verify that this is true for $f(x) = x^3 + 1$ at $x = 2$. *[6 marks]*

(b) Given that $g(x) = xe^{x^2}$, show that $g'(x) > 0$ for all values of x. *[3 marks]*

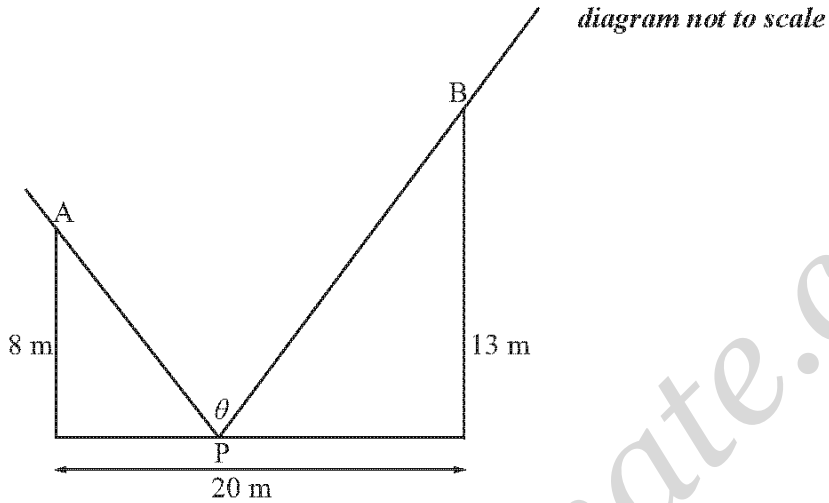
(c) Using the result given at the start of the question, find the value of the gradient function of $y = g^{-1}(x)$ at $x = 2$. *[4 marks]*

(d) (i) With f and g as defined in parts (a) and (b), solve $g \circ f(x) = 2$.

(ii) Let $h(x) = (g \circ f)^{-1}(x)$. Find $h'(2)$. *[6 marks]*

8 – (MAT/22_HL_Summer_2013_Q13) – *Differentiation*

A straight street of width 20 metres is bounded on its parallel sides by two vertical walls, one of height 13 metres, the other of height 8 metres. The intensity of light at point P at ground level on the street is proportional to the angle θ where $\theta = \mathrm{A\hat{P}B}$, as shown in the diagram.



(a) Find an expression for θ in terms of x, where x is the distance of P from the base of the wall of height 8m. *[2 marks]*

(b) (i) Calculate the value of θ when $x = 0$.

(ii) Calculate the value of θ when $x = 20$. *[2 marks]*

(c) Sketch the graph of θ, for $0 \le x \le 20$. *[2 marks]*

(d) Show that $\dfrac{\mathrm{d}\theta}{\mathrm{d}x} = \dfrac{5(744 - 64x - x^2)}{(x^2 + 64)(x^2 - 40x + 569)}$. *[6 marks]*

(e) Using the result in part (d), or otherwise, determine the value of x corresponding to the maximum light intensity at P. Give your answer to four significant figures. *[3 marks]*

(f) The point P moves across the street with speed $0.5\ \mathrm{m\,s^{-1}}$. Determine the rate of change of θ with respect to time when P is at the midpoint of the street. *[4 marks]*

9 – (MAT/22_HL_Summer_2014_Q9) – *Differentiation*

[Maximum mark: 5]

Sand is being poured to form a cone of height h cm and base radius r cm. The height remains equal to the base radius at all times. The height of the cone is increasing at a rate of $0.5\,\text{cm}\,\text{min}^{-1}$.

Find the rate at which sand is being poured, in $\text{cm}^3\,\text{min}^{-1}$, when the height is 4 cm.

10 – (MAT/21_HL_Summer_2014_Q10) – *Functions - Roots, Differentiation*

Let $f(x)=\dfrac{e^{2x}+1}{e^{x}-2}$.

(a) Find the equations of the horizontal and vertical asymptotes of the curve $y=f(x)$. *[4]*

(b) (i) Find $f'(x)$.

(ii) Show that the curve has exactly one point where its tangent is horizontal.

(iii) Find the coordinates of this point. *[8]*

(c) Find the equation of L_1, the normal to the curve at the point where it crosses the y-axis. *[4]*

The line L_2 is parallel to L_1 and tangent to the curve $y=f(x)$.

(d) Find the equation of the line L_2. *[5]*

11 – (MAT/22_HL_Summer_2014_Q10) – *Differentiation*

Consider the curve with equation $\left(x^2 + y^2\right)^2 = 4xy^2$.

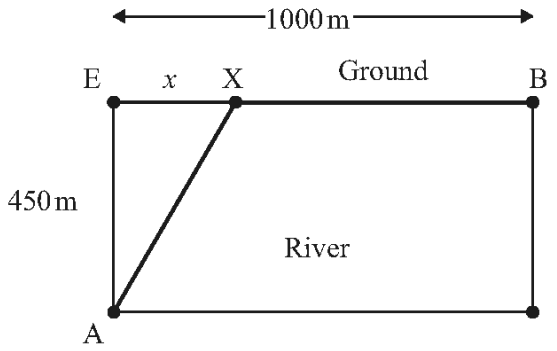
(a) Use implicit differentiation to find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$. *[5]*

(b) Find the equation of the normal to the curve at the point $(1, 1)$. *[3]*

12 – (MAT/22_HL_Summer_2014_Q12) – *Differentiation*

Engineers need to lay pipes to connect two cities A and B that are separated by a river of width 450 metres as shown in the following diagram. They plan to lay the pipes under the river from A to X and then under the ground from X to B. The cost of laying the pipes under the river is five times the cost of laying the pipes under the ground.

Let $\mathrm{EX} = x$.



Let k be the cost, in dollars per metre, of laying the pipes under the ground.

(a) Show that the total cost C, in dollars, of laying the pipes from A to B is given by $C = 5k\sqrt{202500 + x^2} + (1000 - x)k$. *[2]*

(b) (i) Find $\frac{\mathrm{d}C}{\mathrm{d}x}$.

(ii) Hence find the value of x for which the total cost is a minimum, justifying that this value is a minimum. *[7]*

(c) Find the minimum total cost in terms of k. *[1]*

The angle at which the pipes are joined is $\mathrm{A\hat{X}B} = \theta$.

(d) Find θ for the value of x calculated in (b). *[2]*

For safety reasons θ must be at least 120°.

Given this new requirement,

(e) (i) find the new value of x which minimises the total cost;

(ii) find the percentage increase in the minimum total cost. *[3]*

13 - (MAT/20_HL_Winter_2014_Q10) - *Differentiation*

Consider the triangle PQR where $\mathrm{Q\hat{P}R} = 30^\circ$, $\mathrm{PQ} = (x+2)\,\mathrm{cm}$ and $\mathrm{PR} = (5-x)^2\,\mathrm{cm}$, where $-2 < x < 5$.

(a) Show that the area, A cm^2, of the triangle is given by $A = \frac{1}{4}\left(x^3 - 8x^2 + 5x + 50\right)$. *[2]*

(b) (i) State $\frac{\mathrm{d}A}{\mathrm{d}x}$.

(ii) Verify that $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$ when $x = \frac{1}{3}$. *[3]*

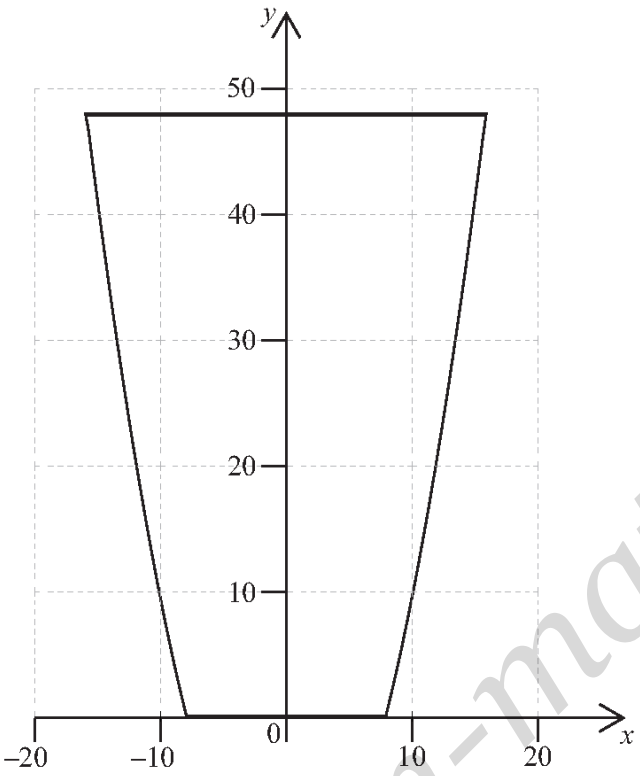
(c) (i) Find $\frac{\mathrm{d}^2A}{\mathrm{d}x^2}$ and hence justify that $x = \frac{1}{3}$ gives the maximum area of triangle PQR.

(ii) State the maximum area of triangle PQR.

(iii) Find QR when the area of triangle PQR is a maximum. *[7]*

14 – (MAT/20_HL_Winter_2014_Q13) – *Differentiation, Integration*

The vertical cross-section of a container is shown in the following diagram.



The curved sides of the cross-section are given by the equation $y = 0.25x^2 - 16$. The horizontal cross-sections are circular. The depth of the container is $48\,\text{cm}$.

(a) If the container is filled with water to a depth of h cm, show that the volume, $V\,\text{cm}^3$, of the water is given by $V = 4\pi\left(\frac{h^2}{2} + 16h\right)$. *[3]*

The container, initially full of water, begins leaking from a small hole at a rate given by $\frac{dV}{dt} = -\frac{250\sqrt{h}}{\pi(h+16)}$ where t is measured in seconds.

(b) (i) Show that $\frac{dh}{dt} = -\frac{250\sqrt{h}}{4\pi^2(h+16)^2}$.

(ii) State $\frac{dt}{dh}$ and hence show that $t = \frac{-4\pi^2}{250}\int\left(h^{\frac{3}{2}} + 32h^{\frac{1}{2}} + 256h^{-\frac{1}{2}}\right)dh$.

(iii) Find, correct to the nearest minute, the time taken for the container to become empty. (60 seconds = 1 minute) *[10]*

Once empty, water is pumped back into the container at a rate of $8.5\,\text{cm}^3\,\text{s}^{-1}$. At the same time, water continues leaking from the container at a rate of $\frac{250\sqrt{h}}{\pi(h+16)}\text{cm}^3\,\text{s}^{-1}$.

(c) Using an appropriate sketch graph, determine the depth at which the water ultimately stabilizes in the container. *[3]*

15 – (MAT/21_HL_Summer_2015_Q5) – *Differentiation*

[Maximum mark: 5]

A bicycle inner tube can be considered as a joined up cylinder of fixed length $200\,\text{cm}$ and radius $r\,\text{cm}$. The radius r increases as the inner tube is pumped up. Air is being pumped into the inner tube so that the volume of air in the tube increases at a constant rate of $30\,\text{cm}^3\,\text{s}^{-1}$. Find the rate at which the radius of the inner tube is increasing when $r = 2\,\text{cm}$.

16 – (MAT/21_HL_Summer_2015_Q6) – *Differentiation*

[Maximum mark: 4]

A function f is defined by $f(x) = x^3 + e^x + 1$, $x \in \mathbb{R}$. By considering $f'(x)$ determine whether f is a one-to-one or a many-to-one function.

17 – (MAT/21_HL_Summer_2015_Q9) – *Differentiation*

[Maximum mark: 7]

Find the equation of the normal to the curve $y = \dfrac{e^x \cos x \ln(x+e)}{\left(x^{17}+1\right)^5}$ at the point where $x = 0$.

In your answer give the value of the gradient, of the normal, to three decimal places.

18 – (MAT/22_HL_Summer_2015_Q11) – *Differentiation*

A curve is defined by $x^2 - 5xy + y^2 = 7$.

(a) Show that $\frac{dy}{dx} = \frac{5y - 2x}{2y - 5x}$. [3]

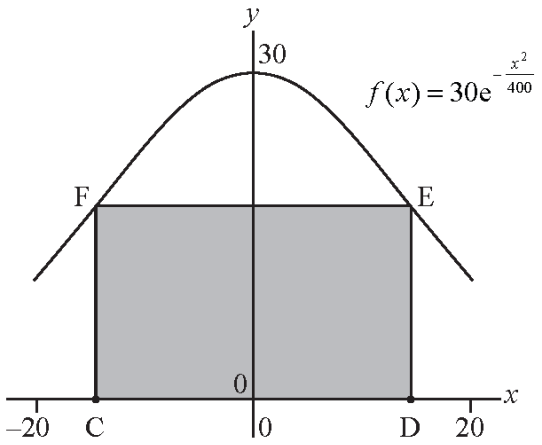
(b) Find the equation of the normal to the curve at the point $(6, 1)$. [4]

(c) Find the distance between the two points on the curve where each tangent is parallel to the line $y = x$. [8]

19 – (MAT/20_HL_Winter_2015_Q13) – *Differentiation*

The following diagram shows a vertical cross section of a building. The cross section of the roof of the building can be modelled by the curve $f(x) = 30e^{-\frac{x^2}{400}}$, where $-20 \le x \le 20$.

Ground level is represented by the x-axis.



(a) Find $f''(x)$. [4]

(b) Show that the gradient of the roof function is greatest when $x = -\sqrt{200}$. [3]

The cross section of the living space under the roof can be modelled by a rectangle CDEF with points $C(-a, 0)$ and $D(a, 0)$, where $0 < a \le 20$.

(c) Show that the maximum area A of the rectangle CDEF is $600\sqrt{2}e^{-\frac{1}{2}}$. [5]

(d) A function I is known as the Insulation Factor of CDEF. The function is defined as $I(a) = \frac{P(a)}{A(a)}$ where P = Perimeter and A = Area of the rectangle.

(i) Find an expression for P in terms of a.

(ii) Find the value of a which minimizes I.

(iii) Using the value of a found in part (ii) calculate the percentage of the cross sectional area under the whole roof that is not included in the cross section of the living space. [9]

20 – (MAT/22_HL_Summer_2016_Q7) – *Differentiation*

Consider the curve with equation $x^3 + y^3 = 4xy$.

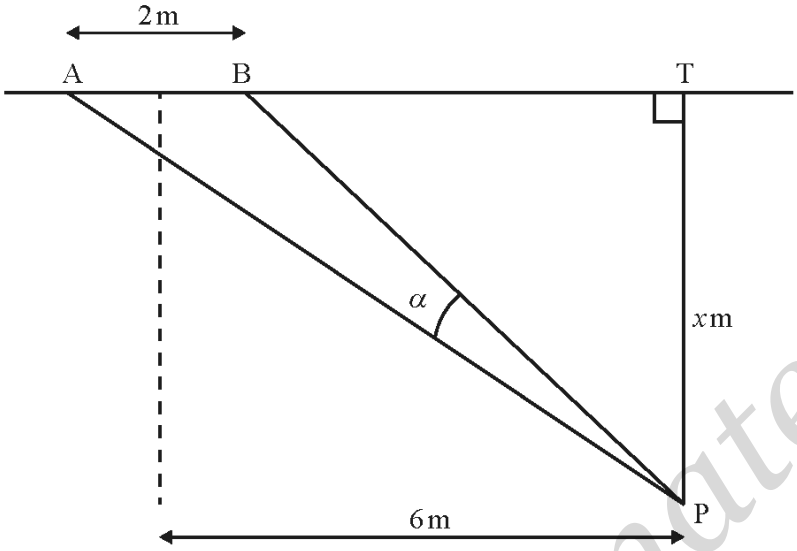
(a) Use implicit differentiation to show that $\frac{dy}{dx} = \frac{4y - 3x^2}{3y^2 - 4x}$. [3]

The tangent to this curve is parallel to the x-axis at the point where $x = k$, $k > 0$.

(b) Find the value of k. [5]

21 – (MAT/22_HL_Summer_2016_Q11) – *Differentiation*

Points A, B and T lie on a line on an indoor soccer field. The goal, $[\mathrm{AB}]$, is 2 metres wide. A player situated at point P kicks a ball at the goal. $[\mathrm{PT}]$ is perpendicular to (AB) and is 6 metres from a parallel line through the centre of $[\mathrm{AB}]$. Let PT be x metres and let $\alpha = \mathrm{A\hat{P}B}$ measured in degrees. Assume that the ball travels along the floor.



(a) Find the value of α when $x = 10$. [4]

(b) Show that $\tan\alpha = \dfrac{2x}{x^2 + 35}$. [4]

The maximum for $\tan\alpha$ gives the maximum for α.

(c) (i) Find $\dfrac{\mathrm{d}}{\mathrm{d}x}(\tan\alpha)$.

(ii) Hence or otherwise find the value of α such that $\dfrac{\mathrm{d}}{\mathrm{d}x}(\tan\alpha) = 0$.

(iii) Find $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}(\tan\alpha)$ and hence show that the value of α never exceeds 10°. [11]

(d) Find the set of values of x for which $\alpha \geq 7^\circ$. [3]

22 – (MAT/21_HL_Summer_2016_Q12) – *Differentiation*

Consider the curve, C defined by the equation $y^2 - 2xy = 5 - e^x$. The point A lies on C and has coordinates $(0, a)$, $a > 0$.

(a) Find the value of a. [2]

(b) Show that $\frac{dy}{dx} = \frac{2y - e^x}{2(y - x)}$. [4]

(c) Find the equation of the normal to C at the point A. [3]

(d) Find the coordinates of the second point at which the normal found in part (c) intersects C. [4]

(e) Given that $v = y^3$, $y > 0$, find $\frac{dv}{dx}$ at $x = 0$. [3]

23 – (MAT/22_HL_Summer_2016_Q12) – *Differentiation, Integration*

The functions f and g are defined by

$$f(x) = \frac{e^x + e^{-x}}{2}, x \in \mathbb{R}$$

$$g(x) = \frac{e^x - e^{-x}}{2}, x \in \mathbb{R}$$

(a) (i) Show that $\frac{1}{4f(x) - 2g(x)} = \frac{e^x}{e^{2x} + 3}$.

(ii) Use the substitution $u = e^x$ to find $\int_0^{\ln 3} \frac{1}{4f(x) - 2g(x)} dx$. Give your answer in the form $\frac{\pi\sqrt{a}}{b}$ where $a, b \in \mathbb{Z}^+$. [9]

Let $h(x) = nf(x) + g(x)$ where $n \in \mathbb{R}, n > 1$.

(b) (i) By forming a quadratic equation in e^x, solve the equation $h(x) = k$, where $k \in \mathbb{R}^+$.

(ii) Hence or otherwise show that the equation $h(x) = k$ has two real solutions provided that $k > \sqrt{n^2 - 1}$ and $k \in \mathbb{R}^+$. [8]

Let $t(x) = \frac{g(x)}{f(x)}$.

(c) (i) Show that $t'(x) = \frac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$ for $x \in \mathbb{R}$.

(ii) Hence show that $t'(x) > 0$ for $x \in \mathbb{R}$. [6]

24 – (MAT/20_HL_Winter_2016_Q6) – *Differentiation*

An earth satellite moves in a path that can be described by the curve $72.5x^2 + 71.5y^2 = 1$ where $x = x(t)$ and $y = y(t)$ are in thousands of kilometres and t is time in seconds.

Given that $\frac{dx}{dt} = 7.75 \times 10^{-5}$ when $x = 3.2 \times 10^{-3}$, find the possible values of $\frac{dy}{dt}$.

Give your answers in standard form.

25 – (MAT/20_HL_Winter_2016_Q10) – *Functions - Roots, Differentiation, Integration*

Let the function f be defined by $f(x) = \dfrac{2 - e^x}{2e^x - 1}$, $x \in D$.

(a) Determine D, the largest possible domain of f. [2]

(b) Show that the graph of f has three asymptotes and state their equations. [5]

(c) Show that $f'(x) = -\dfrac{3e^x}{\left(2e^x - 1\right)^2}$. [3]

(d) Use your answers from parts (b) and (c) to justify that f has an inverse and state its domain. [4]

(e) Find an expression for $f^{-1}(x)$. [4]

(f) Consider the region R enclosed by the graph of $y = f(x)$ and the axes.
Find the volume of the solid obtained when R is rotated through 2π about the y-axis. [4]

26 – (MAT/21_HL_Summer_2017_Q2) – *Differentiation*

The curve C is defined by equation $xy - \ln y = 1$, $y > 0$.

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. [4]

(b) Determine the equation of the tangent to C at the point $\left(\frac{2}{\mathrm{e}}, \mathrm{e}\right)$. [3]

27 – (MAT/22_HL_Summer_2017_Q2) – *Differentiation, Integration*

Consider the curve defined by the equation $4x^2 + y^2 = 7$.

(a) Find the equation of the normal to the curve at the point $\left(1, \sqrt{3}\right)$. [6]

(b) Find the volume of the solid formed when the region bounded by the curve, the x-axis for $x \geq 0$ and the y-axis for $y \geq 0$ is rotated through 2π about the x-axis. [3]

28 – (MAT/21_HL_Summer_2017_Q8) – *Differentiation*

A water trough which is 10 metres long has a uniform cross-section in the shape of a semicircle with radius 0.5 metres. It is partly filled with water as shown in the following diagram of the cross-section. The centre of the circle is O and the angle KOL is θ radians.

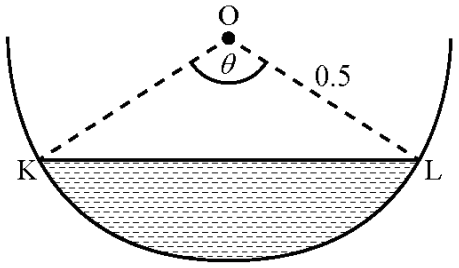


diagram not to scale

(a) Find an expression for the volume of water $V\left(m^3\right)$ in the trough in terms of θ. [3]

The volume of water is increasing at a constant rate of $0.0008\,\mathrm{m}^3\,\mathrm{s}^{-1}$.

(b) Calculate $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ when $\theta = \frac{\pi}{3}$. [4]

29 – (MAT/21_HL_Summer_2017_Q12) – *Functions - Roots, Graphs, Differentiation*

Consider $f(x) = -1 + \ln\left(\sqrt{x^2 - 1}\right)$.

(a) Find the largest possible domain D for f to be a function. [2]

The function f is defined by $f(x) = -1 + \ln\left(\sqrt{x^2 - 1}\right),\ x \in D$.

(b) Sketch the graph of $y = f(x)$ showing clearly the equations of asymptotes and the coordinates of any intercepts with the axes. [3]

(c) Explain why f is an even function. [1]

(d) Explain why the inverse function f^{-1} does not exist. [1]

The function g is defined by $g(x) = -1 + \ln\left(\sqrt{x^2 - 1}\right),\ x \in]1, \infty[$.

(e) Find the inverse function g^{-1} and state its domain. [4]

(f) Find $g'(x)$. [3]

(g) Hence, show that there are no solutions to

(i) $g'(x) = 0$;

(ii) $\left(g^{-1}\right)'(x) = 0$. [4]

30 – (MAT/20_HL_Winter_2017_Q10) – *Differentiation, Integration*

[Maximum mark: 17]

Consider the function $f(x) = \frac{\sqrt{x}}{\sin x}$, $0 < x < \pi$.

(a) (i) Show that the x-coordinate of the minimum point on the curve $y = f(x)$ satisfies the equation $\tan x = 2x$.

(ii) Determine the values of x for which $f(x)$ is a decreasing function. [7]

(b) Sketch the graph of $y = f(x)$ showing clearly the minimum point and any asymptotic behaviour. [3]

(c) Find the coordinates of the point on the graph of f where the normal to the graph is parallel to the line $y = -x$. [4]

Consider the region bounded by the curve $y = f(x)$, the x-axis and the lines $x = \frac{\pi}{6}$, $x = \frac{\pi}{3}$.

(d) This region is now rotated through 2π radians about the x-axis. Find the volume of revolution. [3]

31 – (MAT/20_HL_Winter_2017_Q11) – *Trigonometry, Differentiation*

[Maximum mark: 18]

Consider the function $f(x) = 2\sin^2 x + 7\sin 2x + \tan x - 9,\ 0 \le x < \frac{\pi}{2}$.

(a) (i) Determine an expression for $f'(x)$ in terms of x.

(ii) Sketch a graph of $y = f'(x)$ for $0 \le x < \frac{\pi}{2}$.

(iii) Find the x-coordinate(s) of the point(s) of inflexion of the graph of $y = f(x)$, labelling these clearly on the graph of $y = f'(x)$. [8]

(b) Let $u = \tan x$.

(i) Express $\sin x$ in terms of u.

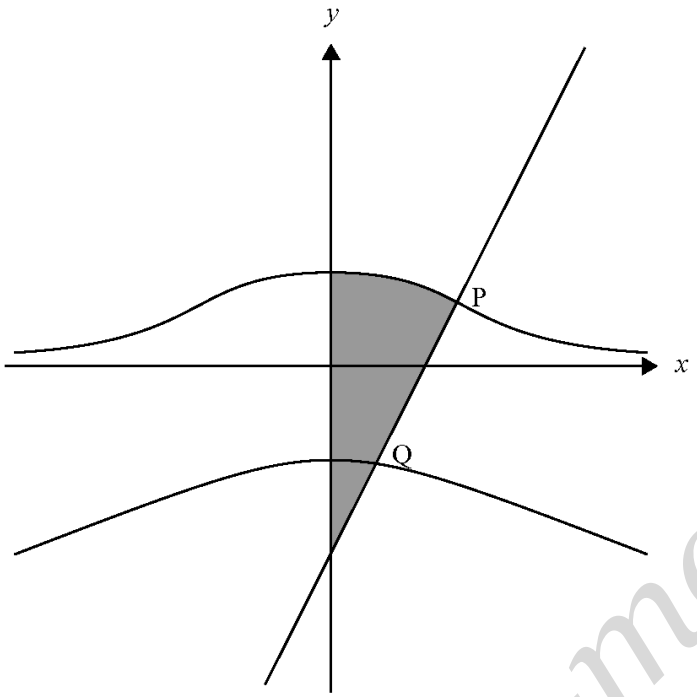
(ii) Express $\sin 2x$ in terms of u.

(iii) Hence show that $f(x) = 0$ can be expressed as $u^3 - 7u^2 + 15u - 9 = 0$. [7]

(c) Solve the equation $f(x) = 0$, giving your answers in the form $\arctan k$ where $k \in \mathbb{Z}$. [3]

32 – (MAT/21_HL_Summer_2018_Q9) – *Differentiation, Integration*

The following graph shows the two parts of the curve defined by the equation $x^2y = 5 - y^4$, and the normal to the curve at the point $\mathrm{P}(2, 1)$.



(a) Show that there are exactly two points on the curve where the gradient is zero. [7]

(b) Find the equation of the normal to the curve at the point P. [5]

(c) The normal at P cuts the curve again at the point Q. Find the x-coordinate of Q. [3]

(d) The shaded region is rotated by 2π about the y-axis. Find the volume of the solid formed. [7]

33 - (MAT/22_HL_Summer_2018_Q11) - *Differentiation*

A curve C is given by the implicit equation $x + y - \cos(xy) = 0$.

(a) Show that $\frac{dy}{dx} = -\left(\frac{1 + y\sin(xy)}{1 + x\sin(xy)}\right)$. [5]

(b) The curve $xy = -\frac{\pi}{2}$ intersects C at P and Q.

(i) Find the coordinates of P and Q.

(ii) Given that the gradients of the tangents to C at P and Q are m_1 and m_2 respectively, show that $m_1 \times m_2 = 1$. [7]

(c) Find the coordinates of the three points on C, nearest the origin, where the tangent is parallel to the line $y = -x$. [7]

34 – (MAT/20_HL_Winter_2018_Q5) – *Differentiation*

[Maximum mark: 5]

Differentiate from first principles the function $f(x) = 3x^3 - x$.

35 – (MAT/20_HL_Winter_2018_Q9) – *Functions - Roots, Differentiation*

The function f is defined by $f(x) = \dfrac{2\ln x + 1}{x - 3}$, $0 < x < 3$.

(a) Find $f'(x)$. [4]

(b) Hence, or otherwise, find the coordinates of the point of inflexion on the graph of $y = f(x)$. [4]

(c) Draw a set of axes showing x and y values between -3 and 3. On these axes

(i) sketch the graph of $y = f(x)$, showing clearly any axis intercepts and giving the equations of any asymptotes.

(ii) sketch the graph of $y = f^{-1}(x)$, showing clearly any axis intercepts and giving the equations of any asymptotes. [8]

(d) Hence, or otherwise, solve the inequality $f(x) > f^{-1}(x)$. [3]

36 – (MAT/21_HL_Summer_2019_Q1) – *Differentiation*

[Maximum mark: 4]

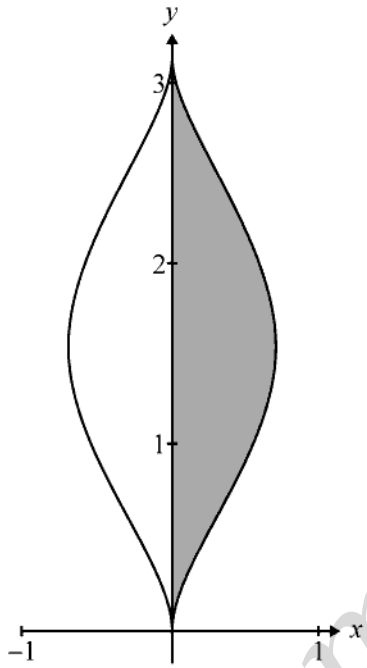
Let l be the tangent to the curve $y = xe^{2x}$ at the point $(1 , e^2)$.

Find the coordinates of the point where l meets the x-axis.

37 – (MAT/20_HL_Winter_2019_Q11) – *Differentiation, Integration*

[Maximum mark: 17]

The following diagram shows part of the graph of $2x^2 = \sin^3 y$ for $0 \le y \le \pi$.



(a) (i) Using implicit differentiation, find an expression for $\frac{dy}{dx}$.

(ii) Find the equation of the tangent to the curve at the point $\left(\frac{1}{4}, \frac{5\pi}{6}\right)$. [8]

The shaded region R is the area bounded by the curve, the y-axis and the lines $y = 0$ and $y = \pi$.

(b) Find the area of R. [3]

The region R is now rotated about the y-axis, through 2π radians, to form a solid.

(c) By writing $\sin^3 y$ as $(1 - \cos^2 y)\sin y$, show that the volume of the solid formed is $\frac{2\pi}{3}$. [6]

38 – (MAT/20_HL_Winter_2020_Q8) – *Differentiation*

A small bead is free to move along a smooth wire in the shape of the curve $y = \frac{10}{3 - 2e^{0.5x}}$ $(x \geq 0)$.

(a) Find an expression for $\frac{dy}{dx}$. [3]

At the point on the curve where $x = 4$, it is given that $\frac{dy}{dt} = -0.1 \text{ m s}^{-1}$.

(b) Find the value of $\frac{dx}{dt}$ at this exact same instant. [3]

39 – (MathAA/21_HL_Summer_2021_Q9) – *Differentiation, Kinematics, Relative Velocity*

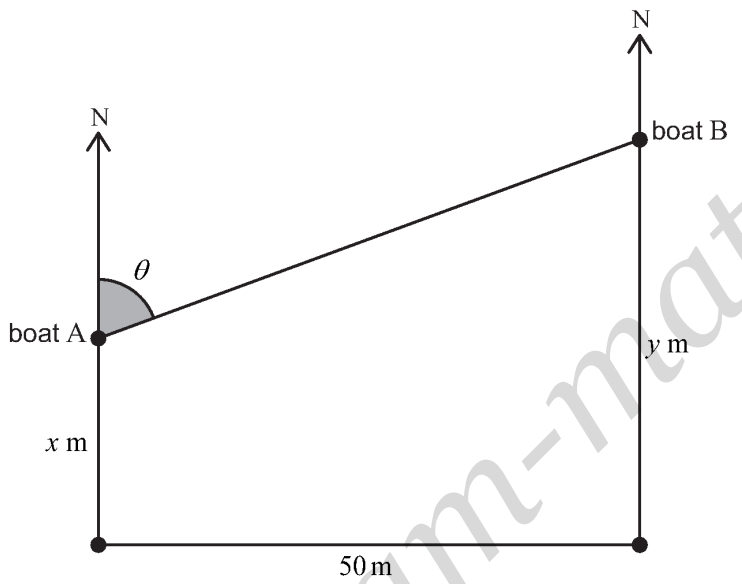
[Maximum mark: 7]

Two boats A and B travel due north.

Initially, boat B is positioned 50 metres due east of boat A.

The distances travelled by boat A and boat B, after t seconds, are x metres and y metres respectively. The angle θ is the radian measure of the bearing of boat B from boat A.
This information is shown on the following diagram.

diagram not to scale



(a) Show that $y = x + 50\cot\theta$. [1]

At time T, the following conditions are true.

Boat B has travelled 10 metres further than boat A.
Boat B is travelling at double the speed of boat A.
The rate of change of the angle θ is -0.1 radians per second.

(b) Find the speed of boat A at time T. [6]

40 – (MathAA/21_HL_Summer_2021_Q10) – *Exponentials - Logarithms, Differentiation*

[Maximum mark: 15]

Consider the function f defined by $f(x) = 90\,\mathrm{e}^{-0.5x}$ for $x \in \mathbb{R}^+$.

The graph of f and the line $y = x$ intersect at point P.

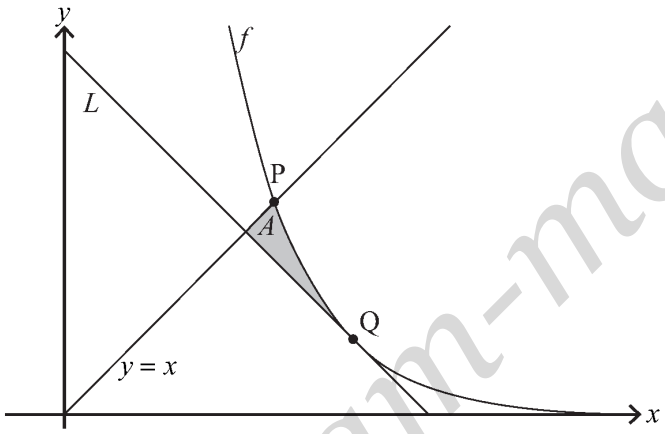
(a) Find the x-coordinate of P. [2]

The line L has a gradient of -1 and is a tangent to the graph of f at the point Q.

(b) Find the exact coordinates of Q. [4]

(c) Show that the equation of L is $y = -x + 2\ln 45 + 2$. [2]

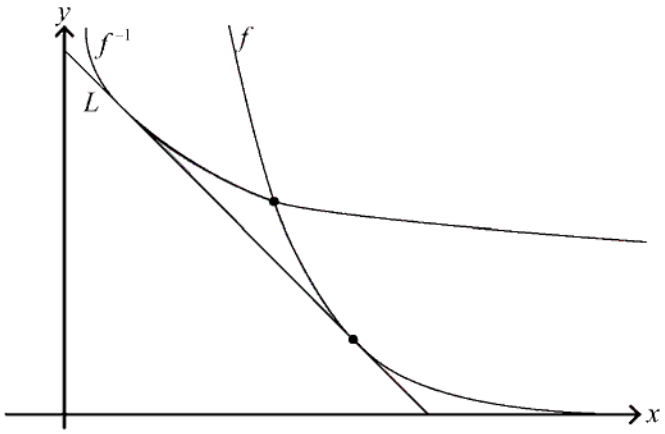
The shaded region A is enclosed by the graph of f and the lines $y = x$ and L.



(d) (i) Find the x-coordinate of the point where L intersects the line $y = x$.

(ii) Hence, find the area of A. [5]

The line L is tangent to the graphs of both f and the inverse function f^{-1}.



(e) Find the shaded area enclosed by the graphs of f and f^{-1} and the line L. [2]

41 – (MathAA/22_HL_Summer_2021_Q11) – *Differentiation*

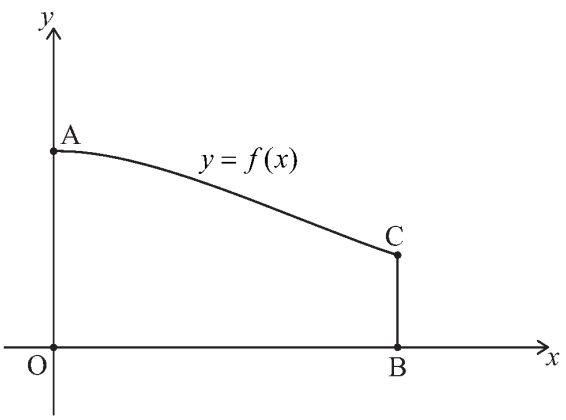
[Maximum mark: 18]

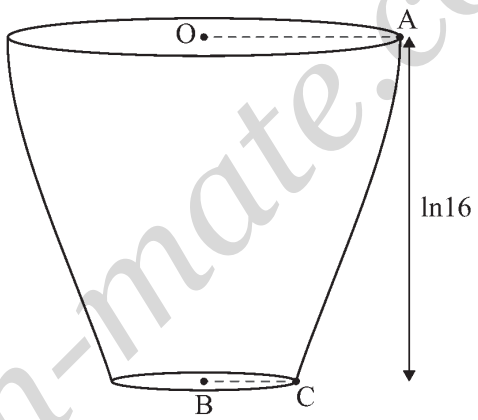
A function f is defined by $f(x)=\dfrac{ke^{\frac{x}{2}}}{1+e^{x}}$ where $x \in \mathbb{R}$, $x \geq 0$ and $k \in \mathbb{R}^{+}$.

The region enclosed by the graph of $y=f(x)$, the x-axis, the y-axis and the line $x=\ln 16$ is rotated $360°$ about the x-axis to form a solid of revolution.

(a) Show that the volume of the solid formed is $\dfrac{15k^2\pi}{34}$ cubic units. [6]

Pedro wants to make a small bowl with a volume of $300\,\text{cm}^3$ based on the result from part (a). Pedro's design is shown in the following diagrams.





The vertical height of the bowl, BO, is measured along the x-axis. The radius of the bowl's top is OA and the radius of the bowl's base is BC. All lengths are measured in cm.

(b) Find the value of k that satisfies the requirements of Pedro's design. [2]

(c) Find

(i) OA;

(ii) BC. [4]

For design purposes, Pedro investigates how the cross-sectional radius of the bowl changes.

(d) (i) By sketching the graph of a suitable derivative of f, find where the cross-sectional radius of the bowl is decreasing most rapidly.

(ii) State the cross-sectional radius of the bowl at this point. [6]

42 – (MathAA/22_HL_Summer_2021_Q12) – *Algebra-functions - Roots, Differentiation*

[Maximum mark: 21]

A function f is defined by $f(x) = \arcsin\left(\frac{x^2-1}{x^2+1}\right)$, $x \in \mathbb{R}$.

(a) Show that f is an even function. [1]

(b) By considering limits, show that the graph of $y = f(x)$ has a horizontal asymptote and state its equation. [2]

(c) (i) Show that $f'(x) = \frac{2x}{\sqrt{x^2}\left(x^2+1\right)}$ for $x \in \mathbb{R}$, $x \neq 0$.

(ii) By using the expression for $f'(x)$ and the result $\sqrt{x^2} = |x|$, show that f is decreasing for $x < 0$. [9]

A function g is defined by $g(x) = \arcsin\left(\frac{x^2-1}{x^2+1}\right)$, $x \in \mathbb{R}$, $x \geq 0$.

(d) Find an expression for $g^{-1}(x)$, justifying your answer. [5]

(e) State the domain of g^{-1}. [1]

(f) Sketch the graph of $y = g^{-1}(x)$, clearly indicating any asymptotes with their equations and stating the values of any axes intercepts. [3]

43 – (MathAA/20_HL_Winter_2021_Q8) – *Differentiation*

[Maximum mark: 8]

Consider the curve C given by $y - x - xy\ln(xy)$ where $x > 0$, $y > 0$.

(a) Show that $\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right)\left(1 + \ln(xy)\right) = 1$. [3]

(b) Hence find the equation of the tangent to C at the point where $x = 1$. [5]

44 – (MathAA/21_HL_Summer_2022_Q10) – *Algebra-functions - Roots, Differentiation, Integration*

[Maximum mark: 20]

Consider the function $f(x)=\sqrt{x^2-1}$, where $1 \le x \le 2$.

(a) Sketch the curve $y=f(x)$, clearly indicating the coordinates of the endpoints. [2]

(b) (i) Show that the inverse function of f is given by $f^{-1}(x)=\sqrt{x^2+1}$.

(ii) State the domain and range of f^{-1}. [5]

The curve $y=f(x)$ is rotated 2π about the y-axis to form a solid of revolution that is used to model a water container.

(c) (i) Show that the volume, $V\,\mathrm{m}^3$, of water in the container when it is filled to a height of h metres is given by $V=\pi\left(\frac{1}{3}h^3+h\right)$.

(ii) Hence, determine the maximum volume of the container. [5]

At $t=0$, the container is empty. Water is then added to the container at a constant rate of $0.4\,\mathrm{m}^3\,\mathrm{s}^{-1}$.

(d) Find the time it takes to fill the container to its maximum volume. [2]

(e) Find the rate of change of the height of the water when the container is filled to half its maximum volume. [6]

45 – (MathAA/22_HL_Summer_2022_Q10) – *Trigonometry, Differentiation*

[Maximum mark: 15]

A scientist conducted a nine-week experiment on two plants, A and B, of the same species. He wanted to determine the effect of using a new plant fertilizer. Plant A was given fertilizer regularly, while Plant B was not.

The scientist found that the height of Plant A, h_A cm, at time t weeks can be modelled by the function $h_A(t) = \sin(2t + 6) + 9t + 27$, where $0 \le t \le 9$.

The scientist found that the height of Plant B, h_B cm, at time t weeks can be modelled by the function $h_B(t) = 8t + 32$, where $0 < t < 9$.

(a) Use the scientist's models to find the initial height of

(i) Plant B;

(ii) Plant A correct to three significant figures. [3]

(b) Find the values of t when $h_A(t) = h_B(t)$. [3]

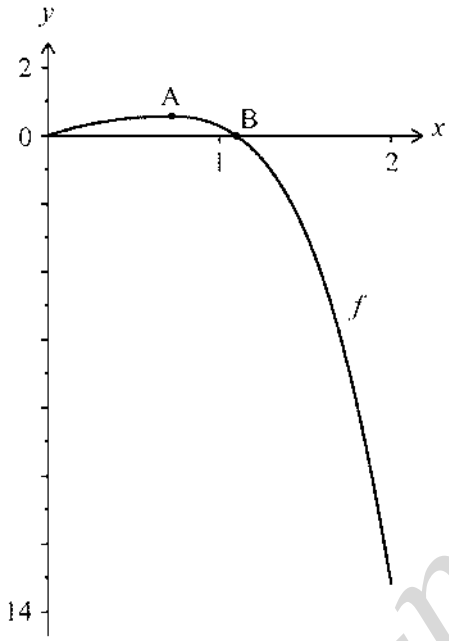
(c) For $t > 6$, prove that Plant A was always taller than Plant B. [3]

(d) For $0 \le t \le 9$, find the total amount of time when the rate of growth of Plant B was greater than the rate of growth of Plant A. [6]

46 – (MathAA/20_HL_Winter_2022_Q2) – *Differentiation, Integration*

[Maximum mark: 6]

The function f is defined as $f(x) = \ln(xe^x + 1) - x^4$, for $0 < x < 2$. The graph of f is shown in the following diagram.



The graph of f has a local maximum at point A. The graph intersects the x-axis at the origin and at point B.

(a) Find the coordinates of A. [2]

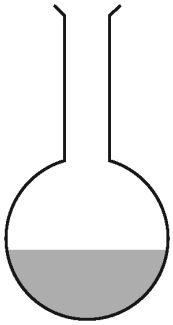
(b) Find the x-coordinate of B. [1]

(c) Find the total area enclosed by the graph of f, the x-axis and the line $x = 2$. [3]

47 – (MathAA/20_HL_Winter_2022_Q8) – *Differentiation*

[Maximum mark: 6]

The following diagram shows liquid in a round-bottomed glass flask, which is made of a sphere and a cylindrical neck.



Initially, the flask is empty. Liquid is poured into the flask at a rate of $2\,\text{cm}^3\text{s}^{-1}$. You may assume that the liquid does not reach the cylindrical neck.

The volume $V\,\text{cm}^3$ and the height $h\,\text{cm}$ of the liquid in the flask satisfy the equation $V = 5\pi h^2 - \frac{1}{3}\pi h^3$.

Find the rate of change of the height of the liquid in the flask at the instant when the volume of the liquid is $200\,\text{cm}^3$.

48 – (MathAA/20_HL_Winter_2022_Q11) – *Algebra-functions - Roots, Differentiation*

[Maximum mark: 15]

The function f is defined by $f(x) = e^{-2x}(3x - 4)$, where $x \in \mathbb{R}$.

(a) Find $f'(x)$. [3]

(b) Hence or otherwise, find the coordinates of the point on the graph of $y = f(x)$ where the tangent is parallel to the line $y = x$. [3]

The region enclosed by the curve $y = f(x)$, the x-axis and the y-axis is rotated through 2π radians about the x-axis to form a solid of revolution.

(c) Find the volume of this solid. [4]

Consider a function g such that $g(0) = 1$ and $g'(0) = 2$.

(d) Find the value of

(i) $(f \circ g)(0)$;

(ii) $(f \circ g)'(0)$. [5]

1 – (MAT/21_HL_Summer_2012_Q8) – *Integration*

[Maximum mark: 9]

A cone has height h and base radius r. Deduce the formula for the volume of this cone by rotating the triangular region, enclosed by the line $y = h - \frac{h}{r}x$ and the coordinate axes, through 2π about the y-axis.

2 – (MAT/20_HL_Winter_2012_Q8) – *Integration*

[Maximum mark: 7]

By using the substitution $x = \sin t$, find $\int \frac{x^3}{\sqrt{1-x^2}}\,\mathrm{d}x$.

3 – (MAT/20_HL_Winter_2012_Q9) – *Integration*

[Maximum mark: 7]

Find the area of the region enclosed by the curves $y = x^3$ and $x = y^2 - 3$.

4 – (MAT/21_HL_Summer_2013_Q4) – *Integration*

[Maximum mark: 6]

Let $f(x) = \sin(x-1)$, $0 \le x \le \frac{\pi}{2} + 1$

Find the volume of the solid formed when the region bounded by $y = f(x)$, and the lines $x = 0$, $y = 0$ and $y = 1$ is rotated by 2π about the y-axis.

5 – (MAT/22_HL_Summer_2013_Q4) – *Integration*

(a) Find $\int x\sec^2 x \, dx$. *[4 marks]*

(b) Determine the value of m if $\int_0^m x\sec^2 x \, dx = 0.5$, where $m > 0$. *[2 marks]*

6 – (MAT/22_HL_Summer_2013_Q12) – *Differentiation, Integration*

Consider the differential equation $y\frac{\mathrm{d}y}{\mathrm{d}x} = \cos 2x$.

(a) (i) Show that the function $y = \cos x + \sin x$ satisfies the differential equation.

(ii) Find the general solution of the differential equation. Express your solution in the form $y = f(x)$, involving a constant of integration.

(iii) For which value of the constant of integration does your solution coincide with the function given in part (i)? *[10 marks]*

(b) A different solution of the differential equation, satisfying $y = 2$ when $x = \frac{\pi}{4}$, defines a curve C.

(i) Determine the equation of C in the form $y = g(x)$, and state the range of the function g.

A region R in the xy plane is bounded by C, the x-axis and the vertical lines $x = 0$ and $x = \frac{\pi}{2}$.

(ii) Find the area of R.

(iii) Find the volume generated when that part of R above the line $y = 1$ is rotated about the x-axis through 2π radians. *[12 marks]*

7 – (MAT/20_HL_Winter_2013_Q10) – *Integration*

[Maximum mark: 7]

By using the substitution $x = 2\tan u$, show that $\int \frac{\mathrm{d}x}{x^2\sqrt{x^2+4}} = \frac{-\sqrt{x^2+4}}{4x} + C$.

8 – (MAT/20_HL_Winter_2013_Q13) – *Functions - Roots, Exponentials - Logarithms, Integration*

A function f is defined by $f(x)=\frac{1}{2}\left(e^{x}+e^{-x}\right)$, $x \in \mathbb{R}$.

(a) (i) Explain why the inverse function f^{-1} does not exist.

(ii) Show that the equation of the normal to the curve at the point P where $x=\ln 3$ is given by $9x+12y-9\ln 3-20=0$.

(iii) Find the x-coordinates of the points Q and R on the curve such that the tangents at Q and R pass through $(0, 0)$. *[14]*

(b) The domain of f is now restricted to $x \geq 0$.

(i) Find an expression for $f^{-1}(x)$.

(ii) Find the volume generated when the region bounded by the curve $y=f(x)$ and the lines $x=0$ and $y=5$ is rotated through an angle of 2π radians about the y-axis. *[8]*

9 – (MAT/22_HL_Summer_2014_Q3) – *Integration*

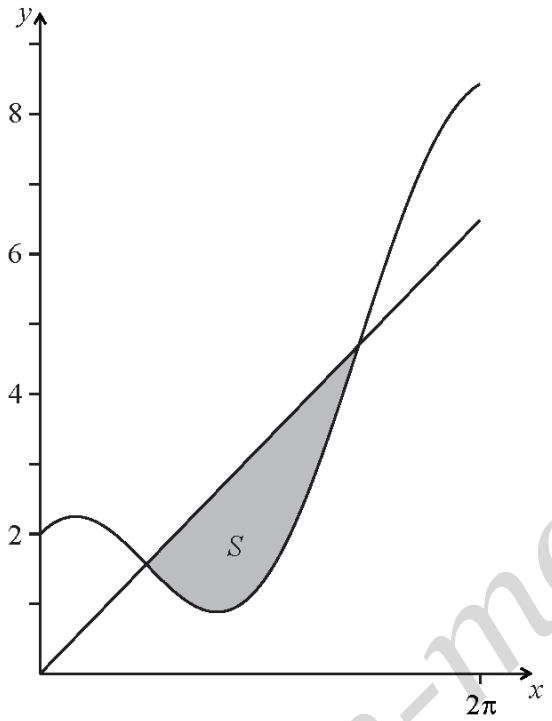
The graphs of $y = x^2 e^{-x}$ and $y = 1 - 2\sin x$ for $2 \le x \le 7$ intersect at points A and B. The x-coordinates of A and B are x_A and x_B.

(a) Find the value of x_A and the value of x_B. *[2]*

(b) Find the area enclosed between the two graphs for $x_A \le x \le x_B$. *[3]*

10 – (MAT/21_HL_Summer_2014_Q5) – *Integration*

The shaded region S is enclosed between the curve $y = x + 2\cos x$, for $0 \le x \le 2\pi$, and the line $y = x$, as shown in the diagram below.



(a) Find the coordinates of the points where the line meets the curve. *[3]*

The region S is rotated by 2π about the x-axis to generate a solid.

(b) (i) Write down an integral that represents the volume V of the solid.

(ii) Find the volume V. *[5]*

11 – (MAT/21_HL_Summer_2014_Q6) – *Integration*

Let $f(x) = x(x+2)^6$.

(a) Solve the inequality $f(x) > x$. *[5]*

(b) Find $\int f(x)\,\mathrm{d}x$. *[5]*

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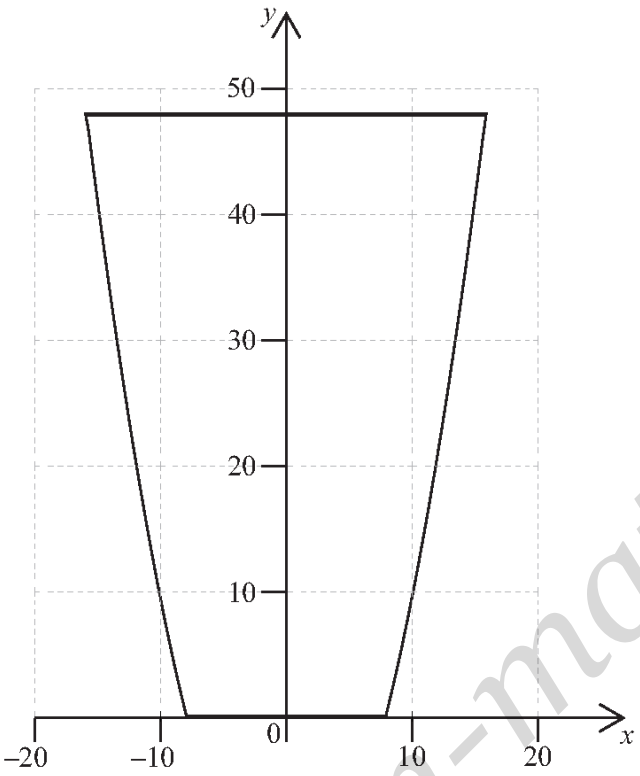
12 – (MAT/20_HL_Winter_2014_Q13) – *Differentiation, Integration*

The vertical cross-section of a container is shown in the following diagram.



The curved sides of the cross-section are given by the equation $y = 0.25x^2 - 16$. The horizontal cross-sections are circular. The depth of the container is $48\,\text{cm}$.

(a) If the container is filled with water to a depth of h cm, show that the volume, $V\,\text{cm}^3$, of the water is given by $V = 4\pi\left(\frac{h^2}{2} + 16h\right)$. *[3]*

The container, initially full of water, begins leaking from a small hole at a rate given by $\frac{dV}{dt} = -\frac{250\sqrt{h}}{\pi(h+16)}$ where t is measured in seconds.

(b) (i) Show that $\frac{dh}{dt} = -\frac{250\sqrt{h}}{4\pi^2(h+16)^2}$.

(ii) State $\frac{dt}{dh}$ and hence show that $t = \frac{-4\pi^2}{250}\int\left(h^{\frac{3}{2}} + 32h^{\frac{1}{2}} + 256h^{-\frac{1}{2}}\right)dh$.

(iii) Find, correct to the nearest minute, the time taken for the container to become empty. (60 seconds = 1 minute) *[10]*

Once empty, water is pumped back into the container at a rate of $8.5\,\text{cm}^3\,\text{s}^{-1}$. At the same time, water continues leaking from the container at a rate of $\frac{250\sqrt{h}}{\pi(h+16)}\text{cm}^3\,\text{s}^{-1}$.

(c) Using an appropriate sketch graph, determine the depth at which the water ultimately stabilizes in the container. *[3]*

13 – (MAT/21_HL_Summer_2015_Q1) – *Integration*

[Maximum mark: 4]

The region R is enclosed by the graph of $y = e^{-x^2}$, the x-axis and the lines $x = -1$ and $x = 1$. Find the volume of the solid of revolution that is formed when R is rotated through 2π about the x-axis.

14 – (MAT/22_HL_Summer_2015_Q6) – *Integration*

(a) Find the value of a and the value of b. [4]

(b) The region bounded by the graph of $y = \ln(5x + 10)$, the x-axis and the lines $x = \mathrm{e}$ and $x = 2\mathrm{e}$, is rotated through 2π radians about the x-axis. Find the volume generated. [2]

15 – (MAT/20_HL_Winter_2015_Q5) – *Integration*

A function is defined by $f(x) = x^2 + 2$, $x \geq 0$. A region R is enclosed by $y = f(x)$, the y-axis and the line $y = 4$.

(a) (i) Express the area of the region R as an integral with respect to y.

(ii) Determine the area of R, giving your answer correct to four significant figures. [3]

(b) Find the exact volume generated when the region R is rotated through 2π radians about the y-axis. [3]

16 – (MAT/22_HL_Summer_2016_Q12) – *Differentiation, Integration*

The functions f and g are defined by

$$f(x) = \frac{e^x + e^{-x}}{2}, x \in \mathbb{R}$$

$$g(x) = \frac{e^x - e^{-x}}{2}, x \in \mathbb{R}$$

(a) (i) Show that $\frac{1}{4f(x) - 2g(x)} = \frac{e^x}{e^{2x} + 3}$.

(ii) Use the substitution $u = e^x$ to find $\int_0^{\ln 3} \frac{1}{4f(x) - 2g(x)} dx$. Give your answer in the form $\frac{\pi\sqrt{a}}{b}$ where $a, b \in \mathbb{Z}^+$. [9]

Let $h(x) = nf(x) + g(x)$ where $n \in \mathbb{R}, n > 1$.

(b) (i) By forming a quadratic equation in e^x, solve the equation $h(x) = k$, where $k \in \mathbb{R}^+$.

(ii) Hence or otherwise show that the equation $h(x) = k$ has two real solutions provided that $k > \sqrt{n^2 - 1}$ and $k \in \mathbb{R}^+$. [8]

Let $t(x) = \frac{g(x)}{f(x)}$.

(c) (i) Show that $t'(x) = \frac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$ for $x \in \mathbb{R}$.

(ii) Hence show that $t'(x) > 0$ for $x \in \mathbb{R}$. [6]

17 – (MAT/20_HL_Winter_2016_Q10) – *Functions - Roots, Differentiation, Integration*

Let the function f be defined by $f(x) = \dfrac{2 - e^x}{2e^x - 1}$, $x \in D$.

(a) Determine D, the largest possible domain of f. [2]

(b) Show that the graph of f has three asymptotes and state their equations. [5]

(c) Show that $f'(x) = -\dfrac{3e^x}{\left(2e^x - 1\right)^2}$. [3]

(d) Use your answers from parts (b) and (c) to justify that f has an inverse and state its domain. [4]

(e) Find an expression for $f^{-1}(x)$. [4]

(f) Consider the region R enclosed by the graph of $y = f(x)$ and the axes.
Find the volume of the solid obtained when R is rotated through 2π about the y-axis. [4]

18 – (MAT/22_HL_Summer_2017_Q2) – *Differentiation, Integration*

Consider the curve defined by the equation $4x^2 + y^2 = 7$.

(a) Find the equation of the normal to the curve at the point $\left(1, \sqrt{3}\right)$. [6]

(b) Find the volume of the solid formed when the region bounded by the curve, the x-axis for $x \geq 0$ and the y-axis for $y \geq 0$ is rotated through 2π about the x-axis. [3]

19 – (MAT/21_HL_Summer_2017_Q4) – *Integration*

The region A is enclosed by the graph of $y = 2\arcsin(x-1) - \frac{\pi}{4}$, the y-axis and the line $y = \frac{\pi}{4}$.

(a) Write down a definite integral to represent the area of A. [4]

(b) Calculate the area of A. [2]

20 – (MAT/20_HL_Winter_2017_Q8) – *Integration*

[Maximum mark: 7]

By using the substitution $x^2 = 2\sec\theta$, show that $\int \frac{\mathrm{d}x}{x\sqrt{x^4 - 4}} = \frac{1}{4}\arccos\left(\frac{2}{x^2}\right) + c$.

21 – (MAT/20_HL_Winter_2017_Q10) – *Differentiation, Integration*

[Maximum mark: 17]

Consider the function $f(x) = \dfrac{\sqrt{x}}{\sin x}$, $0 < x < \pi$.

(a) (i) Show that the x-coordinate of the minimum point on the curve $y = f(x)$ satisfies the equation $\tan x = 2x$.

(ii) Determine the values of x for which $f(x)$ is a decreasing function. [7]

(b) Sketch the graph of $y = f(x)$ showing clearly the minimum point and any asymptotic behaviour. [3]

(c) Find the coordinates of the point on the graph of f where the normal to the graph is parallel to the line $y = -x$. [4]

Consider the region bounded by the curve $y = f(x)$, the x-axis and the lines $x = \dfrac{\pi}{6}$, $x = \dfrac{\pi}{3}$.

(d) This region is now rotated through 2π radians about the x-axis. Find the volume of revolution. [3]

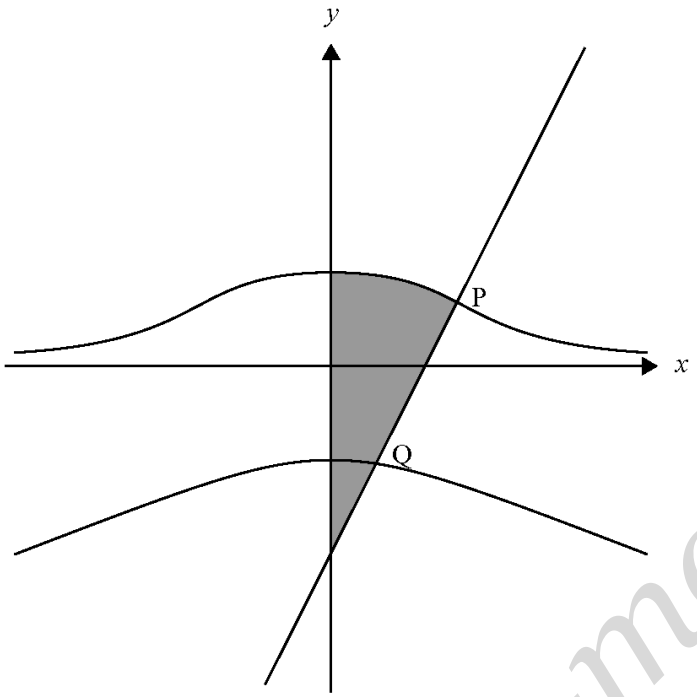
22 – (MAT/21_HL_Summer_2018_Q5) – *Integration*

(a) Given that $2x^3 - 3x + 1$ can be expressed in the form $Ax(x^2+1) + Bx + C$, find the values of the constants A, B and C. [2]

(b) Hence find $\int \frac{2x^3 - 3x + 1}{x^2 + 1} \mathrm{d}x$. [5]

23 – (MAT/21_HL_Summer_2018_Q9) – *Differentiation, Integration*

The following graph shows the two parts of the curve defined by the equation $x^2y = 5 - y^4$, and the normal to the curve at the point $\mathrm{P}(2, 1)$.



(a) Show that there are exactly two points on the curve where the gradient is zero. [7]

(b) Find the equation of the normal to the curve at the point P. [5]

(c) The normal at P cuts the curve again at the point Q. Find the x-coordinate of Q. [3]

(d) The shaded region is rotated by 2π about the y-axis. Find the volume of the solid formed. [7]

24 – (MAT/20_HL_Winter_2018_Q2) – *Integration*

[Maximum mark: 7]

A function f satisfies the conditions $f(0) = -4$, $f(1) = 0$ and its second derivative is $f''(x) = 15\sqrt{x} + \frac{1}{(x+1)^2}$, $x \geq 0$.

Find $f(x)$.

25 – (MAT/21_HL_Summer_2019_Q5) – *Integration*

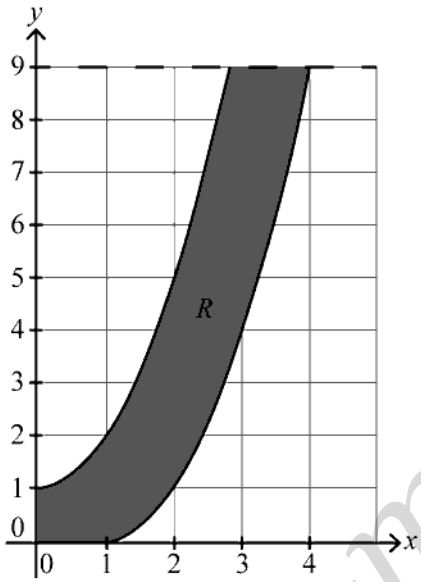
[Maximum mark: 6]

Use integration by parts to find $\int (\ln x)^2 \mathrm{d}x$.

26 – (MAT/21_HL_Summer_2019_Q7) – *Integration*

[Maximum mark: 7]

The function f is defined by $f(x) = (x-1)^2, x \geq 1$ and the function g is defined by $g(x) = x^2 + 1, x \geq 0$.
The region R is bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $y = 0$, $x = 0$ and $y = 9$ as shown on the following diagram.



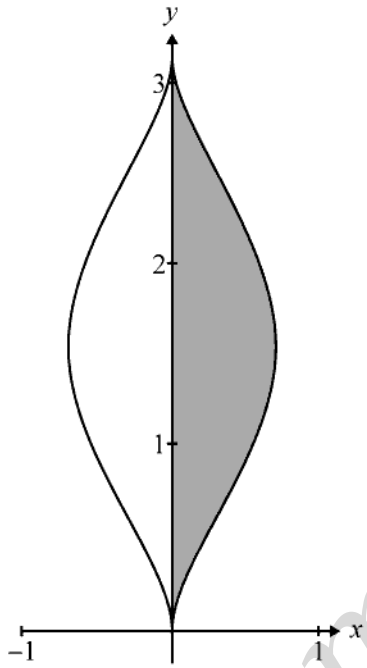
The shape of a clay vase can be modelled by rotating the region R through 360° about the y-axis.

Find the volume of clay used to make the vase.

27 – (MAT/20_HL_Winter_2019_Q11) – *Differentiation, Integration*

[Maximum mark: 17]

The following diagram shows part of the graph of $2x^2 = \sin^3 y$ for $0 \le y \le \pi$.



(a) (i) Using implicit differentiation, find an expression for $\frac{dy}{dx}$.

(ii) Find the equation of the tangent to the curve at the point $\left(\frac{1}{4}, \frac{5\pi}{6}\right)$. [8]

The shaded region R is the area bounded by the curve, the y-axis and the lines $y = 0$ and $y = \pi$.

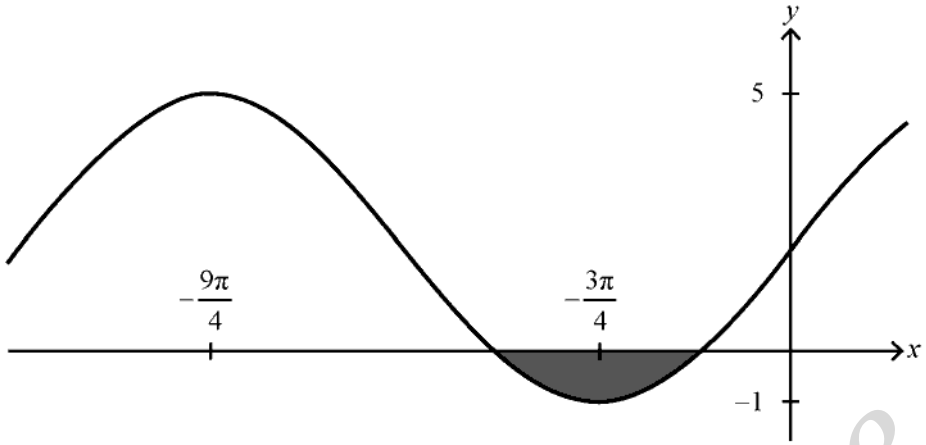
(b) Find the area of R. [3]

The region R is now rotated about the y-axis, through 2π radians, to form a solid.

(c) By writing $\sin^3 y$ as $(1 - \cos^2 y)\sin y$, show that the volume of the solid formed is $\frac{2\pi}{3}$. [6]

28 – (MAT/20_HL_Winter_2020_Q3) – *Trigonometry, Integration*

The following diagram shows part of the graph of $y = p + q\sin(rx)$. The graph has a local maximum point at $\left(-\frac{9\pi}{4}, 5\right)$ and a local minimum point at $\left(-\frac{3\pi}{4}, -1\right)$.



(a) Determine the values of p, q and r. [4]

(b) Hence find the area of the shaded region. [4]

29 – (MathAA/20_HL_Winter_2021_Q10) – *Algebra-functions - Roots, Graphs, Integration*

[Maximum mark: 18]

Consider the function $f(x) = \dfrac{x^2 - x - 12}{2x - 15}$, $x \in \mathbb{R}$, $x \neq \dfrac{15}{2}$.

(a) Find the coordinates where the graph of f crosses the

(i) x-axis;

(ii) y-axis. [3]

(b) Write down the equation of the vertical asymptote of the graph of f. [1]

(c) The oblique asymptote of the graph of f can be written as $y = ax + b$ where $a, b \in \mathbb{Z}$.

Find the value of a and the value of b. [4]

(d) Sketch the graph of f for $-30 \leq x \leq 30$, clearly indicating the points of intersection with each axis and any asymptotes. [3]

(e) (i) Express $\dfrac{1}{f(x)}$ in partial fractions.

(ii) Hence find the exact value of $\displaystyle\int_0^3 \frac{1}{f(x)}\,\mathrm{d}x$, expressing your answer as a single logarithm. [7]

30 – (MathAA/21_HL_Summer_2022_Q10) – *Algebra-functions - Roots, Differentiation, Integration*

[Maximum mark: 20]

Consider the function $f(x)=\sqrt{x^2-1}$, where $1 \le x \le 2$.

(a) Sketch the curve $y=f(x)$, clearly indicating the coordinates of the endpoints. [2]

(b) (i) Show that the inverse function of f is given by $f^{-1}(x)=\sqrt{x^2+1}$.

(ii) State the domain and range of f^{-1}. [5]

The curve $y=f(x)$ is rotated 2π about the y-axis to form a solid of revolution that is used to model a water container.

(c) (i) Show that the volume, $V\,\mathrm{m}^3$, of water in the container when it is filled to a height of h metres is given by $V=\pi\left(\frac{1}{3}h^3+h\right)$.

(ii) Hence, determine the maximum volume of the container. [5]

At $t=0$, the container is empty. Water is then added to the container at a constant rate of $0.4\,\mathrm{m}^3\,\mathrm{s}^{-1}$.

(d) Find the time it takes to fill the container to its maximum volume. [2]

(e) Find the rate of change of the height of the water when the container is filled to half its maximum volume. [6]

31 – (MathAA/22_HL_Summer_2022_Q2) – *Integration*

[Maximum mark: 5]

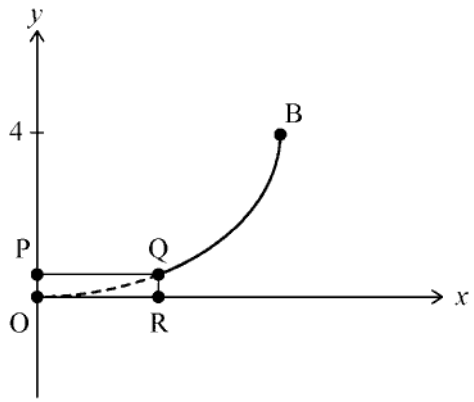
The derivative of a function g is given by $g'(x) = 3x^2 + 5e^x$, where $x \in \mathbb{R}$. The graph of g passes through the point $(0, 4)$. Find $g(x)$.

32 – (MathAA/22_HL_Summer_2022_Q6) – *Integration*

[Maximum mark: 5]

The following diagram shows the curve $\frac{x^2}{36}+\frac{(y-4)^2}{16}=1$, where $h \le y \le 4$.

diagram not to scale



The curve from point Q to point B is rotated $360°$ about the y-axis to form the interior surface of a bowl. The rectangle OPQR, of height h cm, is rotated $360°$ about the y-axis to form a solid base.

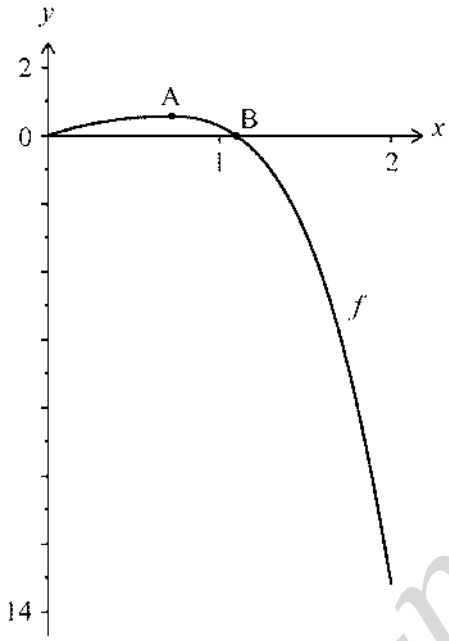
The bowl is assumed to have negligible thickness.

Given that the interior volume of the bowl is to be $285\,\mathrm{cm}^3$, determine the height of the base.

33 – (MathAA/20_HL_Winter_2022_Q2) – *Differentiation, Integration*

[Maximum mark: 6]

The function f is defined as $f(x) = \ln(xe^x + 1) - x^4$, for $0 < x < 2$. The graph of f is shown in the following diagram.



The graph of f has a local maximum at point A. The graph intersects the x-axis at the origin and at point B.

(a) Find the coordinates of A. [2]

(b) Find the x-coordinate of B. [1]

(c) Find the total area enclosed by the graph of f, the x-axis and the line $x = 2$. [3]

1 – (MathAA/21_HL_Summer_2021_Q12) – *Differential Equations*

[Maximum mark: 20]

The function f has a derivative given by $f'(x)=\frac{1}{x(k-x)}$, $x \in \mathbb{R}$, $x \neq 0$, $x \neq k$ where k is a positive constant.

(a) The expression for $f'(x)$ can be written in the form $\frac{a}{x}+\frac{b}{k-x}$, where a, $b \in \mathbb{R}$. Find a and b in terms of k. [3]

(b) Hence, find an expression for $f(x)$. [3]

Consider P, the population of a colony of ants, which has an initial value of 1200.

The rate of change of the population can be modelled by the differential equation $\frac{\mathrm{d}P}{\mathrm{d}t}=\frac{P(k-P)}{5k}$,

where t is the time measured in days, $t \geq 0$, and k is the upper bound for the population.

(c) By solving the differential equation, show that $P=\frac{1200k}{(k-1200)\mathrm{e}^{-\frac{t}{5}}+1200}$. [8]

At $t=10$ the population of the colony has doubled in size from its initial value.

(d) Find the value of k, giving your answer correct to four significant figures. [3]

(e) Find the value of t when the rate of change of the population is at its maximum. [3]

2 – (exam-mate QID138653) – *What Is Science* [0]

Which of the following statements is true of science?

A) Science is an investigation in a laboratory.

B) Science enables us to know the past and the future.

C) Science is a systematic study of natural phenomena.

D) Science can explain every phenomenon around us.

3 – (exam-mate QID138654) – *What Is Science* [0]

Which of the following is a natural phenomenon?

A) Air pollution

B) Formation of a rainbow

C) Construction of a dam

D) Invention of wheels

4 – (MathAA/21_HL_Summer_2022_Q12) – *Differential Equations*

[Maximum mark: 19]

Consider the differential equation $x^2\frac{dy}{dx} = y^2 - 2x^2$ for $x > 0$ and $y > 2x$. It is given that $y = 3$ when $x = 1$.

(a) Use Euler's method, with a step length of 0.1, to find an approximate value of y when $x = 1.5$. [4]

(b) Use the substitution $y = vx$ to show that $x\frac{dv}{dx} = v^2 - v - 2$. [3]

(c) (i) By solving the differential equation, show that $y = \frac{8x + x^4}{4 - x^3}$.

(ii) Find the actual value of y when $x = 1.5$.

(iii) Using the graph of $y = \frac{8x + x^4}{4 - x^3}$, suggest a reason why the approximation given by Euler's method in part (a) is not a good estimate to the actual value of y at $x = 1.5$. [12]

5 – (MathAA/22_HL_Summer_2022_Q12) – *Differential Equations*

[Maximum mark: 21]

The population, P, of a particular species of marsupial on a small remote island can be modelled by the logistic differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{N}\right)$$

where t is the time measured in years and k, N are positive constants.

The constant N represents the maximum population of this species of marsupial that the island can sustain indefinitely.

(a) In the context of the population model, interpret the meaning of $\frac{\mathrm{d}P}{\mathrm{d}t}$. [1]

(b) Show that $\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = k^2P\left(1 - \frac{P}{N}\right)\left(1 - \frac{2P}{N}\right)$. [4]

(c) Hence show that the population of marsupials will increase at its maximum rate when $P = \frac{N}{2}$. Justify your answer. [5]

(d) Hence determine the maximum value of $\frac{\mathrm{d}P}{\mathrm{d}t}$ in terms of k and N. [2]

Let P_0 be the initial population of marsupials.

(e) By solving the logistic differential equation, show that its solution can be expressed in the form

$$kt = \ln\frac{P}{P_0}\left(\frac{N - P_0}{N - P}\right).$$ [7]

After 10 years, the population of marsupials is $3P_0$. It is known that $N = 4P_0$.

(f) Find the value of k for this population model. [2]

1 – (MAT/22_HL_Summer_2012_Q12) – *Kinematics*

A particle moves in a straight line with velocity v metres per second. At any time t seconds, $0 \le t < \frac{3\pi}{4}$, the velocity is given by the differential equation $\frac{dv}{dt} + v^2 + 1 = 0$.

It is also given that $v = 1$ when $t = 0$.

(a) Find an expression for v in terms of t. *[7 marks]*

(b) Sketch the graph of v against t, clearly showing the coordinates of any intercepts, and the equations of any asymptotes. *[3 marks]*

(c) (i) Write down the time T at which the velocity is zero.

(ii) Find the distance travelled in the interval $[0, T]$. *[3 marks]*

(d) Find an expression for s, the displacement, in terms of t, given that $s = 0$ when $t = 0$. *[5 marks]*

(e) Hence, or otherwise, show that $s = \frac{1}{2}\ln\frac{2}{1+v^2}$. *[4 marks]*

2 – (MAT/20_HL_Winter_2012_Q6) – *Kinematics*

[Maximum mark: 6]

A particle moves along a straight line so that after t seconds its displacement s, in metres, satisfies the equation $s^2 + s - 2t = 0$. Find, in terms of s, expressions for its velocity and its acceleration.

3 – (MAT/22_HL_Summer_2013_Q10) – *Kinematics*

[Maximum mark: 6]

The acceleration of a car is $\frac{1}{40}(60 - v)\ \text{m s}^{-2}$, when its velocity is $v\ \text{m s}^{-1}$. Given the car starts from rest, find the velocity of the car after 30 seconds.

4 – (MAT/21_HL_Summer_2013_Q12) – *Kinematics*

A particle, A, is moving along a straight line. The velocity, v_A ms^{-1}, of A t seconds after its motion begins is given by

$$v_A = t^3 - 5t^2 + 6t.$$

(a) Sketch the graph of $v_A = t^3 - 5t^2 + 6t$ for $t \geq 0$, with v_A on the vertical axis and t on the horizontal. Show on your sketch the local maximum and minimum points, and the intercepts with the t-axis. *[3 marks]*

(b) Write down the times for which the velocity of the particle is increasing. *[2 marks]*

(c) Write down the times for which the magnitude of the velocity of the particle is increasing. *[3 marks]*

At $t = 0$ the particle is at point O on the line.

(d) Find an expression for the particle's displacement, x_A m, from O at time t. *[3 marks]*

A second particle, B, moving along the same line, has position x_B m, velocity v_B ms^{-1} and acceleration, a_B ms^{-2}, where $a_B = -2v_B$ for $t \geq 0$. At $t = 0$, $x_B = 20$ and $v_B = -20$.

(e) Find an expression for v_B in terms of t. *[4 marks]*

(f) Find the value of t when the two particles meet. *[6 marks]*

5 – (MAT/20_HL_Winter_2013_Q12) – *Kinematics*

(a) A particle P moves in a straight line with velocity $v\,\text{ms}^{-1}$. At time $t=0$, P is at the point O and has velocity $12\,\text{ms}^{-1}$. Its acceleration at time t seconds is given by $\frac{\mathrm{d}v}{\mathrm{d}t}=3\cos\frac{t}{4}\text{ms}^{-2}$, $(t\geq 0)$.

(i) Find an expression for the particle's velocity v, in terms of t.

(ii) Sketch a velocity/time graph for the particle for $0\leq t\leq 8\pi$, showing clearly where the curve meets the axes and any maximum or minimum points.

(iii) Find the distance travelled by the particle before first coming to rest. *[8]*

(b) Another particle Q moves in a straight line with displacement s metres and velocity $v\,\text{ms}^{-1}$. Its acceleration is given by $a=-\left(v^2+4\right)\text{ms}^{-2}$, $(0\leq t\leq 1)$. At time $t=0$, Q is at the point O and has velocity $2\,\text{ms}^{-1}$.

(i) Show that the velocity v at time t is given by $v=2\tan\left(\frac{\pi-8t}{4}\right)$.

(ii) Show that $\frac{\mathrm{d}v}{\mathrm{d}s}=-\frac{\left(v^2+4\right)}{v}$.

(iii) Find the distance travelled by the particle before coming to rest. *[12]*

6 – (MAT/22_HL_Summer_2014_Q14) – *Kinematics*

Particle A moves such that its velocity v ms^{-1}, at time t seconds, is given by $v(t)=\dfrac{t}{12+t^4}$, $t \geq 0$.

(a) Sketch the graph of $y=v(t)$. Indicate clearly the local maximum and write down its coordinates. *[2]*

(b) Use the substitution $u=t^2$ to find $\int \dfrac{t}{12+t^4}\,\mathrm{d}t$. *[4]*

(c) Find the exact distance travelled by particle A between $t=0$ and $t=6$ seconds. Give your answer in the form $k\arctan(b)$, $k, b \in \mathbb{R}$. *[3]*

Particle B moves such that its velocity v ms^{-1} is related to its displacement s m, by the equation $v(s)=\arcsin\left(\sqrt{s}\right)$.

(d) Find the acceleration of particle B when $s=0.1\text{m}$. *[3]*

7 – (MAT/20_HL_Winter_2014_Q4) – *Kinematics*

[Maximum mark: 5]

Two cyclists are at the same road intersection. One cyclist travels north at $20\,\text{km}\,\text{h}^{-1}$. The other cyclist travels west at $15\,\text{km}\,\text{h}^{-1}$.

Use calculus to show that the rate at which the distance between the two cyclists changes is independent of time.

8 – (MAT/20_HL_Winter_2014_Q8) – *Kinematics*

A particle moves in a straight line such that its velocity, $v\,\mathrm{m\,s^{-1}}$, at time t seconds, is given by

$$v(t)=\begin{cases}5-(t-2)^2, & 0\le t\le 4\\ 3-\dfrac{t}{2}, & t>4\end{cases}.$$

(a) Find the value of t when the particle is instantaneously at rest. *[2]*

The particle returns to its initial position at $t=T$.

(b) Find the value of T. *[5]*

9 – (MAT/22_HL_Summer_2015_Q12) – *Kinematics*

A particle moves in a straight line, its velocity $v\,\mathrm{ms}^{-1}$ at time t seconds is given by $v = 9t - 3t^2$, $0 \le t \le 5$.

At time $t = 0$, the displacement s of the particle from an origin O is $3\,\mathrm{m}$.

(a) Find the displacement of the particle when $t = 4$. [3]

(b) Sketch a displacement/time graph for the particle, $0 \le t \le 5$, showing clearly where the curve meets the axes and the coordinates of the points where the displacement takes greatest and least values. [5]

For $t > 5$, the displacement of the particle is given by $s = a + b\cos\frac{2\pi t}{5}$ such that s is continuous for all $t \ge 0$.

(c) Given further that $s = 16.5$ when $t = 7.5$, find the values of a and b. [3]

(d) Find the times t_1 and t_2 $(0 < t_1 < t_2 < 8)$ when the particle returns to its starting point. [4]

10 – (MAT/21_HL_Summer_2015_Q13) – *Kinematics*

Richard, a marine soldier, steps out of a stationary helicopter, $1000\,\text{m}$ above the ground, at time $t = 0$. Let his height, in metres, above the ground be given by $s(t)$. For the first 10 seconds his velocity, $v(t)\,\text{ms}^{-1}$, is given by $v(t) = -10t$.

(a) (i) Find his acceleration $a(t)$ for $t < 10$.

(ii) Calculate $v(10)$.

(iii) Show that $s(10) = 500$. [6]

At $t = 10$ his parachute opens and his acceleration $a(t)$ is subsequently given by $a(t) = -10 - 5v$, $t \geq 10$.

(b) Given that $\frac{\mathrm{d}t}{\mathrm{d}v} = \frac{1}{\frac{\mathrm{d}v}{\mathrm{d}t}}$, write down $\frac{\mathrm{d}t}{\mathrm{d}v}$ in terms of v. [1]

You are told that Richard's acceleration, $a(t) = -10 - 5v$, is always positive, for $t \geq 10$.

(c) Hence show that $t = 10 + \frac{1}{5}\ln\left(\frac{98}{-2-v}\right)$. [5]

(d) Hence find an expression for the velocity, v, for $t \geq 10$. [2]

(e) Find an expression for his height, s, above the ground for $t \geq 10$. [5]

(f) Find the value of t when Richard lands on the ground. [2]

11 – (MAT/20_HL_Winter_2015_Q9) – *Kinematics*

A particle can move along a straight line from a point O. The velocity v, in $\mathrm{m\,s^{-1}}$, is given by the function $v(t) = 1 - \mathrm{e}^{-\sin t^2}$ where time $t \geq 0$ is measured in seconds.

(a) Write down the first two times $t_1, t_2 > 0$, when the particle changes direction. [2]

(b) (i) Find the time $t < t_2$ when the particle has a maximum velocity.

(ii) Find the time $t < t_2$ when the particle has a minimum velocity. [4]

(c) Find the distance travelled by the particle between times $t = t_1$ and $t = t_2$. [2]

12 – (MAT/21_HL_Summer_2016_Q3) – *Kinematics*

The displacement, s, in metres, of a particle t seconds after it passes through the origin is given by the expression $s = \ln\left(2 - e^{-t}\right)$, $t \geq 0$.

(a) Find an expression for the velocity, v, of the particle at time t. [2]

(b) Find an expression for the acceleration, a, of the particle at time t. [2]

(c) Find the acceleration of the particle at time $t = 0$. [1]

13 – (MAT/22_HL_Summer_2016_Q8) – *Kinematics*

A particle moves such that its velocity $v\,\mathrm{ms}^{-1}$ is related to its displacement s m, by the equation $v(s) = \arctan(\sin s),\ 0 \le s \le 1$. The particle's acceleration is $a\,\mathrm{ms}^{-2}$.

(a) Find the particle's acceleration in terms of s. [4]

(b) Using an appropriate sketch graph, find the particle's displacement when its acceleration is $0.25\,\mathrm{ms}^{-2}$. [2]

14 – (MAT/21_HL_Summer_2017_Q11) – *Kinematics*

Xavier, the parachutist, jumps out of a plane at a height of h metres above the ground. After free falling for 10 seconds his parachute opens.

His velocity, $v\,\mathrm{ms}^{-1}$, t seconds after jumping from the plane, can be modelled by the function

$$v(t) = \begin{cases} 9.8t, & 0 \le t \le 10 \\ \dfrac{98}{\sqrt{1+(t-10)^2}}, & t > 10 \end{cases}.$$

(a) Find his velocity when $t = 15$. [2]

(b) Calculate the vertical distance Xavier travelled in the first 10 seconds. [2]

His velocity when he reaches the ground is $2.8\,\mathrm{ms}^{-1}$.

(c) Determine the value of h. [5]

15 - (MAT/22_HL_Summer_2018_Q7) - *Kinematics*

A point P moves in a straight line with velocity $v\,\mathrm{ms}^{-1}$ given by $v(t) = \mathrm{e}^{-t} - 8t^2\mathrm{e}^{-2t}$ at time t seconds, where $t \geq 0$.

(a) Determine the first time t_1 at which P has zero velocity. [2]

(b) (i) Find an expression for the acceleration of P at time t.

(ii) Find the value of the acceleration of P at time t_1. [3]

16 – (MAT/22_HL_Summer_2019_Q6) – *Kinematics*

[Maximum mark: 6]

A particle moves along a horizontal line such that at time t seconds, $t \geq 0$, its acceleration a is given by $a = 2t - 1$. When $t = 6$, its displacement s from a fixed origin O is 18.25 m. When $t = 15$, its displacement from O is 922.75 m. Find an expression for s in terms of t.

17 – (MAT/20_HL_Winter_2020_Q11) – *Kinematics, Sequences - Series*

A particle P moves in a straight line such that after time t seconds, its velocity, v in $\mathrm{m\,s^{-1}}$, is given by $v = \mathrm{e}^{-3t}\sin 6t$, where $0 < t < \frac{\pi}{2}$.

(a) Find the times when P comes to instantaneous rest. [2]

At time t, P has displacement $s(t)$; at time $t = 0$, $s(0) = 0$.

(b) Find an expression for s in terms of t. [7]

(c) Find the maximum displacement of P, in metres, from its initial position. [2]

(d) Find the total distance travelled by P in the first 1.5 seconds of its motion. [2]

At successive times when the acceleration of P is $0\,\mathrm{m\,s^{-2}}$, the velocities of P form a geometric sequence. The acceleration of P is zero at times t_1, t_2, t_3 where $t_1 < t_2 < t_3$ and the respective velocities are v_1, v_2, v_3.

(e) (i) Show that, at these times, $\tan 6t = 2$.

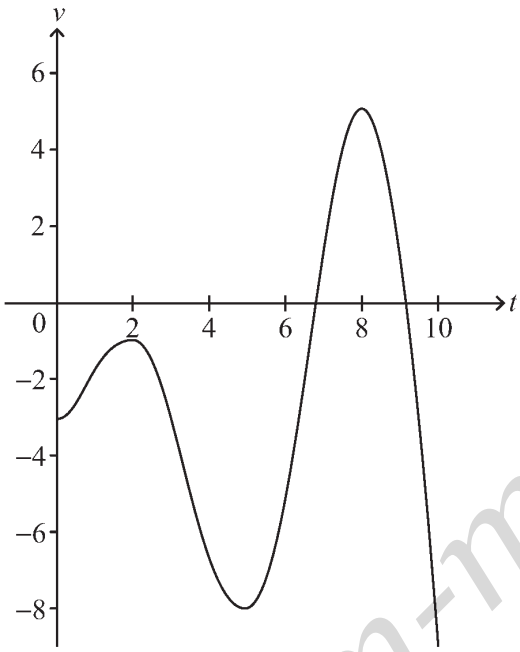
(ii) Hence show that $\frac{v_2}{v_1} = \frac{v_3}{v_2} = -\mathrm{e}^{\frac{\pi}{2}}$. [7]

18 – (MathAA/21_HL_Summer_2021_Q4) – *Kinematics, Relative Velocity*

[Maximum mark: 6]

A particle moves in a straight line. The velocity, $v\,\mathrm{ms}^{-1}$, of the particle at time t seconds is given by $v(t) = t\sin t - 3$, for $0 \le t \le 10$.

The following diagram shows the graph of v.



(a) Find the smallest value of t for which the particle is at rest. [2]

(b) Find the total distance travelled by the particle. [2]

(c) Find the acceleration of the particle when $t = 7$. [2]

19 – (MathAA/21_HL_Summer_2021_Q9) – *Differentiation, Kinematics, Relative Velocity*

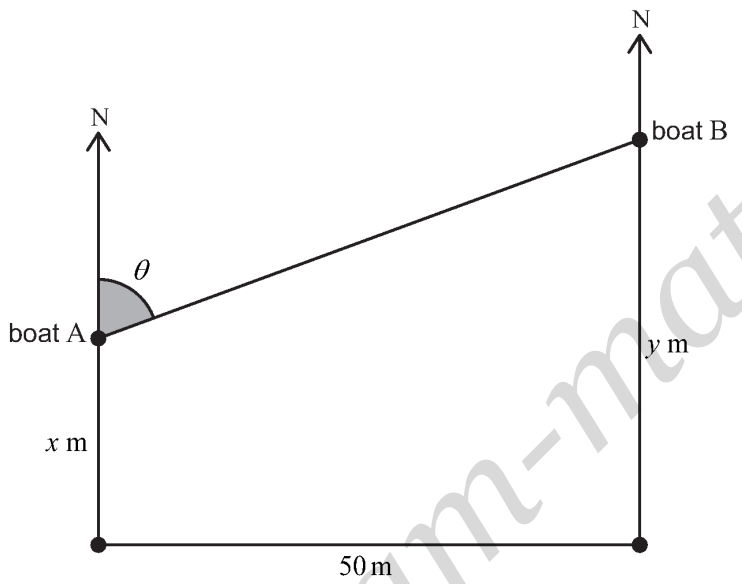
[Maximum mark: 7]

Two boats A and B travel due north.

Initially, boat B is positioned 50 metres due east of boat A.

The distances travelled by boat A and boat B, after t seconds, are x metres and y metres respectively. The angle θ is the radian measure of the bearing of boat B from boat A.
This information is shown on the following diagram.

diagram not to scale



(a) Show that $y = x + 50\cot\theta$. [1]

At time T, the following conditions are true.

Boat B has travelled 10 metres further than boat A.
Boat B is travelling at double the speed of boat A.
The rate of change of the angle θ is -0.1 radians per second.

(b) Find the speed of boat A at time T. [6]

20 – (MathAA/21_HL_Summer_2022_Q4) – *Kinematics, Relative Velocity*

[Maximum mark: 7]

A particle moves along a straight line so that its velocity, $v\,\mathrm{m\,s^{-1}}$, after t seconds is given by $v(t) = e^{\sin t} + 4\sin t$ for $0 \le t \le 6$.

(a) Find the value of t when the particle is at rest. [2]

(b) Find the acceleration of the particle when it changes direction. [3]

(c) Find the total distance travelled by the particle. [2]

21 – (MathAA/22_HL_Summer_2022_Q5) – *Kinematics, Relative Velocity*

[Maximum mark: 7]

A particle moves in a straight line such that its velocity, $v\,\mathrm{m\,s^{-1}}$, at time t seconds is given by $v = \frac{(t^2+1)\cos t}{4}$, $0 \le t \le 3$.

(a) Determine when the particle changes its direction of motion. [2]

(b) Find the times when the particle's acceleration is $-1.9\,\mathrm{m\,s^{-2}}$. [3]

(c) Find the particle's acceleration when its speed is at its greatest. [2]

ANSWERS

1 - (MAT/21_HL_Summer_2012_Q6) - *Functions - Roots*

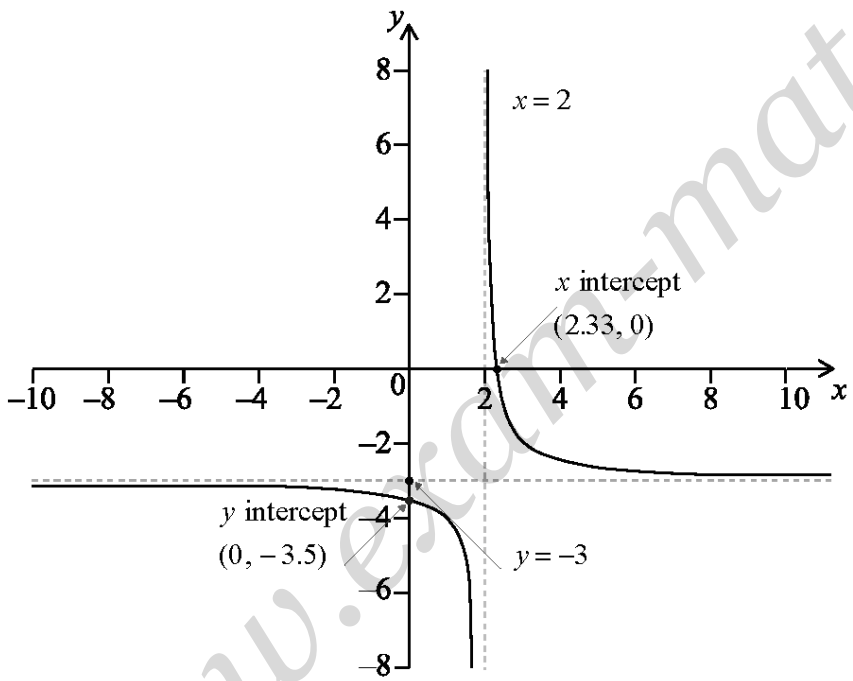
$$h(x) = f(x-3) - 2 = \ln(x-3) - 2$$
$$g(x) = -h(x) = 2 - \ln(x-3)$$

Note: Award **M1** only if it is clear the effect of the reflection in the x-axis: the expression is correct ***OR*** there is a change of signs of the previous expression ***OR*** there's a graph or an explanation making it explicit

$$= \ln e^2 - \ln(x-3)$$
$$= \ln\left(\frac{e^2}{x-3}\right)$$

2 - (MAT/22_HL_Summer_2014_Q7) - *Functions - Roots, Graphs*

(a)



A1A1A1

Note: Award ***A1*** for correct shape, ***A1*** for $x = 2$ clearly stated and ***A1*** for $y = -3$ clearly stated.

x intercept $(2.33, 0)$ and y intercept $(0, -3.5)$ ***A1***

(b) $x = -3 + \dfrac{1}{y-2}$

Note: Award ***M1*** for interchanging x and y (can be done at a later stage).

$$x + 3 = \frac{1}{y-2}$$

$$y - 2 = \frac{1}{x+3}$$

Note: Award ***M1*** for attempting to make y the subject.

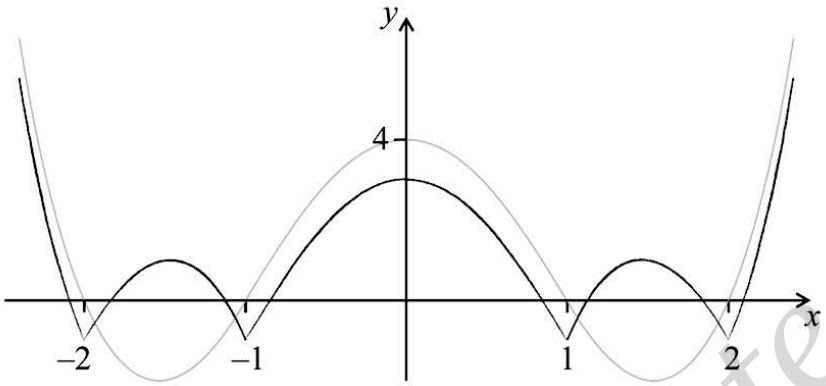
$$f^{-1}(x) = 2 + \frac{1}{x+3} \left(= \frac{2x+7}{x+3} \right), \ x \neq -3$$

3 – (MAT/21_HL_Summer_2014_Q12) – *Functions - Roots, Graphs*

(a) (i) $f(0) = -1$

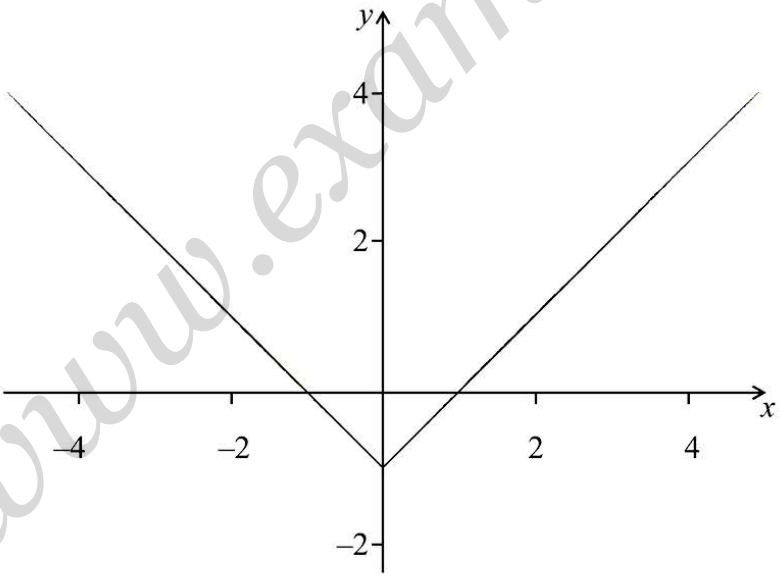
(ii) $(f \circ g)(0) = f(4) = 3$

(iii)



Note: Award ***M1*** for evidence that the lower part of the graph has been reflected and ***A1*** correct shape with y-intercept below 4.

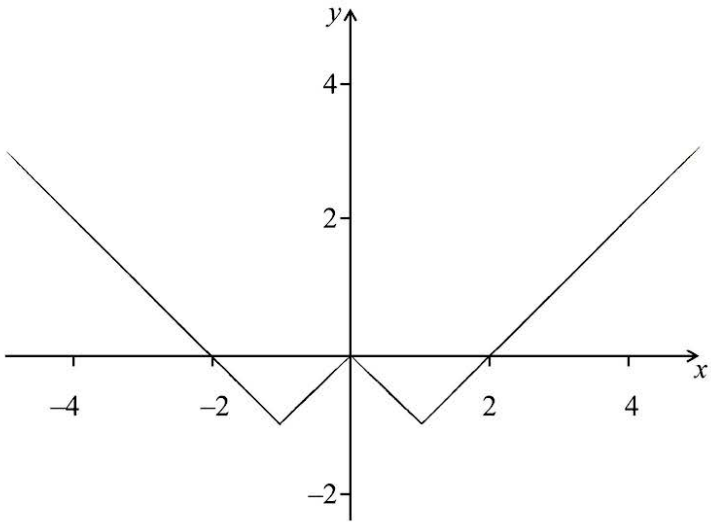
(b) (i)



Note: Award ***M1*** for any translation of $y = |x|$.

(ii) ± 1

(c) (i)



Note:	Award ***M1*** for evidence that lower part of (b) has been reflected in the x-axis and translated.

(ii) $0,\ \pm 2$

(d) (i) $\pm 1,\ \pm 3$

(ii) $0,\ \pm 2,\ \pm 4$

(iii) $0,\ \pm 2,\ \pm 4,\ \pm 6,\ \pm 8$

(e) (i) $(1, 3), (2, 5), \ldots$
$N = 2n+1$

(ii) Using the formula of the sum of an arithmetic series

EITHER

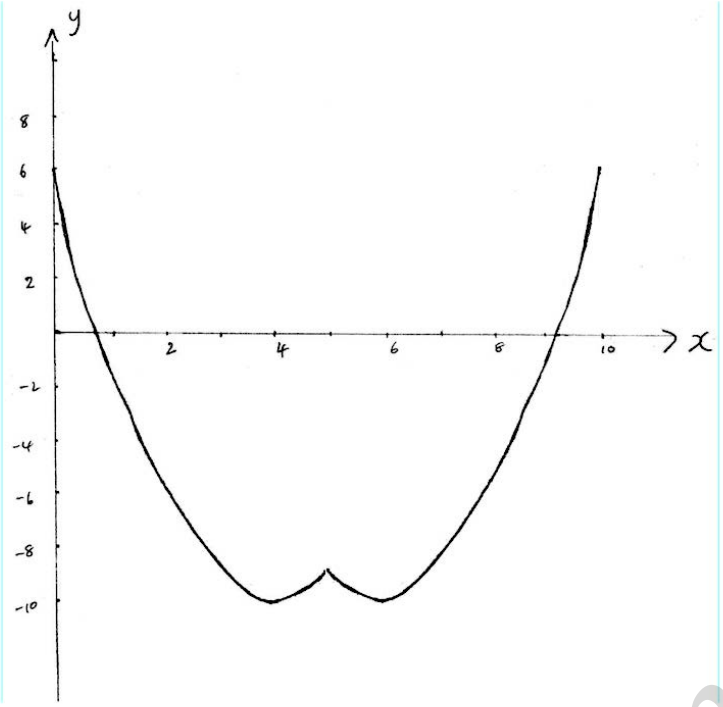
$$4(1+2+3+\ldots+n) = \frac{4}{2}n(n+1)$$
$$= 2n(n+1)$$

OR

$$2(2+4+6+\ldots+2n) = \frac{2}{2}n(2n+2)$$
$$= 2n(n+1)$$

4 – (MAT/22_HL_Summer_2015_Q3) – *Graphs, Functions - Roots*

(a)

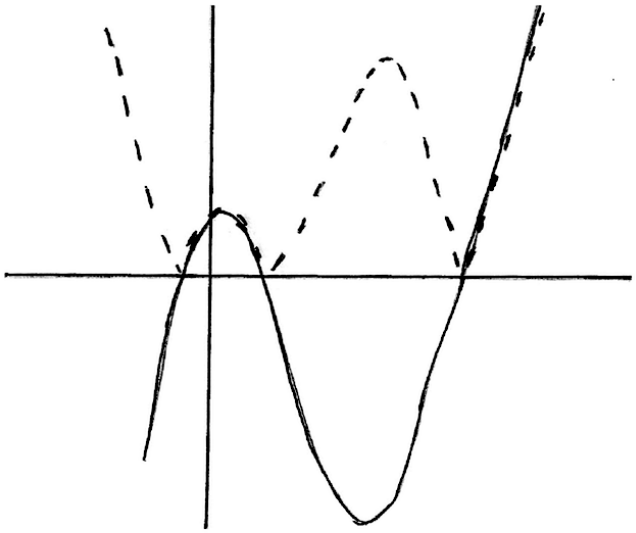


general shape including 2 minimums, cusp
correct domain and symmetrical about the middle $(x = 5)$

(b) $x = 9.16$ or $x = 0.838$

5 – (MAT/21_HL_Summer_2015_Q10) – *Functions - Roots*

(a)

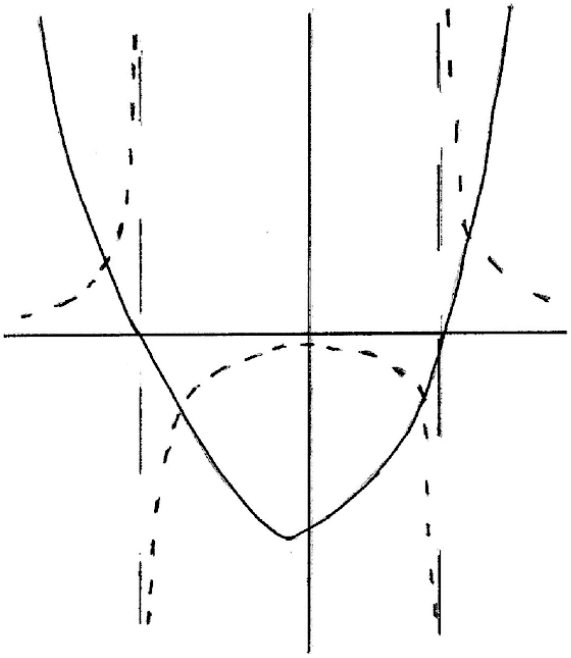


as roots of $f(x)=0$ are $-1, 1, 5$

solution is $]-\infty,-1[\cup]1,5[$ $(x<-1$ or $1<x<5)$

(b) METHOD 1

(graphs of $g(x)$ and $\frac{1}{g(x)}$)



roots of $g(x)=0$ are -3 and 2

Notes: Award ***M1*** if quadratic graph is drawn or two roots obtained. Roots may be indicated anywhere *eg* asymptotes on graph or in inequalities below.

the intersections of the graphs $g(x)$ and of $1/g(x)$ are $-3.19, -2.79, 1.79, 2.19$

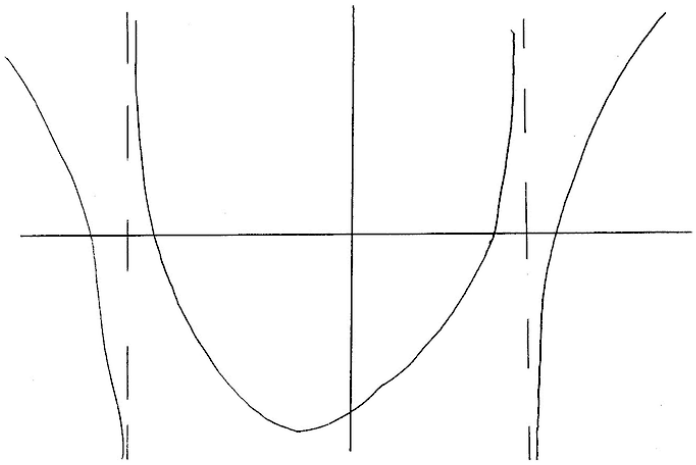
Note: Award ***A1*** for at least one of the values above seen anywhere.

solution is $]-3.19,-3[\cup]-2.79,1.79[\cup]2,2.19[$

($-3.19<x<-3$ or $-2.79<x<1.79$ or $2<x<2.19$)

METHOD 2

(graph of $g(x) - \frac{1}{g(x)}$)



asymptotes at $x = -3$ and $x = 2$

Note: May be indicated on the graph.

roots of graph are $-3.19, -2.79, 1.79, 2.19$

Note: Award ***A1*** for at least one of the values above seen anywhere.

solution is (when graph is negative)
$]-3.19, -3[\cup]-2.79, 1.79[\cup]2, 2.19[$
($-3.19 < x < -3$ or $-2.79 < x < 1.79$ or $2 < x < 2.19$)

6 – (MAT/20_HL_Winter_2015_Q12) – *Functions - Roots*

(a) (i) $-4 \le y \le -2$

(ii) $-5 \le y \le -1$

(iii) $-3 \le 2x - 6 \le 5$

Note: Award ***M1*** for $f(2x-6)$.

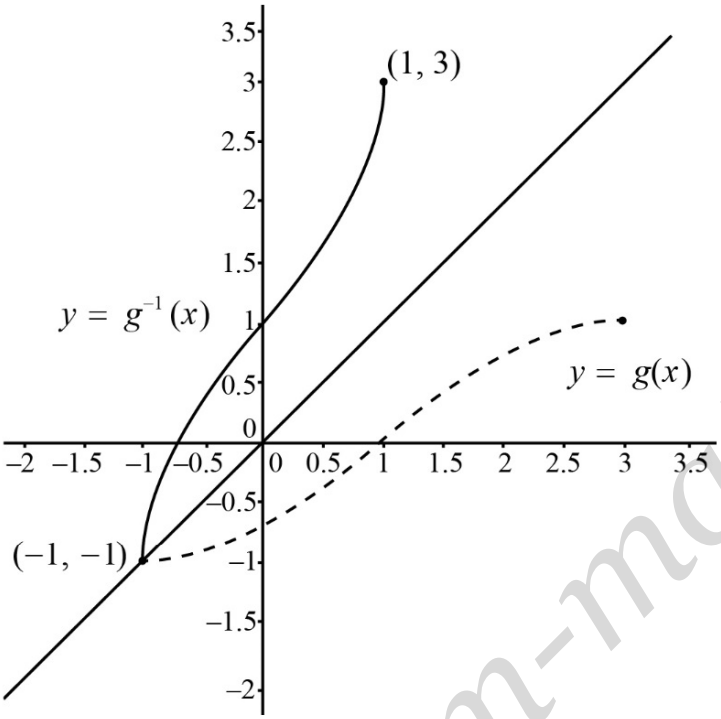
$3 \le 2x \le 11$

$\frac{3}{2} \le x \le \frac{11}{2}$

(b) (i) any valid argument *eg* f is not one to one, f is many to one, fails horizontal line test, not injective ***R1***

(ii) largest domain for the function $g(x)$ to have an inverse is $[-1, 3]$ ***A1A1***

(iii)



y-intercept indicated (coordinates not required) ***A1***
correct shape ***A1***
coordinates of end points $(1, 3)$ and $(-1, -1)$ ***A1***

(c) (i) $y = \dfrac{2x-5}{x+d}$

$(x+d)\,y = 2x-5$

Note: Award ***M1*** for attempting to rearrange x and y in a linear expression.

$x(y-2) = -dy-5$

$x = \dfrac{-dy-5}{y-2}$

Note: x and y can be interchanged at any stage

$h^{-1}(x) = \dfrac{-dx-5}{x-2}$

Note: Award ***A1*** only if $h^{-1}(x)$ is seen.

(ii) self Inverse $\Rightarrow h(x) = h^{-1}(x)$

$\dfrac{2x-5}{x+d} \equiv \dfrac{-dx-5}{x-2}$

$d = -2$

(iii) **METHOD 1**

$\dfrac{2k(x)-5}{k(x)-2} = \dfrac{2x}{x+1}$

$k(x) = \dfrac{x+5}{2}$

METHOD 2

$h^{-1}\left(\dfrac{2x}{x+1}\right) = \dfrac{2\left(\dfrac{2x}{x+1}\right)-5}{\dfrac{2x}{x+1}-2}$

$k(x) = \dfrac{x+5}{2}$

7 – (MAT/21_HL_Summer_2016_Q2) – *Quadratics, Functions - Roots*

(a) $(x+2)^2-6$

(b) $(g\circ f)(x)=(x+2)^2-6$
$\Rightarrow g(x)=x^2-6$

8 – (MAT/22_HL_Summer_2016_Q5) – *Functions - Roots*

$$x=\sqrt{\frac{1-y}{1+y}}$$

Note: Award ***M1*** for interchanging x and y (can be done at a later stage).

$$x^2=\frac{1-y}{1+y}$$

$$x^2+x^2y=1-y$$

Note: Award ***M1*** for attempting to make y the subject.

$$y\left(1+x^2\right)=1-x^2$$

$$f^{-1}(x)=\frac{1-x^2}{1+x^2},\ x\geq 0$$

Note: Award ***A1*** only if $f^{-1}(x)$ is seen. Award ***A1*** for the domain.

the range of f^{-1} is $-1<f^{-1}(x)\leq 1$

Note: Accept correct alternative notation *eg.* $-1<y\leq 1$.

9 – (MAT/21_HL_Summer_2016_Q11) – *Functions - Roots*

(a) valid method *eg*, sketch of curve or critical values found ***(M1)***
$x < -2.24,\ x > 2.24,$ ***A1***
$-1 < x < 0.8$ ***A1***

Note: Award ***M1A1A0*** for correct intervals but with inclusive inequalities.

[3 marks]

(b) (i) $(1.67, -5.14), (-1.74, -3.71)$ ***A1A1***

Note: Award ***A1A0*** for any two correct terms.

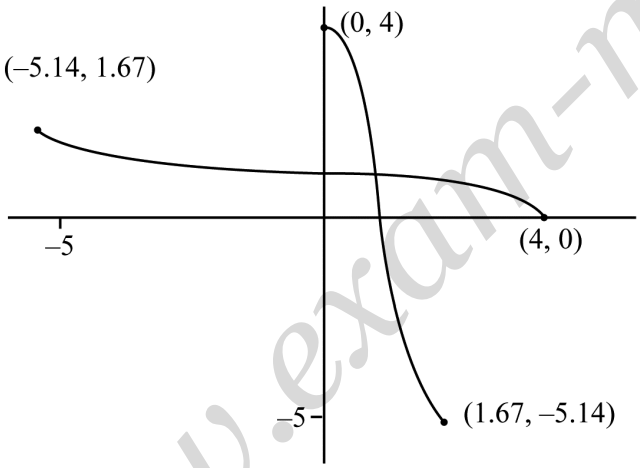
(ii) $f'(x) = 4x^3 + 0.6x^2 - 11.6x - 1$
$f''(x) = 12x^2 + 1.2x - 11.6 = 0$ ***(M1)***
$-1.03,\ 0.934$ ***A1A1***

Note: ***M1*** should be awarded if graphical method to find zeros of $f''(x)$ or turning points of $f'(x)$ is shown.

[5 marks]

(c) (i) 1.67 ***A1***

(ii)



M1A1A1

Note: Award ***M1*** for reflection of their $y = f(x)$ in the line $y = x$ provided their f is one-one.

A1 for $(0, 4)$, $(4,0)$ (Accept axis intercept values) ***A1*** for the other two sets of coordinates of other end points

(iii) $x = f(1)$ ***M1***
$= -1.6$ ***A1***

[6 marks]

(d) (i) $y = 2\sin(x-1) - 3$

$x = 2\sin(y-1) - 3$ ***(M1)***

$\left(g^{-1}(x) =\right) \arcsin\left(\frac{x+3}{2}\right) + 1$ ***A1***

$-5 \le x \le -1$ ***A1A1***

Note: Award ***A1*** for -5 and -1, and ***A1*** for correct inequalities if numbers are reasonable.

(ii) $f^{-1}\left(g(x)\right) < 1$

$g(x) > -1.6$ ***(M1)***

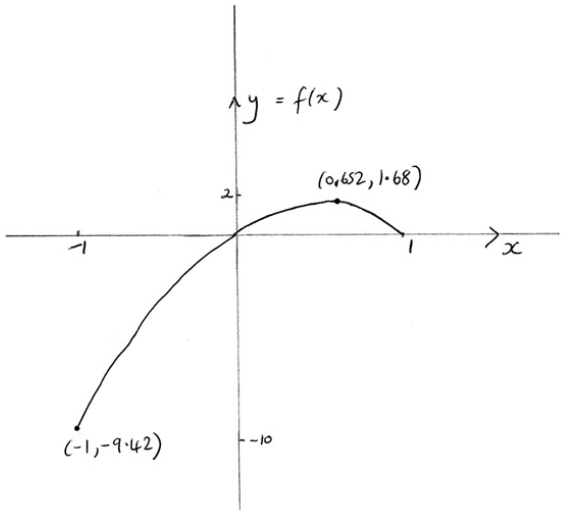
$x > g^{-1}(-1.6) = 1.78$ ***(A1)***

Note: Accept = in the above.

$1.78 < x \le \frac{\pi}{2} + 1$ ***A1A1***

10 – (MAT/20_HL_Winter_2016_Q5) – *Functions - Roots, Graphs*

(a)



correct shape passing through the origin and correct domain ***A1***

Note: Endpoint coordinates are not required. The domain can be indicated by -1 and 1 marked on the x- axis.

$(0.652, 1.68)$ ***A1***

two correct intercepts (coordinates not required) ***A1***

Note: A graph passing through the origin is sufficient for $(0, 0)$.

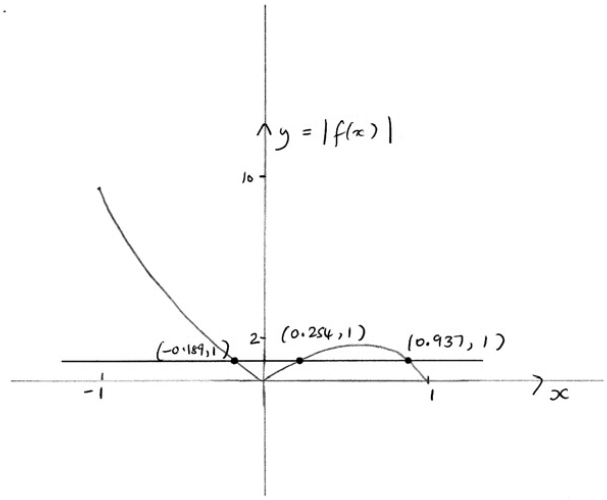
[3 marks]

(b) $[-9.42, 1.68]$ $(\text{or } [-3\pi, 1.68])$ ***A1A1***

Note: Award ***A1A0*** for open or semi-open intervals with correct endpoints. Award ***A1A0*** for closed intervals with one correct endpoint.

[2 marks]

(c) attempting to solve either $\left|3x\arccos(x)\right| > 1$ (or equivalent) or $\left|3x\arccos(x)\right| = 1$ (or equivalent) (*eg*. graphically) ***(M1)***



$x = -0.189,\ 0.254,\ 0.937$ ***(A1)***

$-1 \le x < -0.189$ or $0.254 < x < 0.937$ ***A1A1***

Note: Award ***A0*** for $x < -0.189$.

[4 marks]

11 – (MAT/22_HL_Summer_2017_Q11) – *Functions - Roots, Graphs*

(a) $g(x) = 3x^4 + ax^3 + bx^2 - 7x - 4$

$g(1) = 0 \Rightarrow a + b = 8$ ***M1A1***

$g(-1) = 0 \Rightarrow -a + b = -6$ ***A1***

$\Rightarrow a = 7,\ \mathrm{b} = 1$ ***A1***

[4 marks]

(b) $3x^4 + 7x^3 + x^2 - 7x - 4 = \left(x^2 - 1\right)\left(px^2 + qx + r\right)$

attempt to equate coefficients ***(M1)***

$p = 3,\ q = 7,\ r = 4$ ***(A1)***

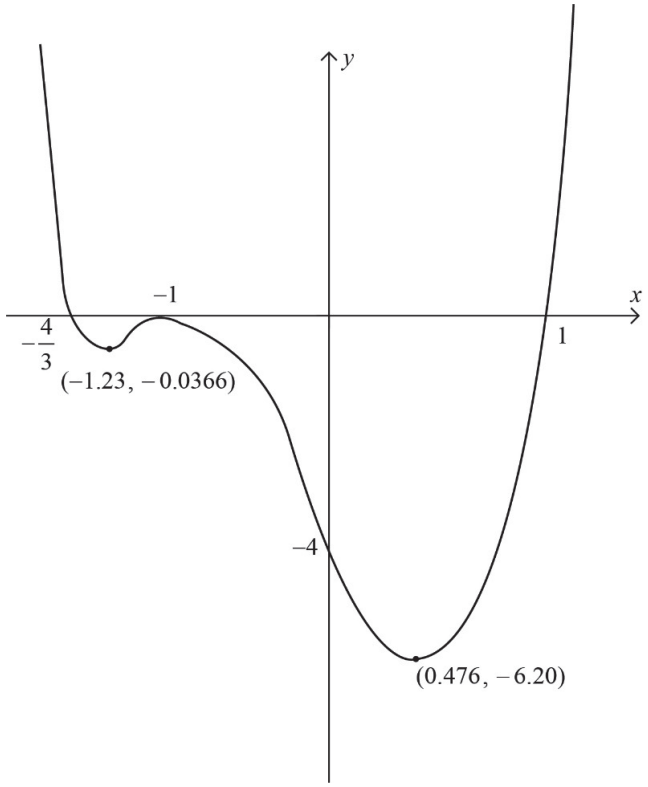
$3x^4 + 7x^3 + x^2 - 7x - 4 = \left(x^2 - 1\right)\left(3x^2 + 7x + 4\right)$

$= (x-1)(x+1)^2(3x+4)$ ***A1***

Note: Accept any equivalent valid method.

[3 marks]

(c)



A1 for correct shape (*ie* with correct number of max/min points)
A1 for correct x and y intercepts
A1 for correct maximum and minimum points

[3 marks]

(d) $c > 0$ ***A1***

$-6.20 < c < -0.0366$ ***A1A1***

Note: Award ***A1*** for correct end points and ***A1*** for correct inequalities.

Note: If the candidate has misdrawn the graph and omitted the first minimum point, the maximum mark that may be awarded is ***A1FTA0A0*** for $c > -6.20$ seen.

[3 marks]

12 - (MAT/21_HL_Summer_2018_Q2) - *Functions - Roots*

METHOD 1

$\alpha+\beta=5,\ \alpha\beta=-7$ ***(M1)(A1)***

Note: Award ***M1A0*** if only one equation obtained.

$(\alpha+1)+(\beta+1)=5+2=7$ ***A1***

$(\alpha+1)(\beta+1)=\alpha\beta+(\alpha+\beta)+1$ ***(M1)***

$=-7+5+1=-1$

$p=-7,\ q=-1$ ***A1A1***

METHOD 2

$\alpha=\dfrac{5+\sqrt{53}}{2}=6.1\ldots;\ \beta=\dfrac{5-\sqrt{53}}{2}=-1.1\ldots$ ***(M1)(A1)***

$\alpha+1=\dfrac{7+\sqrt{53}}{2}=7.1\ldots;\ \beta+1=\dfrac{7-\sqrt{53}}{2}=-0.1\ldots$ ***A1***

$(x-7.14\ldots)(x+0.14\ldots)=x^2-7x-1$ ***(M1)***

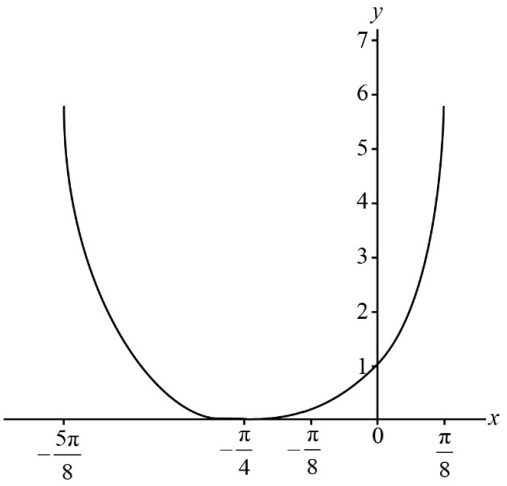
$p=-7,\ q=-1$ ***A1A1***

Note: Exact answers only.

[6 marks]

13 – (MAT/22_HL_Summer_2018_Q10) – *Functions - Roots*

(a) (i)



A1A1

A1 for correct concavity, many to one graph, symmetrical about the midpoint of the domain and with two axes intercepts.

Note: Axes intercepts and scales not required.

A1 for correct domain

(ii) for each value of x there is a unique value of $f(x)$ ***A1***

Note: Accept "passes the vertical line test" or equivalent.

(iii) no inverse because the function fails the horizontal line test or equivalent ***R1***

Note: No **FT** if the graph is in degrees (one-to-one).

(iv) the expression is not valid at either of $x = \frac{\pi}{4}$ $\left(\text{or } -\frac{3\pi}{4}\right)$ ***R1***

[5 marks]

(b) **METHOD 1**

$$f(x) = \frac{\tan\left(x + \frac{\pi}{4}\right)}{\tan\left(\frac{\pi}{4} - x\right)}$$ ***M1***

$$= \frac{\dfrac{\tan x + \tan\frac{\pi}{4}}{1 - \tan x \tan\frac{\pi}{4}}}{\dfrac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\tan x}}$$ ***M1A1***

$$= \left(\frac{1+t}{1-t}\right)^2$$ ***AG***

METHOD 2

$$f(x) = \tan\left(x + \frac{\pi}{4}\right)\tan\left(\frac{\pi}{2} - \frac{\pi}{4} + x\right)$$ ***(M1)***

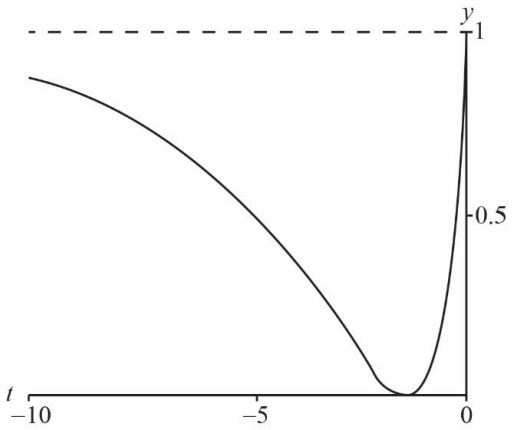
$$= \tan^2\left(x + \frac{\pi}{4}\right)$$ ***A1***

$$g(t) = \left(\frac{\tan x + \tan\frac{\pi}{4}}{1 - \tan x \tan\frac{\pi}{4}}\right)^2$$ ***A1***

$$= \left(\frac{1+t}{1-t}\right)^2$$ ***AG***

[3 marks]

(c)



for $t \le 0$, correct concavity with two axes intercepts and with asymptote $y = 1$ ***A1***

t intercept at $(-1, 0)$ ***A1***

y intercept at $(0, 1)$ ***A1***

[3 marks]

(d) (i) **METHOD 1**

α, β satisfy $\dfrac{(1+t)^2}{(1-t)^2} = k$ ***M1***

$1 + t^2 + 2t = k(1 + t^2 - 2t)$ ***A1***

$(k-1)t^2 - 2(k+1)t + (k-1) = 0$ ***A1***

attempt at using quadratic formula ***M1***

$\alpha, \beta = \dfrac{k + 1 \pm 2\sqrt{k}}{k-1}$ or equivalent ***A1***

METHOD 2

α, β satisfy $\dfrac{1+t}{1-t} = (\pm)\sqrt{k}$ ***M1***

$t + \sqrt{k}t = \sqrt{k} - 1$ ***M1***

$t = \dfrac{\sqrt{k}-1}{\sqrt{k}+1}$ (or equivalent) ***A1***

$t - \sqrt{k}t = -\left(\sqrt{k}+1\right)$ ***M1***

$t = \dfrac{\sqrt{k}+1}{\sqrt{k}-1}$ (or equivalent) ***A1***

so for *eg*, $\alpha = \dfrac{\sqrt{k}-1}{\sqrt{k}+1}$, $\beta = \dfrac{\sqrt{k}+1}{\sqrt{k}-1}$

(ii) $\alpha + \beta = 2\frac{(k+1)}{(k-1)}\left(= -2\frac{(1+k)}{(1-k)}\right)$ ***A1***

since $1 + k > 1 - k$ ***R1***

$\alpha + \beta < -2$ ***AG***

Note: Accept a valid graphical reasoning.

[7 marks]

14 – (MathAA/22_HL_Summer_2021_Q4) – *Algebra-functions - Roots*

(a) attempting to find the vertex ***(M1)***

$x = 1$ OR $y = -5$ OR $f(x) = 6(x-1)^2 - 5$

range is $y \geq -5$ ***A1***

[2 marks]

(b) **METHOD 1**

$(g \circ f)(x) = -(6x^2 - 12x + 1) + c \ \left(= -\left(6(x-1)^2 - 5\right) + c\right)$ ***(A1)***

EITHER

relating to the range of f OR attempting to find $g(-5)$ ***(M1)***

$5 + c \leq 0$ ***(A1)***

OR

attempting to find the discriminant of $(g \circ f)(x)$ ***(M1)***

$144 + 24(c-1) \leq 0 \ (120 + 24c \leq 0)$ ***(A1)***

THEN

$c \leq -5$ ***A1***

[4 marks]

METHOD 2

vertical reflection followed by vertical shift ***(M1)***

new vertex is $(1, 5 + c)$ ***(A1)***

$5 + c \leq 0$ ***(A1)***

$c \leq -5$ ***A1***

[4 marks]

15 – (MathAA/21_HL_Summer_2021_Q11) – *Algebra-functions - Roots, Graphs*

(a) attempt to solve $4x^2 - 1 = 0$ e.g. by factorising $4x^2 - 1$ ***(M1)***

$p = \frac{1}{2}, q = -\frac{1}{2}$ or vice versa ***A1***

[2 marks]

(b) attempt to use quotient rule or product rule ***(M1)***

EITHER

$$f'(x) = \frac{3(4x^2-1) - 8x(3x+2)}{(4x^2-1)^2} \left(= \frac{-12x^2 - 16x - 3}{(4x^2-1)^2} \right)$$ ***A1A1***

Note: Award ***A1*** for each term in the numerator with correct signs, provided correct denominator is seen.

OR

$$f'(x) = -8x(3x+2)(4x^2-1)^{-2} + 3(4x^2-1)^{-1}$$ ***A1A1***

Note: Award ***A1*** for each term.

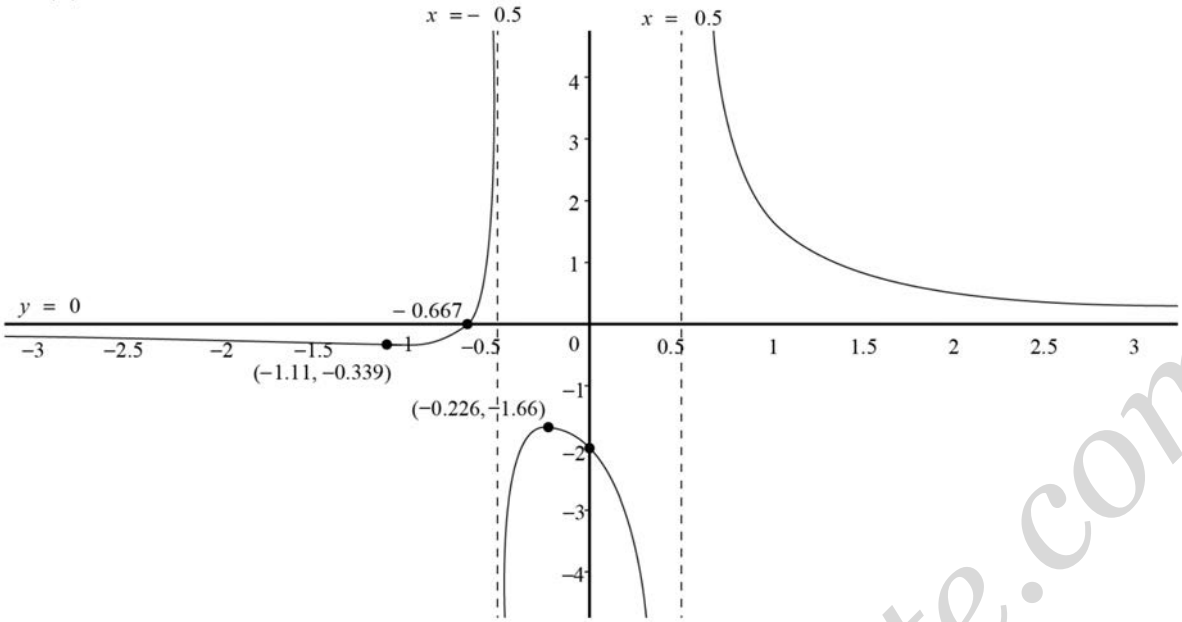
[3 marks]

(c) attempt to find the local min point on $y = f'(x)$ OR solve $f''(x) = 0$ ***(M1)***

$x = -1.60$ ***A1***

[2 marks]

(d)



A1A1A1A1A1

Note: Award ***A1*** for both vertical asymptotes with their equations, award ***A1*** for horizontal asymptote with equation, award ***A1*** for each correct branch including asymptotic behaviour, coordinates of minimum and maximum points (may be seen next to the graph) and values of axes intercepts.
If vertical asymptotes are absent (or not vertical) and the branches overlap as a consequence, award maximum ***A0A1A0A1A1.***

[5 marks]

e) $x = -\frac{2}{3}\left(= -0.667\right)$ ***A1***

(oblique asymptote has) gradient $\frac{4}{3}\left(= 1.33\right)$ ***(A1)***

appropriate method to find complete equation of oblique asymptote ***M1***

$$\begin{array}{r} \frac{4}{3}x - \frac{8}{9} \\ 3x+2\overline{\big)\,4x^2 + 0x - 1} \\ \underline{4x^2 + \frac{8}{3}x} \\ -\frac{8}{3}x - 1 \\ \underline{-\frac{8}{3}x - \frac{16}{9}} \\ \frac{7}{9} \end{array}$$

$y = \frac{4}{3}x - \frac{8}{9}\left(= 1.33x - 0.889\right)$ ***A1***

Note: Do not award the final ***A1*** if the answer is not given as an equation.

[4 marks]

f) attempting to find at least one critical value $\left(x = -0.568729..., x = 1.31872...\right)$ ***(M1)***

$-\frac{2}{3} < x < -0.569$ OR $-0.5 < x < 0.5$ OR $x > 1.32$ ***A1A1A1***

Note: Only penalize once for use of $\leq$ rather than $<$.

[4 marks]

16 – (MathAA/22_HL_Summer_2021_Q12) – *Algebra-functions - Roots, Differentiation*

(a) **EITHER**

$$f(-x) = \arcsin\left(\frac{(-x)^2 - 1}{(-x)^2 + 1}\right) = \arcsin\left(\frac{x^2 - 1}{x^2 + 1}\right) = f(x)$$ ***R1***

OR

a sketch graph of $y = f(x)$ with line symmetry in the y-axis indicated ***R1***

THEN

so $f(x)$ is an even function ***AG***

[1 mark]

(b) as $x \to \pm\infty$, $f(x) \to \arcsin 1 \left(\to \frac{\pi}{2}\right)$ ***A1***

so the horizontal asymptote is $y = \frac{\pi}{2}$ ***A1***

[2 marks]

(c) (i) attempting to use the quotient rule to find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2-1}{x^2+1}\right)$ ***M1***

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2-1}{x^2+1}\right)=\frac{2x\left(x^2+1\right)-2x\left(x^2-1\right)}{\left(x^2+1\right)^2}\quad\left(=\frac{4x}{\left(x^2+1\right)^2}\right)$$ ***A1***

attempting to use the chain rule to find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\arcsin\left(\frac{x^2-1}{x^2+1}\right)\right)$ ***M1***

let $u=\frac{x^2-1}{x^2+1}$ and so $y=\arcsin u$ and $\frac{\mathrm{d}y}{\mathrm{d}u}=\frac{1}{\sqrt{1-u^2}}$

$$f'(x)=\frac{1}{\sqrt{1-\left(\frac{x^2-1}{x^2+1}\right)^2}}\times\frac{4x}{\left(x^2+1\right)^2}$$ ***M1***

$$=\frac{4x}{\sqrt{\left(x^2+1\right)^2-\left(x^2-1\right)^2}}\times\frac{1}{\left(x^2+1\right)}$$ ***A1***

$$=\frac{4x}{\sqrt{4x^2}}\times\frac{1}{\left(x^2+1\right)}$$ ***A1***

$$=\frac{2x}{\sqrt{x^2}\left(x^2+1\right)}$$ ***AG***

(ii) $f'(x)=\frac{2x}{|x|\left(x^2+1\right)}$

EITHER

for $x<0$, $|x|=-x$ ***(A1)***

so $f'(x)=-\frac{2}{x^2+1}$ ***A1***

OR

$|x|>0$ and $x^2+1>0$ ***A1***

$2x<0,\ x<0$ ***A1***

THEN

$f'(x)<0$ ***R1***

Note: Award ***R1*** for stating that in $f'(x)$, the numerator is negative, and the denominator is positive.

so f is decreasing for $x<0$ ***AG***

Note: Do not accept a graphical solution.

[9 marks]

(d) $x = \arcsin\left(\dfrac{y^2-1}{y^2+1}\right)$ ***M1***

$\sin x = \dfrac{y^2-1}{y^2+1} \Rightarrow y^2 \sin x + \sin x = y^2 - 1$ ***A1***

$y^2 = \dfrac{1+\sin x}{1-\sin x}$ ***A1***

domain of g is $x \in \mathbb{R}, x \geq 0$ and so the range of g^{-1} must be $y \in \mathbb{R}, y \geq 0$

hence the positive root is taken (or the negative root is rejected) ***R1***

Note: The ***R1*** is dependent on the above ***A1***.

so $\left(g^{-1}(x) =\right)\sqrt{\dfrac{1+\sin x}{1-\sin x}}$ ***A1***

Note: The final ***A1*** is not dependent on ***R1*** mark.

[5 marks]

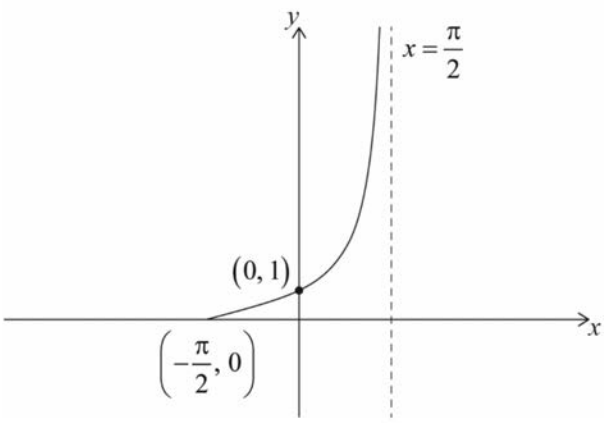
(e) domain is $-\dfrac{\pi}{2} \leq x < \dfrac{\pi}{2}$ ***A1***

Note: Accept correct alternative notations, for example, $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[$ or $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

Accept $[-1.57, 1.57[$ if correct to 3 s.f.

[1 mark]

(f)



A1A1A1

Note: ***A1*** for correct domain and correct range and y-intercept at $y = 1$

A1 for asymptotic behaviour $x \to \frac{\pi}{2}$

A1 for $x = \frac{\pi}{2}$

Coordinates are not required.
Do not accept $x = 1.57$ or other inexact values.

[3 marks]

17 – (MathAA/20_HL_Winter_2021_Q10) – *Algebra-functions - Roots, Graphs, Integration*

(a) (i)

Note: In part (a), penalise once only, if correct values are given instead of correct coordinates.

attempts to solve $x^2 - x - 12 = 0$ ***(M1)***

$(-3, 0)$ and $(4, 0)$ ***A1***

(ii) $\left(0, \frac{4}{5}\right)$ ***A1***

[3 marks]

(b) $x = \frac{15}{2}$ ***A1***

Note: Award ***A0*** for $x \neq \frac{15}{2}$.

Award ***A1*** in part (b), if $x = \frac{15}{2}$ is seen on their graph in part (d).

[1 mark]

(c) **METHOD 1**

$(ax+b)(2x-15) \equiv x^2 - x - 12$

attempts to expand $(ax+b)(2x-15)$ ***(M1)***

$2ax^2 - 15ax + 2bx - 15b \equiv x^2 - x - 12$

$a = \frac{1}{2}$ ***A1***

equates coefficients of x ***(M1)***

$-1 = -\frac{15}{2} + 2b$

$b = \frac{13}{4}$ ***A1***

$\left(y = \frac{x}{2} + \frac{13}{4}\right)$

METHOD 2

attempts division on $\frac{x^2-x-12}{2x-15}$ ***M1***

$\frac{x}{2}+\frac{13}{4}+...$ ***M1***

$a=\frac{1}{2}$ ***A1***

$b=\frac{13}{4}$ ***A1***

$\left(y=\frac{x}{2}+\frac{13}{4}\right)$

METHOD 3

$a=\frac{1}{2}$ ***A1***

$\frac{x^2-x-12}{2x-15}\equiv\frac{x}{2}+b+\frac{c}{2x-15}$ ***M1***

$x^2-x-12\equiv\frac{(2x-15)x}{2}+(2x-15)b+c$

equates coefficients of x : ***(M1)***

$-1=-\frac{15}{2}+2b$

$b=\frac{13}{4}$ ***A1***

$\left(y=\frac{x}{2}+\frac{13}{4}\right)$

METHOD 4

attempts division on $\frac{x^2-x-12}{2x-15}$ ***M1***

$\frac{x^2-x-12}{2x-15}=\frac{x}{2}+\frac{\frac{13x}{2}-12}{2x-15}$

$a=\frac{1}{2}$ ***A1***

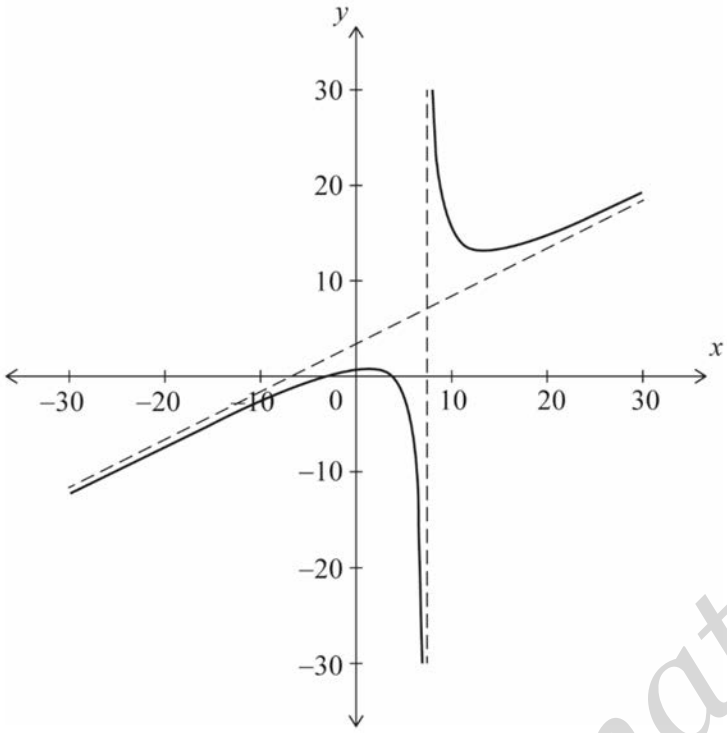
$\frac{\frac{13x}{2}-12}{2x-15}=\frac{13}{4}+...$ ***M1***

$b=\frac{13}{4}$ ***A1***

$\left(y=\frac{x}{2}+\frac{13}{4}\right)$

[4 marks]

(d)



two branches with approximately correct shape (for $-30 \le x \le 30$) ***A1***

their vertical and oblique asymptotes in approximately correct positions with both branches showing correct asymptotic behaviour to these asymptotes ***A1***

their axes intercepts in approximately the correct positions ***A1***

Note: Points of intersection with the axes and the equations of asymptotes are not required to be labelled.

[3 marks]

(e) (i) attempts to split into partial fractions: ***(M1)***

$$\frac{2x-15}{(x+3)(x-4)} \equiv \frac{A}{x+3} + \frac{B}{x-4}$$

$$2x-15 \equiv A(x-4)+B(x+3)$$

$A=3$ ***A1***

$B=-1$ ***A1***

$$\left(\frac{3}{x+3}-\frac{1}{x-4}\right)$$

(ii) $\int_0^3 \left(\frac{3}{x+3}-\frac{1}{x-4}\right) \mathrm{d}x$

attempts to integrate and obtains two terms involving 'ln' ***(M1)***

$=\left[3\ln|x+3|-\ln|x-4|\right]_0^3$ ***A1***

$=3\ln 6-\ln 1-3\ln 3+\ln 4$ ***A1***

$=3\ln 2+\ln 4 \ (=\ln 8+\ln 4)$

$=\ln 32 \ (=5\ln 2)$ ***A1***

Note: The final ***A1*** is dependent on the previous two ***A*** marks.

[7 marks]

Total [18 marks]

18 – (MathAA/21_HL_Summer_2022_Q6) – *Algebra-functions - Roots*

(a) attempt to replace x with $-x$ ***M1***

$f(-x) = 2^{-x} - \frac{1}{2^{-x}}$

EITHER

$= \frac{1}{2^x} - 2^x = -f(x)$ ***A1***

OR

$= -\left(2^x - \frac{1}{2^x}\right)\left(= -f(x)\right)$ ***A1***

Note: Award ***M1A0*** for a graphical approach including evidence that **either** the graph is invariant after rotation by 180° about the origin **or** the graph is invariant after a reflection in the y-axis and then in the x-axis (or vice versa).

so f is an odd function ***AG***

[2 marks]

(b) attempt to find at least one intersection point ***(M1)***

$x = -1.26686..., x = 0.177935..., x = 3.06167...$

$x = -1.27, x = 0.178, x = 3.06$

$-1.27 \le x < -1$, ***A1***

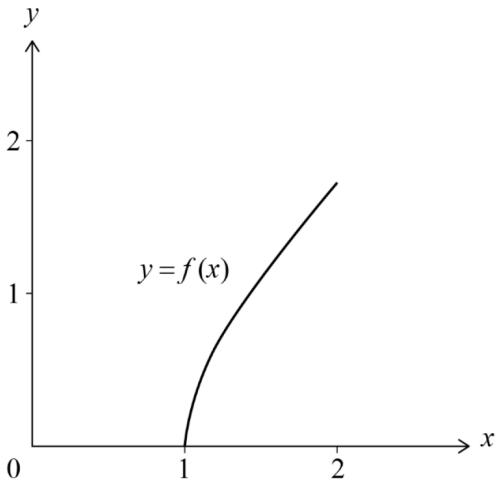
$0.178 \le x < 3$, ***A1***

$x \ge 3.06$ ***A1***

[4 marks]

19 – (MathAA/21_HL_Summer_2022_Q10) – *Algebra-functions - Roots, Differentiation, Integration*

(a)



correct shape (concave down) within the given domain $1 \le x \le 2$ ***A1***

$(1,0)$ and $(2,\sqrt{3})(=(2,1.73))$ ***A1***

Note: The coordinates of endpoints may be seen on the graph or marked on the axes.

(b) (i) interchanging x and y (seen anywhere) ***M1***

$x = \sqrt{y^2 - 1}$

$x^2 = y^2 - 1$ ***A1***

$y = \sqrt{x^2 + 1}$ ***A1***

$f^{-1}(x) = \sqrt{x^2 + 1}$ ***AG***

(ii) $0 \le x \le \sqrt{3}$ OR domain $[0,\sqrt{3}](=[0,1.73])$ ***A1***

$1 \le y \le 2$ OR $1 \le f^{-1}(x) \le 2$ OR range $[1,2]$ ***A1***

[5 marks]

(c) (i) attempt to substitute $x=\sqrt{y^2+1}$ into the correct volume formula ***(M1)***

$V=\pi\int_0^h\left(\sqrt{y^2+1}\right)^2\,\mathrm{d}y\ \left(=\pi\int_0^h\left(y^2+1\right)\,\mathrm{d}y\right)$ ***A1***

$=\pi\left[\frac{1}{3}y^3+y\right]_0^h$ ***A1***

$=\pi\left(\frac{1}{3}h^3+h\right)$ ***AG***

Note: Award marks as appropriate for correct work using a different variable e.g. $\pi\int_0^h\left(\sqrt{x^2+1}\right)^2\,\mathrm{d}x$

(ii) attempt to substitute $h=\sqrt{3}\ \left(=1.732...\right)$ into V ***(M1)***

$V=10.8828...$

$V=10.9$ (m³) $\left(=2\sqrt{3}\pi\right)$ (m³) ***A1***

[5 marks]

(d) **METHOD 1**

time $=\frac{10.8828...}{0.4}\left(=\frac{2\sqrt{3}\pi}{0.4}\right)$ ***(M1)***

$=27.207...$

$=27.2\left(=5\sqrt{3}\pi\right)(s)$ ***A1***

[2 marks]

(e) attempt to find the height of the tank when $V = 5.4414...\ \left(= \sqrt{3}\pi\right)$ ***(M1)***

$\pi\left(\frac{1}{3}h^3 + h\right) = 5.4414...\ \left(= \sqrt{3}\pi\right)$

$h = 1.1818...$ ***(A1)***

attempt to use the chain rule or differentiate $V = \pi\left(\frac{1}{3}h^3 + h\right)$ with respect to t ***(M1)***

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{\pi(h^2+1)} \times \frac{\mathrm{d}V}{\mathrm{d}t}$ OR $\frac{\mathrm{d}V}{\mathrm{d}t} = \pi\left(h^2+1\right)\frac{\mathrm{d}h}{\mathrm{d}t}$ ***(A1)***

attempt to substitute **their** h and $\frac{\mathrm{d}V}{\mathrm{d}t} = 0.4$ ***(M1)***

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{0.4}{\pi\left(1.1818...^2+1\right)} = 0.053124...$

$= 0.0531$ (m s^{-1}) ***A1***

[6 marks]

20 – (MathAA/22_HL_Summer_2022_Q7) – *Algebra-functions - Roots*

(a) (as $\lim_{x \to 0} x^2 = 0$, the indeterminate form $\frac{0}{0}$ is required for the limit to exist)

$\Rightarrow \lim_{x \to 0}\left(\arctan\left(\cos x\right) - k\right) = 0$ ***M1***

$\arctan 1 - k = 0\ \left(k = \arctan 1\right)$ ***A1***

so $k = \frac{\pi}{4}$ ***AG***

Note: Award ***M1A0*** for using $k = \frac{\pi}{4}$ to show the limit is $\frac{0}{0}$.

[2 marks]

(b) $\lim_{x\to 0}\dfrac{\arctan(\cos x)-\frac{\pi}{4}}{x^2}\ \left(=\dfrac{0}{0}\right)$

$=\lim_{x\to 0}\dfrac{\frac{-\sin x}{1+\cos^2 x}}{2x}$ ***A1A1***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator.

recognises to apply l'Hôpital's rule again ***(M1)***

$=\lim_{x\to 0}\dfrac{\frac{-\sin x}{1+\cos^2 x}}{2x}\ \left(=\dfrac{0}{0}\right)$

Note: Award ***M0*** if their limit is not the indeterminate form $\dfrac{0}{0}$.

EITHER

$=\lim_{x\to 0}\dfrac{-\cos x\left(1+\cos^2 x\right)-2\sin^2 x\cos x}{\left(1+\cos^2 x\right)^2}$ ***A1A1***

Note: Award ***A1*** for a correct first term in the numerator and ***A1*** for a correct second term in the numerator.

OR

$\lim_{x\to 0}\dfrac{-\cos x}{2\left(1+\cos^2 x\right)-4x\sin x\cos x}$ ***A1A1***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator.

THEN

substitutes $x=0$ into the correct expression to evaluate the limit ***A1***

Note: The final ***A1*** is dependent on all previous marks.

$=-\dfrac{1}{4}$ ***AG***

[6 marks]

21 – (MathAA/20_HL_Winter_2022_Q11) – *Algebra-functions - Roots, Differentiation*

(a) attempt to use product rule ***(M1)***

$f''(x) = 3e^{2x} + 2e^{2x}(3x-4)\left(= e^{2x}(6x-5)\right)$ ***A2***

Note: Award ***A1*** for 2 out of 3 of $3e^{2x}$, $6xe^{2x}$ and $-8e^{2x}$ seen or implied.

[3 marks]

(b) $f'(x) = 1$ ***(M1)***

$x = 0.86299...$

$x = 0.863$ ***A1***

$y = -7.92719...$

$y = -7.93$ ***A1***

$(0.863, -7.93)$

[3 marks]

(c) x-intercept is at $\frac{4}{3}(1.33)$ ***(A1)***

attempt to use formula for volume of revolution ***(M1)***

Note: Award ***(M1)*** for an integral involving π and $(f(x))^2$. Condone use of 2π and incorrect or absent limits.

$\pi\int_0^{\frac{4}{3}}\left(e^{2x}(3x-4)\right)^2 dx$ ***(A1)***

Note: This ***(A1)*** can be awarded if the dx is omitted.

$= 164.849...$

$= 165$ ***A1***

[4 marks]

(d) (i) attempt to compose functions in the correct order ***(M1)***

$(f \circ g)(0) = f(g(0)) = f(1)$

$= -7.38905...$

$= -7.39 (= -e^{-2})$ ***A1***

(ii) attempt to use the chain rule ***(M1)***

$(f \circ g)'(0) = f'(g(0))g'(0)$

Note: For this ***(M1)*** to be awarded, multiplication of two derivatives should be seen or implied.

$= 2f'(1)(= 2 \times 7.38905...)$ ***(A1)***

$= 14.7781...$

$= 14.8 (= 2e^{-2})$ ***A1***

[5 marks]

1 - (MAT/20_HL_Winter_2012_Q2) - *Quadratics*

$\Delta = (5-k)^2 + 4(k+2)$
$= k^2 - 6k + 33$
$= (k-3)^2 + 24$ which is positive for all k

2 - (MAT/21_HL_Summer_2013_Q9) - *Quadratics*

(a) $\Delta = b^2 - 4ac = 4k^2 - 4 \times 3 \times (k-1) = 4k^2 - 12k + 12$

Note: Award ***M1A1*** if expression seen within quadratic formula.

EITHER

$144 - 4 \times 4 \times 12 < 0$
Δ always positive, therefore the equation always has two distinct real roots
(and cannot be always negative as $a > 0$)

OR

sketch of $y = 4k^2 - 12k + 12$ or $y = k^2 - 3k + 3$ not crossing the x-axis
Δ always positive, therefore the equation always has two distinct real roots

OR

write Δ as $4(k-1.5)^2 + 3$
Δ always positive, therefore the equation always has two distinct real roots

(b) closest together when Δ is least
minimum value occurs when $k = 1.5$

3 - (MAT/21_HL_Summer_2016_Q2) - *Quadratics, Functions - Roots*

(a) $(x+2)^2 - 6$

(b) $(g \circ f)(x) = (x+2)^2 - 6$
$\Rightarrow g(x) = x^2 - 6$

4 – (MAT/22_HL_Summer_2017_Q4) – *Quadratics, Trigonometry*

(a) $k^2 - k - 12 < 0$

$(k-4)(k+3) < 0$ ***(M1)***

$-3 < k < 4$ ***A1***

[2 marks]

(b) $\cos B = \dfrac{2^2 + c^2 - 4^2}{4c}$ $\left(\text{or } 16 = 2^2 + c^2 - 4c\cos B\right)$ ***M1***

$\Rightarrow \dfrac{c^2 - 12}{4c} < \dfrac{1}{4}$ ***A1***

$\Rightarrow c^2 - c - 12 < 0$

from result in (a)

$0 < \text{AB} < 4$ or $-3 < \text{AB} < 4$ ***(A1)***

but AB must be at least 2

$\Rightarrow 2 < AB < 4$ ***A1***

Note: Allow $\leq AB$ for either of the final two ***A*** marks.

[4 marks]

5 – (MAT/20_HL_Winter_2017_Q7) – *Quadratics*

METHOD 1

let roots be α and 3α ***(M1)***

sum of roots $(4\alpha)=\frac{8}{7}$ ***M1***

$\Rightarrow \alpha=\frac{2}{7}$ ***A1***

EITHER

product of roots $\left(3\alpha^2\right)=\frac{p}{7}$ ***M1***

$p=21\alpha^2=21\times\frac{4}{49}$

OR

$7\left(\frac{2}{7}\right)^2-8\left(\frac{2}{7}\right)+p=0$ ***M1***

$\frac{4}{7}-\frac{16}{7}+p=0$

THEN

$\Rightarrow p=\frac{12}{7}(=1.71)$ ***A1***

METHOD 2

$x=\frac{8\pm\sqrt{64-28p}}{14}$ ***(M1)***

$\frac{8+\sqrt{64-28p}}{14}=3\left(\frac{8-\sqrt{64-28p}}{14}\right)$ ***M1A1***

$8+\sqrt{64-28p}=24-3\sqrt{64-28p}\Rightarrow\sqrt{64-28p}=4$ ***(M1)***

$p=\frac{12}{7}(=1.71)$ ***A1***

[5 marks]

6 – (MathAA/20_HL_Winter_2021_Q6) – *Quadratics*

(a) **METHOD 1**

$(p+q)^3 - 3pq(p+q) \equiv p^3 + q^3$

attempts to expand $(p+q)^3$ ***M1***

$p^3 + 3p^2q + 3pq^2 + q^3$

$(p+q)^3 - 3pq(p+q) \equiv p^3 + 3p^2q + 3pq^2 + q^3 - 3pq(p+q)$

$\equiv p^3 + 3p^2q + 3pq^2 + q^3 - 3p^2q - 3pq^2$ ***A1***

$\equiv p^3 + q^3$ ***AG***

Note: Condone the use of equals signs throughout.

METHOD 2

$(p+q)^3 - 3pq(p+q) \equiv p^3 + q^3$

attempts to factorise $(p+q)^3 - 3pq(p+q)$ ***M1***

$\equiv (p+q)\left((p+q)^2 - 3pq\right)$ $\left(\equiv (p+q)\left(p^2 - pq + q^2\right)\right)$

$\equiv p^3 - p^2q + pq^2 + p^2q - pq^2 + q^3$ ***A1***

$\equiv p^3 + q^3$ ***AG***

Note: Condone the use of equals signs throughout.

METHOD 3

$p^3 + q^3 \equiv (p+q)^3 - 3pq(p+q)$

attempts to factorise $p^3 + q^3$ ***M1***

$\equiv (p+q)\left(p^2 - pq + q^2\right)$

$\equiv (p+q)\left((p+q)^2 - 3pq\right)$ ***A1***

$\equiv (p+q)^3 - 3pq(p+q)$ ***AG***

Note: Condone the use of the equals sign throughout.

[2 marks]

(b)

Note: Award a maximum of ***A1M0A0A1M0A0*** for $m=-95$ and $n=8$ found by using $\alpha,\beta=\frac{5\pm\sqrt{17}}{4}$ $(\alpha,\beta=0.219...,2.28...)$.

Condone, as appropriate, solutions that state but clearly do not use the values of α and β.

Special case: Award a maximum of ***A1M1A0A1M0A0*** for $m=-95$ and $n=8$ obtained by solving simultaneously for α and β from product of roots and sum of roots equations.

product of roots of $x^2-\frac{5}{2}x+\frac{1}{2}=0$

$\alpha\beta=\frac{1}{2}$ (seen anywhere) ***A1***

considers $\left(\frac{1}{\alpha^3}\right)\left(\frac{1}{\beta^3}\right)$ by stating $\frac{1}{(\alpha\beta)^3}(=n)$ ***M1***

Note: Award ***M1*** for attempting to substitute their value of $\alpha\beta$ into $\frac{1}{(\alpha\beta)^3}$.

$\frac{1}{(\alpha\beta)^3}=\frac{1}{\left(\frac{1}{2}\right)^3}$

$n=8$ ***A1***

sum of roots of $x^2-\frac{5}{2}x+\frac{1}{2}=0$

$\alpha+\beta=\frac{5}{2}$ (seen anywhere) ***A1***

considers $\frac{1}{\alpha^3}+\frac{1}{\beta^3}$ by stating $\frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{(\alpha\beta)^3}$ $\left(\left(\frac{\alpha+\beta}{\alpha\beta}\right)^3-\frac{3(\alpha+\beta)}{(\alpha\beta)^2}\right)(=-m)$ ***M1***

Note: Award ***M1*** for attempting to substitute their values of $\alpha+\beta$ and $\alpha\beta$ into their expression. Award ***M0*** for use of $(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$ only.

$=\frac{\left(\frac{5}{2}\right)^3-\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)}{\frac{1}{8}}$ $(=125-30=95)$

$m=-95$ ***A1***

$(x^2-95x+8=0)$

[6 marks]

Total [8 marks]

7 – (MathAA/21_HL_Summer_2022_Q8) – *Quadratics*

(a) product of roots $= \frac{2k+9}{k}$ ***A1***

[1 mark]

(b) recognition that the product of the roots will be negative ***(M1)***

$\frac{2k+9}{k} < 0$

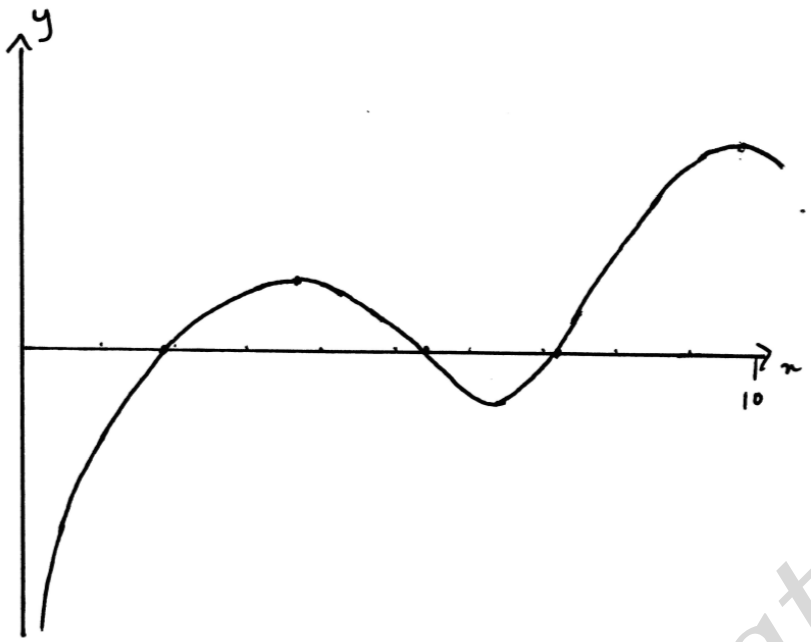
critical values $k = 0, -\frac{9}{2}$ seen ***(A1)***

$-\frac{9}{2} < k < 0$ ***A1***

[3 marks]

1 – (MAT/20_HL_Winter_2013_Q3) – *Graphs, Exponentials - Logarithms*

(a)



A correct graph shape for $0 < x \leq 10$.
maxima $(3.78,\ 0.882)$ and $(9.70,\ 1.89)$
minimum $(6.22,\ -0.885)$
x-axis intercepts $(1.97, 0)$, $(5.24, 0)$ and $(7.11, 0)$

Note: Award *A1* if two x-axis intercepts are correct.

(b) $0 < x \leq 1.97$
$5.24 \leq x \leq 7.11$

2 – (MAT/20_HL_Winter_2013_Q13) – *Functions - Roots, Exponentials - Logarithms, Integration*

(a) (i) either counterexample or sketch or
recognising that $y = k$ $(k > 1)$ intersects the graph of $y = f(x)$ twice .
function is not 1–1 (does not obey horizontal line test)
so f^{-1} does not exist

(ii) $f'(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right)$

$f'(\ln 3) = \frac{4}{3}$ $(= 1.33)$

$m = -\frac{3}{4}$

$f(\ln 3) = \frac{5}{3}$ $(= 1.67)$

EITHER

$$\frac{y - \frac{5}{3}}{x - \ln 3} = -\frac{3}{4}$$

$$4y - \frac{20}{3} = -3x + 3\ln 3$$

OR

$$\frac{5}{3} = -\frac{3}{4}\ln 3 + c$$

$$c = \frac{5}{3} + \frac{3}{4}\ln 3$$

$$y = -\frac{3}{4}x + \frac{5}{3} + \frac{3}{4}\ln 3$$

$$12y = -9x + 20 + 9\ln 3$$

THEN

$$9x + 12y - 9\ln 3 - 20 = 0$$

(iii) The tangent at $(a, f(a))$ has equation $y - f(a) = f'(a)(x - a)$.

$f'(a) = \frac{f(a)}{a}$ (or equivalent)

$e^{a} - e^{-a} = \frac{e^{a} + e^{-a}}{a}$ (or equivalent)

attempting to solve for a

$a = \pm 1.20$

(b) (i) $2y = e^x + e^{-x}$

$e^{2x} - 2ye^x + 1 = 0$ ***M1A1***

Note: Award ***M1*** for either attempting to rearrange or interchanging x and y.

$e^x = \dfrac{2y \pm \sqrt{4y^2 - 4}}{2}$ ***A1***

$e^x = y \pm \sqrt{y^2 - 1}$

$x = \ln\left(y \pm \sqrt{y^2 - 1}\right)$ ***A1***

$f^{-1}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$ ***A1***

Note: Award ***A1*** for correct notation and for stating the positive "branch".

(ii) $V = \pi\int_1^5 \left(\ln\left(y + \sqrt{y^2 - 1}\right)\right)^2 \mathrm{d}y$ ***(M1)(A1)***

Note: Award ***M1*** for attempting to use $V = \pi\int_c^d x^2 \mathrm{d}y$.

$= 37.1\ \left(\text{units}^3\right)$ ***A1***

3 – (MAT/22_HL_Summer_2016_Q3) – *Exponentials - Logarithms*

METHOD 1

$\ln\frac{y}{x} = 2 \Rightarrow -\ln x + \ln y = 2$

$\ln x^2 + \ln y^3 = 7 \Rightarrow 2\ln x + 3\ln y = 7$

attempting to solve for x and y (to obtain $\ln x = \frac{1}{5}$ and $\ln y = \frac{11}{5}$)

$x = e^{\frac{1}{5}}\ (= 1.22)$

$y = e^{\frac{11}{5}}\ (= 9.03)$

METHOD 2

$\ln\frac{y}{x} = 2 \Rightarrow y = e^2 x$

$\ln x^2 + \ln e^6 x^3 = 7$

attempting to solve for x

$x = e^{\frac{1}{5}}\ (= 1.22)$

$y = e^{\frac{11}{5}}\ (= 9.03)$

METHOD 3

$\ln\frac{y}{x} = 2 \Rightarrow y = e^2 x$

$\ln x^2 + \ln y^3 = 7 \Rightarrow \ln\left(x^2 y^3\right) = 7$

$x^2 y^3 = e^7$

substituting $y = e^2 x$ into $x^2 y^3 = e^7$ (to obtain $e^6 x^5 = e^7$)

$x = e^{\frac{1}{5}}\ (= 1.22)$

$y = e^{\frac{11}{5}}\ (= 9.03)$

4 – (MAT/21_HL_Summer_2016_Q7) – *Exponentials - Logarithms*

(a) $a = 420.65$

$390.94 = a \times 2^b$

$2^b = \dfrac{390.94}{420.65} = 0.929...$

$b = -0.10567$

(b) $N = 8 \;\; T = 337.67$

Note: Accept 5sf answers between 337.44 and 337.67.

(c) $N = 8$ Percentage error 1.29%

Note: Accept negative values of the above.

(d) likely not to be a good fit for larger values of N
likely to be quite a good fit for values close to 8

5 – (MAT/22_HL_Summer_2017_Q6) – *Exponentials - Logarithms*

$\log_{10} \dfrac{1}{2\sqrt{2}}(p+2q) = \dfrac{1}{2}\left(\log_{10} p + \log_{10} q\right)$

$\log_{10} \dfrac{1}{2\sqrt{2}}(p+2q) = \dfrac{1}{2}\log_{10} pq$ ***(M1)***

$\log_{10} \dfrac{1}{2\sqrt{2}}(p+2q) = \log_{10}(pq)^{\frac{1}{2}}$ ***(M1)***

$\dfrac{1}{2\sqrt{2}}(p+2q) = (pq)^{\frac{1}{2}}$ ***(A1)***

$(p+2q)^2 = 8pq$

$p^2 + 4pq + 4q^2 = 8pq$

$p^2 - 4pq + 4q^2 = 0$

$(p-2q)^2 = 0$ ***M1***

hence $p = 2q$ ***A1***

[5 marks]

6 – (MAT/21_HL_Summer_2019_Q11) – *Exponentials - Logarithms*

(a) recognition of the other root $= -d\mathrm{i}$ ***(A1)***

$\log_2 a + \log_2 b + \log_2 c + d\mathrm{i} - d\mathrm{i} = 3$ ***M1A1***

Note: Award ***M1*** for sum of the roots, ***A1*** for 3. Award ***A0M1A0*** for just $\log_2 a + \log_2 b + \log_2 c = 3$.

$\log_2 abc = 3$ ***(M1)***

$\Rightarrow abc = 2^3$ ***A1***

$abc = 8$ ***AG***

[5 marks]

(b) **METHOD 1**

let the geometric series be u_1, $u_1 r$, $u_1 r^2$

$(u_1 r)^3 = 8$ ***M1***

$u_1 r = 2$ ***A1***

hence one of the roots is $\log_2 2 = 1$ ***R1***

METHOD 2

$\frac{b}{a} = \frac{c}{b}$

$b^2 = ac \Rightarrow b^3 = abc = 8$ ***M1***

$b = 2$ ***A1***

hence one of the roots is $\log_2 2 = 1$ ***R1***

[3 marks]

(c) **METHOD 1**

product of the roots is $r_1 \times r_2 \times 1 \times d\mathrm{i} \times -d\mathrm{i} = -8d^2$ ***(M1)(A1)***

$r_1 \times r_2 = -8$ ***A1***

sum of the roots is $r_1 + r_2 + 1 + d\mathrm{i} + -d\mathrm{i} = 3$ ***(M1)(A1)***

$r_1 + r_2 = 2$ ***A1***

solving simultaneously ***(M1)***

$r_1 = -2$, $r_2 = 4$ ***A1A1***

7 – (MathAA/22_HL_Summer_2021_Q5) – *Exponentials - Logarithms*

(a) $100 = A_0 e^0$ ***A1***

$A_0 = 100$ ***AG***

[1 mark]

(b) correct substitution of values into exponential equation ***(M1)***

$50 = 100e^{-5730k}$ OR $e^{-5730k} = \frac{1}{2}$

EITHER

$-5730k = \ln\frac{1}{2}$ ***A1***

$\ln\frac{1}{2} = -\ln 2$ OR $-\ln\frac{1}{2} = \ln 2$ ***A1***

OR

$e^{5730k} = 2$ ***A1***

$5730k = \ln 2$ ***A1***

THEN

$k = \frac{\ln 2}{5730}$ ***AG***

Note: There are many different ways of showing that $k = \frac{\ln 2}{5730}$ which involve showing different steps. Award full marks for at least two correct algebraic steps seen.

[3 marks]

(c) if $25\,\%$ of the carbon-14 has decayed, $75\,\%$ remains ie, 75 units remain

$75 = 100e^{-\frac{\ln 2}{5730}t}$ ***(A1)***

EITHER

using an appropriate graph to attempt to solve for t ***(M1)***

OR

manipulating logs to attempt to solve for t ***(M1)***

$\ln 0.75 = -\frac{\ln 2}{5730}t$

$t = 2378.164...$

THEN

$t = 2380$ (years) (correct to the nearest 10 years) ***A1***

[3 marks]

8 – (MathAA/21_HL_Summer_2021_Q10) – *Exponentials - Logarithms, Differentiation*

(a) attempt to find the point of intersection of the graph of f and the line $y = x$ ***(M1)***

$x = 5.56619...$

$= 5.57$ ***A1***

[2 marks]

(b) $f'(x) = -45e^{-0.5x}$ ***A1***

attempt to set the gradient of f equal to -1 ***(M1)***

$-45e^{-0.5x} = -1$

Q has coordinates $(2\ln 45, 2)$ (accept $(-2\ln\frac{1}{45}, 2)$) ***A1A1***

Note: Award ***A1*** for each value, even if the answer is not given as a coordinate pair.

Do not accept $\frac{\ln\frac{1}{45}}{-0.5}$ or $\frac{\ln 45}{0.5}$ as a final value for x. Do not accept 2.0 or 2.00 as a final value for y.

[4 marks]

(c) attempt to substitute coordinates of Q (in any order) into an appropriate equation ***(M1)***

$y - 2 = -(x - 2\ln 45)$ OR $2 = -2\ln 45 + c$ ***A1***

equation of L is $y = -x + 2\ln 45 + 2$ ***AG***

[2 marks]

(d) (i) $x = \ln 45 + 1 (= 4.81)$ ***A1***

(ii) appropriate method to find the sum of two areas using integrals of the difference of two functions ***(M1)***

Note: Allow absent or incorrect limits.

$$\int_{4.806...}^{5.566...} \left(x - \left(-x + 2\ln 45 + 2\right)\right) dx + \int_{5.566...}^{7.613...} \left(90e^{-0.5x} - \left(-x + 2\ln 45 + 2\right)\right) dx$$ ***(A1)(A1)***

Note: Award ***A1*** for one correct integral expression including correct limits and integrand.
Award ***A1*** for a second correct integral expression including correct limits and integrand.

$= 1.51965...$
$= 1.52$ ***A1***

[5 marks]

(e) by symmetry 2×1.52 ***(M1)***
$= 3.03930...$
$= 3.04$ ***A1***

Note: Accept any answer that rounds to 3.0 (but do not accept 3).

[2 marks]

9 – (MathAA/20_HL_Winter_2022_Q4) – *Exponentials - Logarithms*

recognition that initial population is 15000 (seen anywhere) ***(A1)***

$P(0) = 15000$ OR 0.11×15000 OR 0.89×15000

population after 11% decrease is $15000 \times 0.89 (= 13350)$ ***(A1)***

recognizing that $t = 8$ on 1 January 2022 (seen anywhere) ***(A1)***

substitution of their value of t for 1 January 2022 and their value of $P(8)$ into the model ***(M1)***

$15000 \times 0.89 = 15000e^{8k}$ OR $13350 = 15000e^{8k}$

$k = \frac{\ln 0.89}{8} (-0.014566)$ ***(A1)***

substitution of $t = 2041 - 2014 (= 27)$ and their value for k into the model ***(M1)***

$P(27) = 15000e^{-0.0145... \times 27}$

$10122.3...$

$P(27) = 10100$ (10122) ***A1***

1 – (MAT/21_HL_Summer_2016_Q5) – *Graphs*

(a) $f(-x) = \dfrac{3(-x)^2 + 10}{(-x)^2 - 4}$ ***A1***

$= \dfrac{3x^2 + 10}{x^2 - 4} = f(x)$

$f(x) = f(-x)$ ***R1***

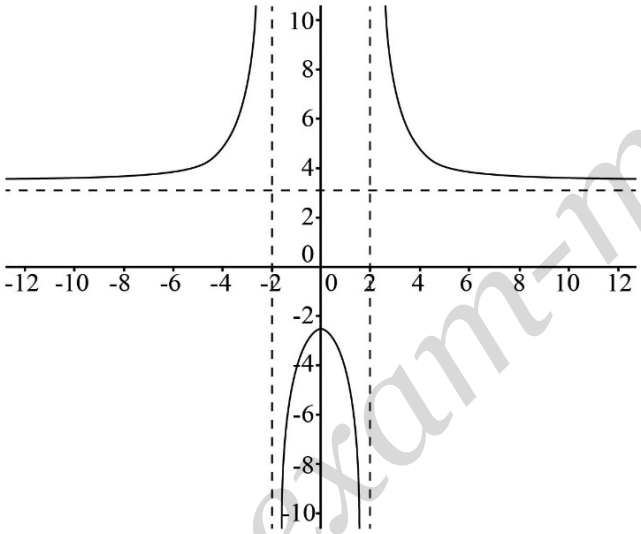
hence this is an even function ***AG***

Note: Award ***A1R1*** for the statement, all the powers are even hence $f(x) = f(-x)$.

Note: Just stating all the powers are even is ***A0R0***.

Note: Do not accept arguments based on the symmetry of the graph.

(b) (i)



correct shape in 3 parts which are asymptotic and symmetrical ***A1***

correct vertical asymptotes clear at 2 and -2 ***A1***

correct horizontal asymptote clear at 3 ***A1***

(ii) $f(x) > 3$

$f(x) \le -2.5$

2 – (MAT/21_HL_Summer_2017_Q6) – *Graphs*

(a) $x = \frac{\pi}{4}$ ***A1***

$x = \frac{5\pi}{4},\ x = -\frac{3\pi}{4}$ ***A1***

[2 marks]

(b) reflection in the y-axis results in $y = \tan\left(-x + \frac{\pi}{4}\right)\left(= \cot\left(x + \frac{\pi}{4}\right)\right)$ ***(A1)***

vertical stretch gives $y = \frac{1}{2}\tan\left(-x + \frac{\pi}{4}\right)\left(= \frac{1}{2}\cot\left(x + \frac{\pi}{4}\right)\right)$ ***(A1)***

translation

$y = \frac{1}{2}\tan\left[-\left(x - \frac{\pi}{4} - \frac{\pi}{4}\right)\right] - 3$

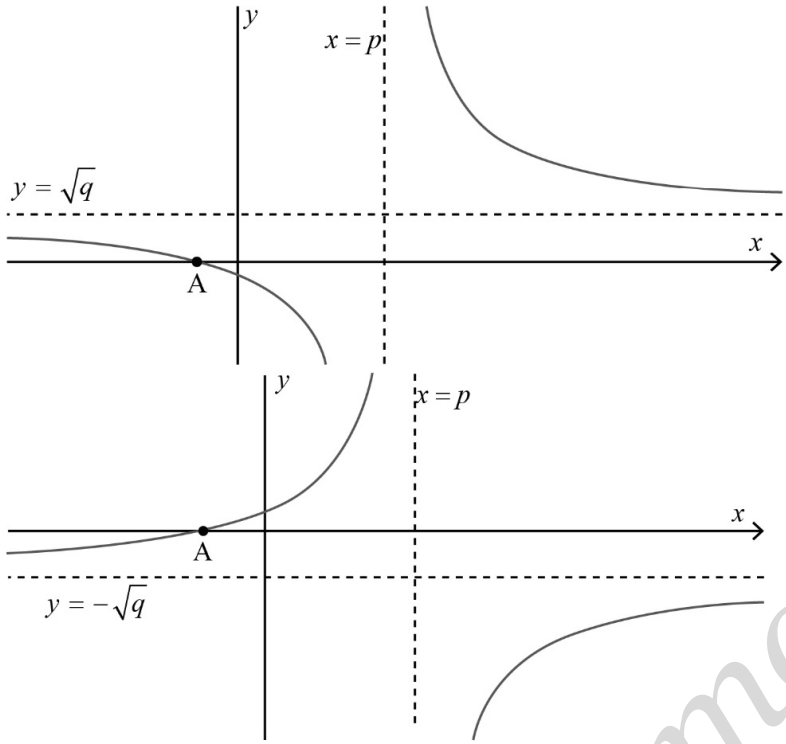
$= \frac{1}{2}\tan\left(-x + \frac{\pi}{2}\right) - 3\left(= \frac{1}{2}\cot(x) - 3\right)$ ***A1A1***

Notes: Award the ***A1***s independently of each other.
Do not penalize the absence of $y =$.

[4 marks]

3 – (MAT/20_HL_Winter_2018_Q8) – *Graphs*

(a)



either graph passing through (or touching) A	***A1***
correct shape and vertical asymptote with correct equation for either graph	***A1***
correct horizontal asymptote with correct equation for either graph	***A1***
two completely correct sketches	***A1***
	[4 marks]

(b) $a\left(-\frac{1}{2}\right)+1=0 \Rightarrow a=2$ ***A1***

from horizontal asymptote, $\left(\frac{a}{b}\right)^2=\frac{4}{9}$ ***(M1)***

$\frac{a}{b}=\pm\frac{2}{3} \Rightarrow b=\pm 3$ ***A1***

from vertical asymptote, $b\left(\frac{4}{3}\right)+c=0$

$b=3,\ c=-4$ or $b=-3,\ c=4$ ***A1***

[4 marks]

4 – (MAT/22_HL_Summer_2019_Q9) – *Graphs*

(a) **METHOD 1**

attempt to find roots or factors ***(M1)***

roots are $-3,\ 1,\ (4+\mathrm{i}),(4-\mathrm{i})$ ***A1A1***

Note: Award ***A1*** for each pair of roots or factors, real and complex.

attempt to form quadratic ***M1***

$(z-1)(z+3)=z^2+2z-3$ ***A1***

$(z-(4+\mathrm{i}))(z-(4-\mathrm{i}))$

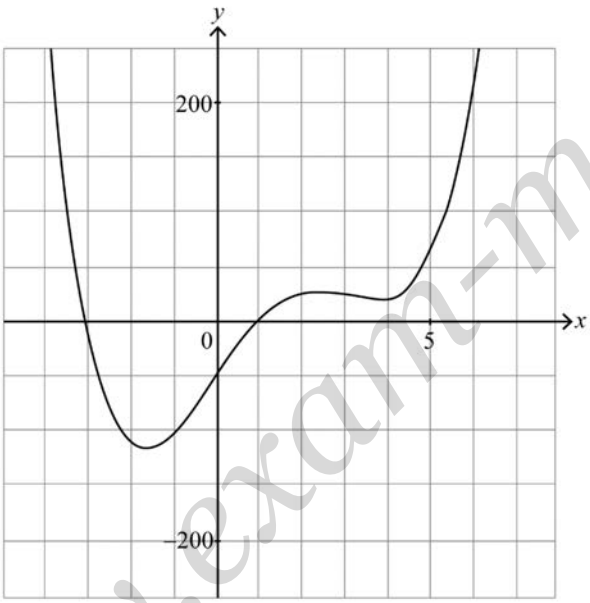
$=z^2-(4-\mathrm{i})z-(4+\mathrm{i})z+17$ ***(A1)***

$=z^2-8z+17$ ***A1***

$z^4-6z^3-2z^2+58z-51=(z^2-8z+17)(z^2+2z-3)$

[7 marks]

(b)



shape ***A1***

x-axis intercepts at $(-3,\ 0)$, $(1,\ 0)$ and y-axis intercept at $(0,\ -51)$ ***A1A1***

minimum points at $(-1.62,\ -118)$ and $(3.72,\ 19.7)$ ***A1A1***

maximum point at $(2.40,\ 26.9)$ ***A1***

Note: Coordinates may be seen on the graph or elsewhere.

Note: Accept -3, 1 and -51 marked on the axes.

[6 marks]

(c) from graph, $19.7 \le k \le 26.9$ ***A1A1***

Note: Award ***A1*** for correct endpoints and ***A1*** for correct inequalities.

[2 marks]

5 – (MAT/20_HL_Winter_2019_Q3) – *Graphs*

(a) $f(1)=0$ ***(A1)***

$f(0)=-1$ ***A1***

[2 marks]

(b) $a=f(3)$ ***(M1)***

$\Rightarrow a=4$ ***A1***

[2 marks]

(c) domain is $-2\le x\le 6$ ***A1***

range is $-6\le y\le 10$ ***A1***

[2 marks]

6 – (MathAA/21_HL_Summer_2021_Q11) – *Algebra-functions - Roots, Graphs*

(a) attempt to solve $4x^2-1=0$ e.g. by factorising $4x^2-1$ ***(M1)***

$p=\frac{1}{2}, q=-\frac{1}{2}$ or vice versa ***A1***

[2 marks]

(b) attempt to use quotient rule or product rule ***(M1)***

EITHER

$$f'(x)=\frac{3(4x^2-1)-8x(3x+2)}{(4x^2-1)^2}\left(=\frac{-12x^2-16x-3}{(4x^2-1)^2}\right)$$ ***A1A1***

Note: Award ***A1*** for each term in the numerator with correct signs, provided correct denominator is seen.

OR

$$f'(x)=-8x(3x+2)(4x^2-1)^{-2}+3(4x^2-1)^{-1}$$ ***A1A1***

Note: Award ***A1*** for each term.

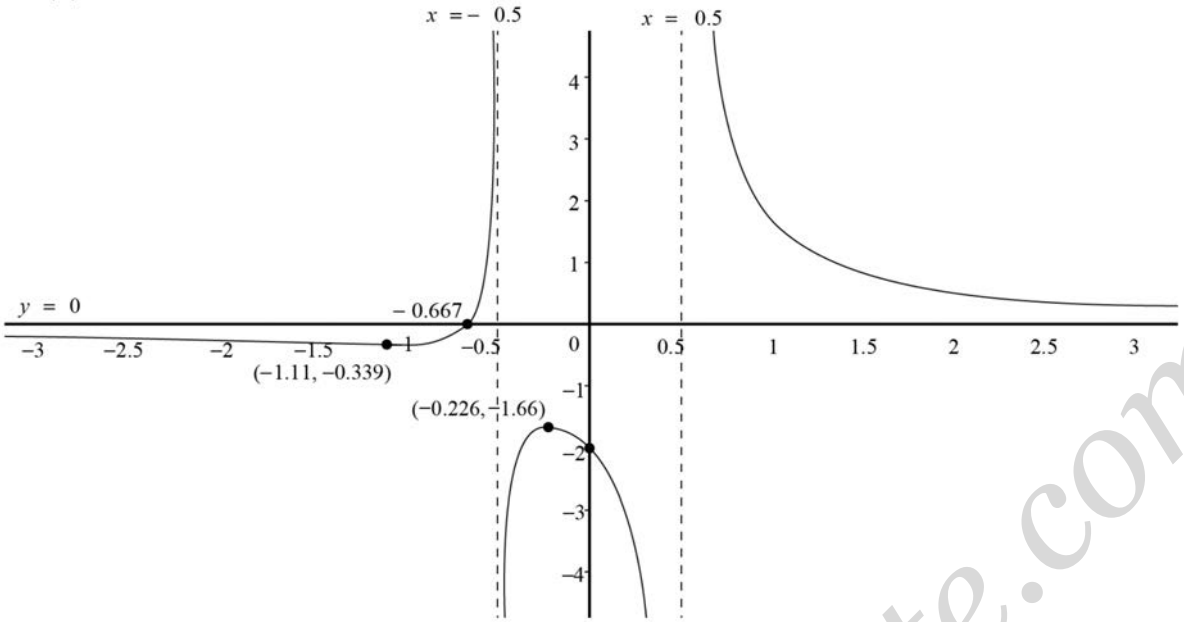
[3 marks]

(c) attempt to find the local min point on $y=f'(x)$ OR solve $f''(x)=0$ ***(M1)***

$x=-1.60$ ***A1***

[2 marks]

(d)



A1A1A1A1A1

Note: Award ***A1*** for both vertical asymptotes with their equations, award ***A1*** for horizontal asymptote with equation, award ***A1*** for each correct branch including asymptotic behaviour, coordinates of minimum and maximum points (may be seen next to the graph) and values of axes intercepts.
If vertical asymptotes are absent (or not vertical) and the branches overlap as a consequence, award maximum ***A0A1A0A1A1.***

[5 marks]

e) $x = -\frac{2}{3}\left(= -0.667\right)$ ***A1***

(oblique asymptote has) gradient $\frac{4}{3}\left(= 1.33\right)$ ***(A1)***

appropriate method to find complete equation of oblique asymptote ***M1***

$$\begin{array}{r} \frac{4}{3}x - \frac{8}{9} \\ 3x+2\overline{\big)\,4x^2 + 0x - 1} \\ \underline{4x^2 + \frac{8}{3}x} \\ -\frac{8}{3}x - 1 \\ \underline{-\frac{8}{3}x - \frac{16}{9}} \\ \frac{7}{9} \end{array}$$

$y = \frac{4}{3}x - \frac{8}{9}\left(= 1.33x - 0.889\right)$ ***A1***

Note: Do not award the final ***A1*** if the answer is not given as an equation.

[4 marks]

f) attempting to find at least one critical value $\left(x = -0.568729..., x = 1.31872...\right)$ ***(M1)***

$-\frac{2}{3} < x < -0.569$ OR $-0.5 < x < 0.5$ OR $x > 1.32$ ***A1A1A1***

Note: Only penalize once for use of $\leq$ rather than $<$.

[4 marks]

7 – (MathAA/20_HL_Winter_2021_Q10) – *Algebra-functions - Roots, Graphs, Integration*

(a) (i)

Note: In part (a), penalise once only, if correct values are given instead of correct coordinates.

attempts to solve $x^2 - x - 12 = 0$ ***(M1)***

$(-3, 0)$ and $(4, 0)$ ***A1***

(ii) $\left(0, \frac{4}{5}\right)$ ***A1***

[3 marks]

(b) $x = \frac{15}{2}$ ***A1***

Note: Award ***A0*** for $x \neq \frac{15}{2}$.

Award ***A1*** in part (b), if $x = \frac{15}{2}$ is seen on their graph in part (d).

[1 mark]

(c) **METHOD 1**

$(ax+b)(2x-15) \equiv x^2 - x - 12$

attempts to expand $(ax+b)(2x-15)$ ***(M1)***

$2ax^2 - 15ax + 2bx - 15b \equiv x^2 - x - 12$

$a = \frac{1}{2}$ ***A1***

equates coefficients of x ***(M1)***

$-1 = -\frac{15}{2} + 2b$

$b = \frac{13}{4}$ ***A1***

$\left(y = \frac{x}{2} + \frac{13}{4}\right)$

METHOD 2

attempts division on $\frac{x^2-x-12}{2x-15}$ ***M1***

$\frac{x}{2}+\frac{13}{4}+...$ ***M1***

$a=\frac{1}{2}$ ***A1***

$b=\frac{13}{4}$ ***A1***

$\left(y=\frac{x}{2}+\frac{13}{4}\right)$

METHOD 3

$a=\frac{1}{2}$ ***A1***

$\frac{x^2-x-12}{2x-15}\equiv\frac{x}{2}+b+\frac{c}{2x-15}$ ***M1***

$x^2-x-12\equiv\frac{(2x-15)x}{2}+(2x-15)b+c$

equates coefficients of x : ***(M1)***

$-1=-\frac{15}{2}+2b$

$b=\frac{13}{4}$ ***A1***

$\left(y=\frac{x}{2}+\frac{13}{4}\right)$

METHOD 4

attempts division on $\frac{x^2-x-12}{2x-15}$ ***M1***

$\frac{x^2-x-12}{2x-15}=\frac{x}{2}+\frac{\frac{13x}{2}-12}{2x-15}$

$a=\frac{1}{2}$ ***A1***

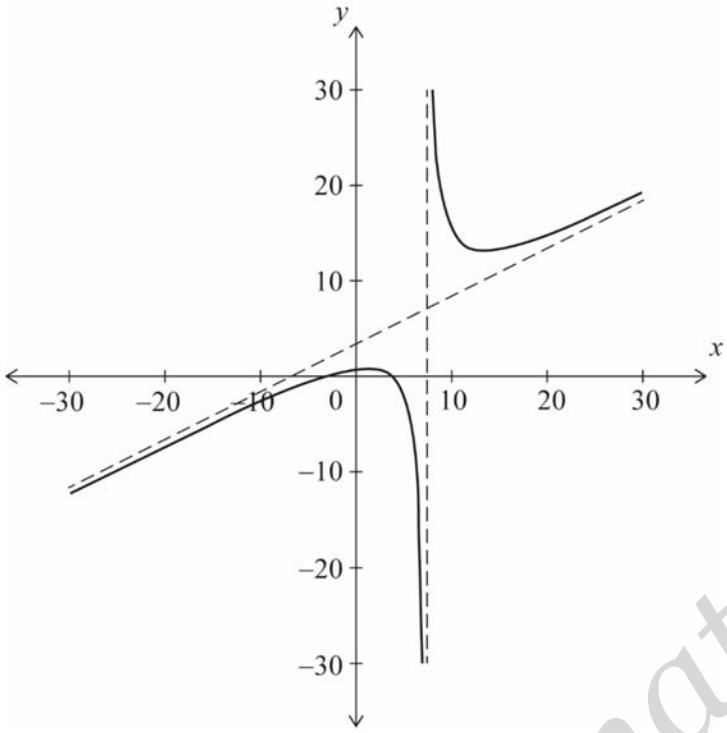
$\frac{\frac{13x}{2}-12}{2x-15}=\frac{13}{4}+...$ ***M1***

$b=\frac{13}{4}$ ***A1***

$\left(y=\frac{x}{2}+\frac{13}{4}\right)$

[4 marks]

(d)



two branches with approximately correct shape (for $-30 \le x \le 30$) ***A1***

their vertical and oblique asymptotes in approximately correct positions with both branches showing correct asymptotic behaviour to these asymptotes ***A1***

their axes intercepts in approximately the correct positions ***A1***

Note: Points of intersection with the axes and the equations of asymptotes are not required to be labelled.

[3 marks]

(e) (i) attempts to split into partial fractions: ***(M1)***

$$\frac{2x-15}{(x+3)(x-4)} \equiv \frac{A}{x+3} + \frac{B}{x-4}$$

$$2x-15 \equiv A(x-4)+B(x+3)$$

$A=3$ ***A1***

$B=-1$ ***A1***

$$\left(\frac{3}{x+3} - \frac{1}{x-4}\right)$$

(ii) $\int_0^3 \left(\frac{3}{x+3} - \frac{1}{x-4}\right) \mathrm{d}x$

attempts to integrate and obtains two terms involving 'ln' ***(M1)***

$= \left[3\ln|x+3| - \ln|x-4|\right]_0^3$ ***A1***

$= 3\ln 6 - \ln 1 - 3\ln 3 + \ln 4$ ***A1***

$= 3\ln 2 + \ln 4 \ (= \ln 8 + \ln 4)$

$= \ln 32 \ (= 5\ln 2)$ ***A1***

Note: The final ***A1*** is dependent on the previous two ***A*** marks.

[7 marks]

Total [18 marks]

1 - (MAT/22_HL_Summer_2012_Q1) - *Sequences - Series*

(a) $S_n = \frac{n}{2}[2a + (n-1)d]$

$212 = \frac{16}{2}(2a + 15d) \quad (= 16a + 120d)$

n^{th} term is $a + (n-1)d$

$8 = a + 4d$

solving simultaneously:

$d = 1.5,\ a = 2$

(b) $\frac{n}{2}[4 + 1.5(n-1)] > 600$

$\Rightarrow 3n^2 + 5n - 2400 > 0$

$\Rightarrow n > 27.4\ldots,\ (n < -29.1\ldots)$

Note: Do not penalize improper use of inequalities.

$\Rightarrow n = 28$

2 – (MAT/22_HL_Summer_2012_Q8) – *Sequences - Series*

(a) $\text{height} = 4 \times 0.95^4$
$= 3.26 \quad \text{(metres)}$

(b) $4 \times 0.95^n < 1$
$0.95^n < 0.25$
$\Rightarrow n > \dfrac{\ln 0.25}{\ln 0.95}$
$\Rightarrow n > 27.0$

Note: Do not penalize improper use of inequalities.

$\Rightarrow n = 28$

Note: If candidates have used $n-1$ rather than n throughout penalise in part (a) and treat as follow through in parts (b) and (c).

(c) **METHOD 1**

recognition of geometric series with sum to infinity, first term of 4×0.95 and common ratio 0.95
recognition of the need to double this series and to add 4
total distance travelled is $2\left(\dfrac{4 \times 0.95}{1-0.95}\right) + 4 = 156$ (metres)

Note: If candidates have used $n-1$ rather than n throughout penalise in part (a) and treat as follow through in parts (b) and (c).

METHOD 2

recognition of a geometric series with sum to infinity, first term of 4 and common ratio 0.95

recognition of the need to double this series and to subtract 4

total distance travelled is $2\left(\dfrac{4}{1-0.95}\right) - 4 = 156$ (metres)

3 – (MAT/20_HL_Winter_2012_Q1) – *Sequences - Series*

METHOD 1

$102+105+\ldots+498$
so number of terms $=133$

EITHER

$=\frac{133}{2}(2\times102+132\times3)$

$=39900$

OR

$=(102+498)\times\frac{133}{2}$

$=39900$

OR

$\sum_{n=34}^{166}3n$

$=39900$

METHOD 2

$500\div3=166.666...$ and $100\div3=33.333...$

$102+105+\ldots+498=\sum_{n=1}^{166}3n-\sum_{n=1}^{33}3n$

$\sum_{n=1}^{166}3n=41583$

$\sum_{n=1}^{33}3n=1683$

the sum is 39 900

4 - (MAT/20_HL_Winter_2012_Q5) - *Sequences - Series*

the pieces have lengths $a, ar, \ldots, ar^9$

$8a = ar^9$ (or $8 = r^9$)

$r = \sqrt[9]{8} = 1.259922\ldots$

$a\dfrac{r^{10}-1}{r-1} = 1 \qquad \left(\text{or } a\dfrac{r^{10}-1}{r-1} = 1000\right)$

$a = \dfrac{r-1}{r^{10}-1} = 0.0286\ldots \qquad \left(\text{or } a = \dfrac{r-1}{r^{10}-1} = 28.6\ldots\right)$

$a = 29\,\text{mm}$ (accept 0.029 m or any correct answer regardless the units)

5 - (MAT/22_HL_Summer_2013_Q5) - *Sequences - Series*

(a) $u_n - v_n = 1.6 + (n-1)\times 1.5 - 3\times 1.2^{n-1}$ $(= 1.5n + 0.1 - 3\times 1.2^{n-1})$

(b) attempting to solve $u_n > v_n$ numerically or graphically.

$n = 2.621\ldots, 9.695\ldots$

So $3 \le n \le 9$

(c) The greatest value of $u_n - v_n$ is 1.642.

6 – (MAT/22_HL_Summer_2013_Q11) – *Sequences - Series, Probability*

(a) (i) $\sum_{k=1}^{n}(2k-1)$ (or equivalent)

Note: Award ***A0*** for $\sum_{n=1}^{n}(2n-1)$ or equivalent.

(ii) **EITHER**

$2\times\frac{n(n+1)}{2}-n$

OR

$\frac{n}{2}\left(2+(n-1)2\right)$ (using $S_n=\frac{n}{2}\left(2u_1+(n-1)d\right)$)

OR

$\frac{n}{2}(1+2n-1)$ (using $S_n=\frac{n}{2}(u_1+u_n)$)

THEN

$=n^2$

(iii) $47^2-14^2=2013$

(b) (i) **EITHER**

a pentagon and five diagonals

OR

five diagonals (circle optional)

(ii) Each point joins to $n-3$ other points.
a correct argument for $n(n-3)$
a correct argument for $\frac{n(n-3)}{2}$

(iii) attempting to solve $\frac{1}{2}n(n-3)>1000000$ for n.

$n>1415.7$
$n=1416$

(c) (i) $np = 4$ and $npq = 3$

attempting to solve for n and p

$n = 16$ and $p = \frac{1}{4}$

(ii) $X \sqcup \mathrm{B}(16, 0.25)$

$\mathrm{P}(X = 1) = 0.0534538\ldots \left(= \binom{16}{1}(0.25)(0.75)^{15}\right)$

$\mathrm{P}(X = 3) = 0.207876\ldots \left(= \binom{16}{3}(0.25)^3(0.75)^{13}\right)$

$\mathrm{P}(X = 1) + \mathrm{P}(X = 3)$

$= 0.261$

7 – (MAT/20_HL_Winter_2013_Q2) – *Sequences - Series*

(a) **METHOD 1**

$34 = a + 3d$ and $76 = a + 9d$

$d = 7$

$a = 13$

METHOD 2

$76 = 34 + 6d$

$d = 7$

$34 = a + 21$

$a = 13$

(b) $\frac{n}{2}\left(26 + 7\left(n - 1\right)\right) > 5000$

$n > 36.463\ldots$

Note: Award ***M1A1A1*** for using either an equation, a graphical approach or a numerical approach.

$n = 37$

8 – (MAT/22_HL_Summer_2014_Q1) – *Sequences - Series*

(a) (i) $n = 27$

METHOD 1

$$S_{27} = \frac{14+196}{2} \times 27$$
$$= 2835$$

METHOD 2

$$S_{27} = \frac{27}{2}(2\times 14 + 26\times 7)$$
$$= 2835$$

METHOD 3

$$S_{27} = \sum_{n=1}^{27} 7 + 7n$$
$$= 2835$$

(ii) $\sum_{n=1}^{27}(7+7n)$ or equivalent

Note: Accept $\sum_{n=2}^{28} 7n$

(b) $\frac{n}{2}\left(2000 - 6(n-1)\right) < 0$

$n > 334.333$

$n = 335$

9 – (MAT/20_HL_Winter_2014_Q7) – *Sequences - Series*

(a) using $r = \frac{u_2}{u_1} = \frac{u_3}{u_2}$ to form $\frac{a+2d}{a+6d} = \frac{a}{a+2d}$

$a(a+6d) = (a+2d)^2$

$2d(2d-a) = 0$ (or equivalent)

since $d \neq 0 \Rightarrow d = \frac{a}{2}$

(b) substituting $d = \frac{a}{2}$ into $a + 6d = 3$ and solving for a and d

$a = \frac{3}{4}$ and $d = \frac{3}{8}$

$r = \frac{1}{2}$

$$\frac{n}{2}\left(2 \times \frac{3}{4} + (n-1)\frac{3}{8}\right) - \frac{3\left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}} \geq 200$$

attempting to solve for n

$n \geq 31.68\ldots$

so the least value of n is 32

10 – (MAT/22_HL_Summer_2016_Q4) – *Sequences - Series*

$ar + ar^2 = 96$

Note: Award ***A1*** for any valid equation involving a and r, *eg*, $\frac{a\left(1-r^3\right)}{1-r} - a = 96$.

$\frac{a}{1-r} = 500$

EITHER

attempting to eliminate a to obtain $500r\left(1-r^2\right) = 96$ (or equivalent in unsimplified form)

OR

attempting to obtain $a = \frac{96}{r+r^2}$ and $a = 500(1-r)$

THEN

attempting to solve for r

$r = 0.2\left(=\frac{1}{5}\right)$ or $r = 0.885\left(=\frac{\sqrt{97}-1}{10}\right)$

11 – (MAT/20_HL_Winter_2016_Q12) – *Sequences - Series*

(a) $A_1 = 1.004x$ ***A1***

$A_2 = 1.004(1.004x + x)$ ***A1***

$= 1.004^2 x + 1.004x$ ***AG***

Note: Accept an argument in words for example, first deposit has been in for two months and second deposit has been in for one month.

[2 marks]

(b) (i) $A_3 = 1.004\left(1.004^2 x + 1.004x + x\right) = 1.004^3 x + 1.004^2 x + 1.004x$ ***(M1)A1***

$A_4 = 1.004^4 x + 1.004^3 x + 1.004^2 x + 1.004x$ ***A1***

(ii) $A_{120} = \left(1.004^{120} + 1.004^{119} + \ldots + 1.004\right)x$ ***(A1)***

$= \dfrac{1.004^{120} - 1}{1.004 - 1} \times 1.004x$ ***M1A1***

$= 251\left(1.004^{120} - 1\right)x$ ***AG***

[6 marks]

(c) $A_{216} = 251\left(1.004^{216} - 1\right)x \quad \left(= x\sum_{t=1}^{216} 1.004^t\right)$ ***A1***

[1 mark]

(d) $251\left(1.004^{216} - 1\right)x = 20000 \Rightarrow x = 58.22\ldots$ ***(A1)(M1)(A1)***

Note: Award ***(A1)*** for $251\left(1.004^{216} - 1\right)x > 20000$, ***(M1)*** for attempting to solve and ***(A1)*** for $x > 58.22\ldots$.

$x = 59$ ***A1***

Note: Accept $x = 58$. Accept $x \geq 59$.

[4 marks]

(e) $r = 1.004^{12} \ (= 1.049\ldots)$ ***(M1)***

$15000\, r^n - 1000\, \dfrac{r^n - 1}{r - 1} = 0 \Rightarrow n = 27.8\ldots$ ***(A1)(M1)(A1)***

Note: Award ***(A1)*** for the equation (with their value of r), ***(M1)*** for attempting to solve for n and ***(A1)*** for $n = 27.8\ldots$.

$n = 28$ ***A1***

Note: Accept $n = 27$.

[5 marks]

12 – (MAT/20_HL_Winter_2017_Q12) – *Sequences - Series*

(a) 150000×1.035^{20} ***(M1)(A1)***

$= \$298468$ ***A1***

Note: Only accept answers to the nearest dollar. Accept $\$298469$.

[3 marks]

(b) attempt to look for a pattern by considering 1 year, 2 years *etc* ***(M1)***
recognising a geometric series with first term P and common ratio 1.02 ***(M1)***

EITHER

$P + 1.02P + \ldots + 1.02^{19}P \left(= P\left(1 + 1.02 + \ldots + 1.02^{19}\right)\right)$ ***A1***

OR

explicitly identify $u_1 = P$, $r = 1.02$ and $n = 20$ (may be seen as S_{20}). ***A1***

THEN

$S_{20} = \dfrac{\left(1.02^{20} - 1\right)P}{\left(1.02 - 1\right)}$ ***AG***

[3 marks]

(c) $24.297\ldots P = 298468$ ***(M1)(A1)***
$P = 12284$ ***A1***

Note: Accept answers which round to 12284.

[3 marks]

(d) (i) **METHOD 1**

$Q\left(1.028^n\right) = 5000\left(1 + 1.028 + 1.028^2 + 1.028^3 + \ldots + 1.028^{n-1}\right)$ ***M1A1***

$Q = \dfrac{5000\left(1 + 1.028 + 1.028^2 + 1.028^3 + \ldots + 1.028^{n-1}\right)}{1.028^n}$ ***A1***

$= \dfrac{5000}{1.028} + \dfrac{5000}{1.028^2} + \ldots + \dfrac{5000}{1.028^n}$ ***AG***

METHOD 2

the initial value of the first withdrawal is $\frac{5000}{1.028}$ ***A1***

the initial value of the second withdrawal is $\frac{5000}{1.028^2}$ ***R1***

the investment required for these two withdrawals is $\frac{5000}{1.028}+\frac{5000}{1.028^2}$ ***R1***

$Q=\frac{5000}{1.028}+\frac{5000}{1.028^2}+\ldots+\frac{5000}{1.028^n}$ ***AG***

(ii) sum to infinity is $\frac{\frac{5000}{1.028}}{1-\frac{1}{1.028}}$ ***(M1)(A1)***

$=178571.428\ldots$

so minimum amount is $\$178572$ ***A1***

Note: Accept answers which round to $\$178571$ or $\$178572$.

[6 marks]

13 – (MAT/21_HL_Summer_2018_Q1) – *Sequences - Series*

(a) $u_1+2d=1407$, $u_1+9d=1183$ ***(M1)(A1)***

$u_1=1471$, $d=-32$ ***A1A1***

[4 marks]

(b) $1471+(n-1)(-32)>0$ ***(M1)***

$\Rightarrow n<\frac{1471}{32}+1$

$n<46.96\ldots$ ***(A1)***

so 46 positive terms ***A1***

[3 marks]

14 – (MAT/21_HL_Summer_2018_Q7) – *Sequences - Series*

(a) **EITHER**

2019: $2500 \times 0.93 + 250 = 2575$ ***(M1)A1***

2020: $2575 \times 0.93 + 250$ ***M1***

OR

2020: $2500 \times 0.93^2 + 250(0.93 + 1)$ ***M1M1A1***

Note: Award ***M1*** for starting with 2500, ***M1*** for multiplying by 0.93 and adding 250 twice. ***A1*** for correct expression. Can be shown in recursive form.

THEN

$(= 2644.75) = 2645$ ***AG***

[3 marks]

(b) 2020: $2500 \times 0.93^2 + 250(0.93 + 1)$

2042: $2500 \times 0.93^{24} + 250\left(0.93^{23} + 0.93^{22} + \ldots + 1\right)$ ***(M1)(A1)***

$= 2500 \times 0.93^{24} + 250\dfrac{\left(0.93^{24} - 1\right)}{(0.93 - 1)}$ ***(M1)(A1)***

$= 3384$ ***A1***

Note: If recursive formula used, award ***M1*** for $u_n = 0.93\, u_{n-1} + 250$ **and** u_0 or u_1 seen (can be awarded if seen in part (a)). Then award ***M1A1*** for attempt to find u_{24} or u_{25} respectively (different term if other than 2500 used) (***M1A0*** if incorrect term is being found) and ***A2*** for correct answer.

Note: Accept all answers that round to 3380.

[5 marks]

15 – (MAT/20_HL_Winter_2018_Q1) – *Sequences - Series*

(a) $u_4 = u_1 r^3 \Rightarrow -2.916 = 4r^3$ ***(A1)***

solving, $r = -0.9$ ***(M1)A1***

[3 marks]

(b) $S_\infty = \dfrac{4}{1 - (-0.9)}$ ***(M1)***

$= \dfrac{40}{19} (= 2.11)$ ***A1***

[2 marks]

16 – (MAT/22_HL_Summer_2019_Q7) – *Sequences - Series*

$n=1 \Rightarrow S_1 = u_1$, so true for $n=1$ ***R1***

assume true for $n=k$, ie. $S_k = \dfrac{u_1\left(1-r^k\right)}{1-r}$ ***M1***

Note: Award ***M0*** for statements such as "let $n=k$".

Note: Subsequent marks after the first ***M1*** are independent of this mark and can be awarded.

$S_{k-1} = S_k + u_1 r^k$ ***M1***

$S_{k+1} = \dfrac{u_1\left(1-r^k\right)}{1-r} + u_1 r^k$ ***A1***

$S_{k+1} = \dfrac{u_1\left(1-r^k\right)}{1-r} + \dfrac{u_1 r^k\left(1-r\right)}{1-r}$

$S_{k+1} = \dfrac{u_1 - u_1 r^k + u_1 r^k - r u_1 r^k}{1-r}$ ***A1***

$S_{k+1} = \dfrac{u_1\left(1-r^{k-1}\right)}{1-r}$ ***A1***

true for $n=1$ and if true for $n=k$ then true for $n=k+1$,
the statement is true for any positive integer (or equivalent). ***R1***

Note: Award the final ***R1*** mark provided at least four of the previous marks are gained.

[7 marks]

17 – (MAT/20_HL_Winter_2019_Q1) – *Sequences - Series*

$u_1 r^3 = -70,\ u_1 r^6 = 8.75$ ***(M1)***

$r^3 = \dfrac{8.75}{-70} = -0.125$ ***(A1)***

$\Rightarrow r = -0.5$ ***(A1)***

valid attempt to find u_2 ***(M1)***

for example: $u_1 = \dfrac{-70}{-0.125} = 560$

$u_2 = 560 \times -0.5$

$= -280$ ***A1***

[5 marks]

18 – (MAT/20_HL_Winter_2020_Q11) – *Kinematics, Sequences - Series*

(a) $\frac{\pi}{6}(=0.524)$ ***A1***

$\frac{\pi}{3}(=1.05)$ ***A1***

[2 marks]

(b) attempt to use integration by parts ***M1***

$s=\int e^{-3t}\sin 6t\ dt$

EITHER

$=-\frac{e^{-3t}\sin 6t}{3}-\int -2e^{-3t}\cos 6t\ dt$ ***A1***

$=-\frac{e^{-3t}\sin 6t}{3}-\left(\frac{2e^{-3t}\cos 6t}{3}-\int -4e^{-3t}\sin 6t\ dt\right)$ ***A1***

$=-\frac{e^{-3t}\sin 6t}{3}-\left(\frac{2e^{-3t}\cos 6t}{3}+4s\right)$

$5s=\frac{-3e^{-3t}\sin 6t-6e^{-3t}\cos 6t}{9}$ ***M1***

OR

$=-\frac{e^{-3t}\cos 6t}{6}-\int \frac{1}{2}e^{-3t}\cos 6t\ dt$ ***A1***

$=-\frac{e^{-3t}\cos 6t}{6}-\left(\frac{e^{-3t}\sin 6t}{12}+\int \frac{1}{4}e^{-3t}\sin 6t\ dt\right)$ ***A1***

$=-\frac{e^{-3t}\cos 6t}{6}-\left(\frac{e^{-3t}\sin 6t}{12}+\frac{1}{4}s\right)$

$\frac{5}{4}s=\frac{-2e^{-3t}\cos 6t-e^{-3t}\sin 6t}{12}$ ***M1***

THEN

$s=-\frac{e^{-3t}(\sin 6t+2\cos 6t)}{15}(+c)$ ***A1***

at $t=0, s=0 \Rightarrow 0=-\frac{2}{15}+c$ ***M1***

$c=\frac{2}{15}$ ***A1***

$$s = \frac{2}{15} - \frac{e^{-3t}\left(\sin 6t + 2\cos 6t\right)}{15}$$

[7 marks]

(c) **EITHER**

substituting $t = \frac{\pi}{6}$ into their equation for s ***(M1)***

$$\left(s = \frac{2}{15} - \frac{e^{-\frac{\pi}{2}}\left(\sin \pi + 2\cos \pi\right)}{15} \right)$$

OR

using GDC to find maximum value ***(M1)***

OR

evaluating $\int_0^{\frac{\pi}{6}} v\,dt$ ***(M1)***

THEN

$$= 0.161 \left(= \frac{2}{15}\left(1 + e^{-\frac{\pi}{2}}\right) \right)$$ ***A1***

[2 marks]

distance required $= \int_0^{1.5} \left| e^{-3t} \sin 6t \right| \, dt$ ***(M1)***

OR

distance required $= \int_0^{\frac{\pi}{6}} e^{-3t} \sin 6t \, dt + \left| \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} e^{-3t} \sin 6t \, dt \right| + \int_{\frac{\pi}{3}}^{1.5} e^{-3t} \sin 6t \, dt$ ***(M1)***

$(= 0.16105... + 0.033479... + 0.006806...)$

THEN

$= 0.201$ (m) ***A1***

METHOD 2

using successive minimum and maximum values on the displacement graph ***(M1)***

$0.16105... + (0.16105... - 0.12757...) + (0.13453... - 0.12757...)$

$= 0.201$ (m) ***A1***

[2 marks]

(e) (i) valid attempt to find $\frac{dv}{dt}$ using product rule and set $\frac{dv}{dt} = 0$ ***M1***

$\frac{dv}{dt} = e^{-3t} 6\cos 6t - 3e^{-3t} \sin 6t$ ***A1***

$\frac{dv}{dt} = 0 \Rightarrow \tan 6t = 2$ ***AG***

(ii) attempt to evaluate t_1, t_2, t_3 in exact form ***M1***

$6t_1 = \arctan 2 \left(\Rightarrow t_1 = \frac{1}{6}\arctan 2 \right)$

$6t_2 = \pi + \arctan 2 \left(\Rightarrow t_2 = \frac{\pi}{6} + \frac{1}{6}\arctan 2 \right)$

$6t_3 = 2\pi + \arctan 2 \left(\Rightarrow t_3 = \frac{\pi}{3} + \frac{1}{6}\arctan 2 \right)$ ***A1***

Note: The ***A1*** is for any two consecutive correct, or showing that $6t_2 = \pi + 6t_1$ or $6t_3 = \pi + 6t_2$.

showing that $\sin 6t_{n+1} = -\sin 6t_n$

eg $\tan 6t = 2 \Rightarrow \sin 6t = \pm \frac{2}{\sqrt{5}}$ ***M1A1***

showing that $\frac{e^{-3t_{n+1}}}{e^{-3t_n}} = e^{-\frac{\pi}{2}}$ ***M1***

eg $e^{-3\left(\frac{\pi}{6}+k\right)} \div e^{-3k} = e^{-\frac{\pi}{2}}$

Note: Award the ***A1*** for any two consecutive terms.

$\frac{v_3}{v_2} = \frac{v_2}{v_1} = -e^{-\frac{\pi}{2}}$ ***AG***

[7 marks]

19 – (MathAA/22_HL_Summer_2021_Q2) – *Sequences - Series*

(a) attempt to use $u_1 + (n-1)d = 0$ ***(M1)***

$60 - 2.5(k-1) = 0$

$k = 25$ ***A1***

[2 marks]

(b) **METHOD 1**

attempting to express S_n in terms of n ***(M1)***

use of a graph or a table to attempt to find the maximum sum ***(M1)***

$= 750$ ***A1***

METHOD 2

EITHER

recognizing maximum occurs at $n = 25$ ***(M1)***

$S_{25} = \frac{25}{2}(60+0)$, $S_{25} = \frac{25}{2}(2\times 60 + 24\times -2.5)$ ***(A1)***

OR

attempting to calculate S_{24} ***(M1)***

$S_{24} = \frac{24}{2}(2\times 60 + 23\times -2.5)$ ***(A1)***

THEN

$= 750$ ***A1***

[3 marks]

20 – (MathAA/22_HL_Summer_2021_Q9) – *Sequences - Series*

(a) $1-t+t^2$ ***A1***

Note: Accept $1, -t$ and t^2.

[1 mark]

(b) $\sec x = \dfrac{1}{1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}(-...)}$ $\left(=\left(1-\dfrac{x^2}{2!}+\left(\dfrac{x^4}{4!}(-...)\right)\right)^{-1}\right)$ ***(M1)***

$t=\cos x-1$ or $\sec x = 1-(\cos x-1)+(\cos x-1)^2$ ***(M1)***

$=1-\left(-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}(-...)\right)+\left(-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}(-...)\right)^2$ ***A1***

$=1+\dfrac{x^2}{2}-\dfrac{x^4}{24}+\dfrac{x^4}{4}$ ***A1***

so the Maclaurin series for $\sec x$ up to and including the term in x^4 is $1+\dfrac{x^2}{2}+\dfrac{5x^4}{24}$ ***AG***

[4 marks]

Note: Condone the absence of '…'

(c) $\arctan 2x = 2x-\dfrac{(2x)^3}{3}+...$

$\lim\limits_{x\to 0}\left(\dfrac{x\arctan 2x}{\sec x-1}\right)=\lim\limits_{x\to 0}\left(\dfrac{x\left(2x-\dfrac{(2x)^3}{3}+...\right)}{\left(1+\dfrac{x^2}{2}+\dfrac{5x^4}{24}\right)-1}\right)$ ***M1***

$=\lim\limits_{x\to 0}\left(\dfrac{2x^2-\dfrac{8x^4}{3}+...}{\dfrac{x^2}{2}+\dfrac{5x^4}{24}}\right)$ ***A1***

$=\lim\limits_{x\to 0}\left(\dfrac{2x^2\left(1-\dfrac{4x^2}{3}\right)}{\dfrac{x^2}{2}\left(1+\dfrac{5x^2}{12}\right)}\right)$

$=4$ ***A1***

Note: Condone missing 'lim' and errors in higher derivatives.
Do not award ***M1*** unless x is replaced by $2x$ in arctan.

[3 marks]

21 – (MathAA/20_HL_Winter_2021_Q5) – *Sequences - Series*

(a) $u_1 = S_1 = \frac{2}{3} \times \frac{7}{8}$ ***(M1)***

$= \frac{14}{24}\left(= \frac{7}{12} = 0.583333\ldots\right)$ ***A1***

[2 marks]

(b) $r = \frac{7}{8}(= 0.875)$ ***(A1)***

substituting their values for u_1 and r into $S_\infty = \frac{u_1}{1-r}$ ***(M1)***

$= \frac{14}{3}(= 4.66666\ldots)$ ***A1***

[3 marks]

(c) attempt to substitute their values into the inequality or formula for S_n ***(M1)***

$\frac{14}{3} - \sum_{r=1}^{n} \frac{2}{3}\left(\frac{7}{8}\right)^r < 0.001$ OR $S_n = \frac{\frac{7}{12}\left(1-\left(\frac{7}{8}\right)^n\right)}{\left(1-\frac{7}{8}\right)}$

attempt to solve their inequality using a table, graph or logarithms
(must be exponential) ***(M1)***

Note: Award ***(M0)*** if the candidate attempts to solve $S_\infty - u_n < 0.001$.

correct critical value or at least one correct crossover value ***(A1)***

$63.2675\ldots$ OR $S_\infty - S_{63} = 0.001036\ldots$ OR $S_\infty - S_{64} = 0.000906\ldots$

OR $S_\infty - S_{63} - 0.001 = 0.0000363683\ldots$ OR $S_\infty - S_{64} - 0.001 = -0.0000931777\ldots$

least value is $n = 64$ ***A1***

[4 marks]

Total [9 marks]

22 – (MathAA/20_HL_Winter_2022_Q3) – *Sequences - Series*

$86.4 = 50r^3$ ***(A1)***

$r = 1.2\left(= \sqrt[3]{\frac{86.4}{50}}\right)$ seen anywhere ***(A1)***

$\frac{50(1.2^n - 1)}{0.2} > 33500$ OR $250(1.2^n - 1) = 33500$ ***(A1)***

attempt to solve their geometric S_n inequality or equation ***(M1)***

sketch OR $n > 26.9045$, $n = 26.9$ OR $S_{26} = 28368.8$ OR $S_{27} = 34092.6$ OR algebraic manipulation involving logarithms

$n = 27$ (accept $n \geq 27$) ***A1***

1 - (MAT/20_HL_Winter_2012_Q10) - *Complex Numbers*

METHOD 1

$$(1-\omega^2)^* = (1-\text{cis } 2\theta)^* = \left((1-\cos 2\theta) - \text{i}\sin 2\theta\right)^*$$
$$= (1-\cos 2\theta) + \text{i}\sin 2\theta$$
$$\left|(1-\omega^2)^*\right| = \sqrt{(1-\cos 2\theta)^2 + \sin^2 2\theta}\left(= \sqrt{(2\sin^2\theta)^2 + \left(2\sin\theta\cos\theta\right)^2}\right)$$
$$= \left|2\sin\theta\right|$$
$$\arg\left((1-\omega^2)^*\right) = \alpha \Rightarrow \tan\alpha = \cot(\theta)$$
$$\alpha = \frac{\pi}{2} - \theta$$

therefore:

modulus is $2\left|\sin\theta\right|$ and argument is $\frac{\pi}{2} - \theta$ or $\frac{\pi}{2} - \theta \pm \pi$

Note: Accept modulus is $2\sin\theta$ and argument is $\frac{\pi}{2} - \theta$

METHOD 2

EITHER

$$(1-\omega^2)^* = (1-\text{cis } 2\theta)^* = \left((1-\cos 2\theta) - \text{i}\sin 2\theta\right)^*$$
$$= (1-\cos 2\theta) + \text{i}\sin 2\theta$$
$$= \left(1 - 1 + 2\sin^2\theta\right) + 2\text{i}\sin\theta\cos\theta$$

OR

$$(1-\omega^2)^* = \left(1 - (\cos\theta + i\sin\theta)^2\right)^*$$
$$= \left(1 - \cos^2\theta + \sin^2\theta - 2\text{i}\sin\theta\cos\theta\right)^*$$
$$= 2\sin^2\theta + 2\text{i}\sin\theta\cos\theta$$

THEN

$$= 2\sin\theta(\sin\theta + \text{i}\cos\theta)$$
$$= 2\sin\theta\left(\cos\left(\frac{\pi}{2} - \theta\right) + \text{i}\sin\left(\frac{\pi}{2} - \theta\right)\right)$$
$$= 2\sin\theta\,\text{cis}\left(\frac{\pi}{2} - \theta\right)$$

therefore:

modulus is $2\left|\sin\theta\right|$ and argument is $\frac{\pi}{2} - \theta$ or $\frac{\pi}{2} - \theta \pm \pi$

2 - (MAT/21_HL_Summer_2013_Q6) - *Complex Numbers*

other root is $2-i$

a quadratic factor is therefore $(x-2+i)(x-2-i)$

$= x^2 - 4x + 5$

$x+1$ is a factor

$(x-2)^2$ is a factor

$p(x) = a(x+1)(x-2)^2(x^2-4x+5)$

$p(0) = 4 \Rightarrow a = \frac{1}{5}$

$p(x) = \frac{1}{5}(x+1)(x-2)^2(x^2-4x+5)$

3 – (MAT/20_HL_Winter_2013_Q6) – *Complex Numbers*

(a) $\dfrac{a+\mathrm{i}}{a-\mathrm{i}} \times \dfrac{a+\mathrm{i}}{a+\mathrm{i}}$ ***M1***

$= \dfrac{a^2-1+2a\mathrm{i}}{a^2+1} \left(= \dfrac{a^2-1}{a^2+1} + \dfrac{2a}{a^2+1}\mathrm{i} \right)$ ***A1***

(i) z is real when $a = 0$ ***A1***

(ii) z is purely imaginary when $a = \pm 1$ ***A1***

Note: Award ***M1A0A1A0*** for $\dfrac{a^2-1+2a\mathrm{i}}{a^2-1} \left(= 1 + \dfrac{2a}{a^2-1}\mathrm{i} \right)$ leading to $a = 0$ in (i).

(b) **METHOD 1**

attempting to find either $|z|$ or $|z|^2$ by expanding and simplifying

$eg\ |z|^2 = \dfrac{\left(a^2-1\right)^2 + 4a^2}{\left(a^2+1\right)^2} = \dfrac{a^4+2a^2+1}{\left(a^2+1\right)^2}$ ***M1***

$= \dfrac{\left(a^2+1\right)^2}{\left(a^2+1\right)^2}$

$|z|^2 = 1 \Rightarrow |z| = 1$ ***A1***

METHOD 2

$|z| = \dfrac{|a+\mathrm{i}|}{|a-\mathrm{i}|}$ ***M1***

$|z| = \dfrac{\sqrt{a^2+1}}{\sqrt{a^2+1}} \Rightarrow |z| = 1$ ***A1***

4 - (MAT/21_HL_Summer_2014_Q1) - *Complex Numbers*

METHOD 1

substituting
$-5+12\text{i}+a(2+3\text{i})+b=0$
equating real or imaginary parts
$12+3a=0 \Rightarrow a=-4$
$-5+2a+b=0 \Rightarrow b=13$

METHOD 2

other root is $2-3\text{i}$
considering either the sum or product of roots or multiplying factors
$4=-a$ (sum of roots) so $a=-4$
$13=b$ (product of roots)

5 – (MAT/22_HL_Summer_2014_Q13) – *Complex Numbers*

(a) let P(n) be the proposition $z^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta),\ n \in \mathbb{Z}^+$

let $n = 1 \Rightarrow$

$\mathrm{LHS} = r(\cos\theta + \mathrm{i}\sin\theta)$

$\mathrm{RHS} = r(\cos\theta + \mathrm{i}\sin\theta),\ \therefore\ \mathrm{P}(1)$ is true

assume true for $n = k \Rightarrow r^k(\cos\theta + \mathrm{i}\sin\theta)^k = r^k\left(\cos(k\theta) + \mathrm{i}\sin(k\theta)\right)$

Note: Only award the ***M1*** if truth is assumed.

now show $n = k$ true implies $n = k+1$ also true

$r^{k+1}(\cos\theta + \mathrm{i}\sin\theta)^{k+1} = r^{k+1}(\cos\theta + \mathrm{i}\sin\theta)^k(\cos\theta + \mathrm{i}\sin\theta)$

$= r^{k+1}\left(\cos(k\theta) + \mathrm{i}\sin(k\theta)\right)(\cos\theta + \mathrm{i}\sin\theta)$

$= r^{k+1}\left(\cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta + \mathrm{i}\left(\sin(k\theta)\cos\theta + \cos(k\theta)\sin\theta\right)\right)$

$= r^{k+1}\left(\cos(k\theta + \theta) + \mathrm{i}\sin(k\theta + \theta)\right)$

$= r^{k+1}\left(\cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta\right) \Rightarrow n = k+1$ is true

$\mathrm{P}(k)$ true implies $\mathrm{P}(k+1)$ true and $\mathrm{P}(1)$ is true, therefore by mathematical induction statement is true for $n \geq 1$

Note: Only award the final ***R1*** if the first 4 marks have been awarded.

(b) (i) $u = 2\mathrm{cis}\left(\frac{\pi}{3}\right)$

$v = \sqrt{2}\mathrm{cis}\left(-\frac{\pi}{4}\right)$

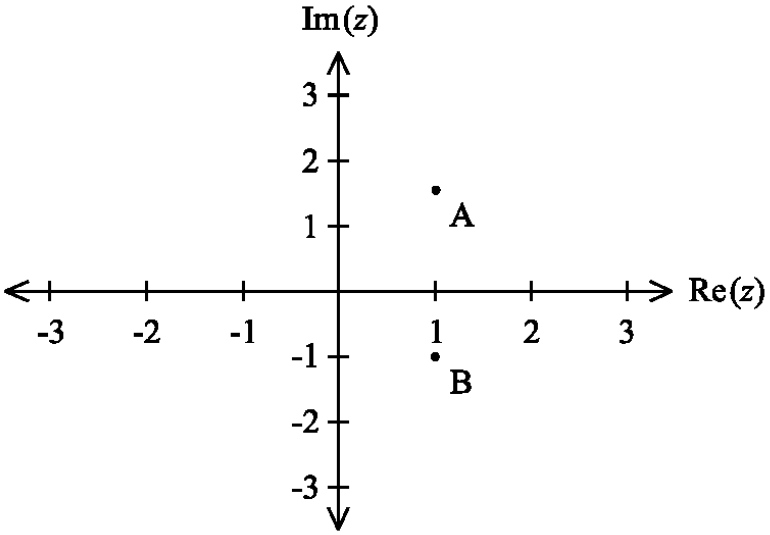
Notes: Accept 3 sf answers only. Accept equivalent forms.
Accept $2e^{\frac{\pi}{3}i}$ and $\sqrt{2}\,e^{-\frac{\pi}{4}i}$.

(ii) $u^3 = 2^3\mathrm{cis}(\pi) = -8$

$v^4 = 4\mathrm{cis}(-\pi) = -4$

$u^3v^4 = 32$

(c)



Note: Award **A1** if A or $1+\sqrt{3}i$ and B or $1-i$ are in their correct quadrants, are aligned vertically and it is clear that $|u|>|v|$.

(d) $\text{Area} = \frac{1}{2}\times 2\times\sqrt{2}\times\sin\left(\frac{5\pi}{12}\right)$

$= 1.37\left(=\frac{\sqrt{2}}{4}\left(\sqrt{6}+\sqrt{2}\right)\right)$

Notes: Award ***M1A0A0*** for using $\frac{7\pi}{12}$.

(e) $(z-1+\mathrm{i})(z-1-\mathrm{i}) = z^2-2z+2$

Note: Award ***M1*** for recognition that a complex conjugate is also a root.

$\left(z-1-\sqrt{3}\mathrm{i}\right)\left(z-1+\sqrt{3}\mathrm{i}\right) = z^2-2z+4$

$\left(z^2-2z+2\right)\left(z^2-2z+4\right) = z^4-4z^3+10z^2-12z+8$

6 – (MAT/21_HL_Summer_2015_Q12) – *Complex Numbers*

(a) (i) $(\cos\theta + \mathrm{i}\sin\theta)^5$

$= \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta + 10\mathrm{i}^2\cos^3\theta\sin^2\theta +$
$10\mathrm{i}^3\cos^2\theta\sin^3\theta + 5\mathrm{i}^4\cos\theta\sin^4\theta + \mathrm{i}^5\sin^5\theta$
$(= \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta -$
$10\mathrm{i}\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + \mathrm{i}\sin^5\theta)$

Note: Award first ***A1*** for correct binomial coefficients.

(ii) $(\mathrm{cis}\,\theta)^5 = \mathrm{cis}\,5\theta = \cos 5\theta + \mathrm{i}\sin 5\theta$

$= \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10\mathrm{i}\cos^2\theta\sin^3\theta +$
$5\cos\theta\sin^4\theta + \mathrm{i}\sin^5\theta$

Note: Previous line may be seen in (i)

equating imaginary terms
$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

(iii) equating real terms
$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

(b) $(r\,\mathrm{cis}\,\alpha)^5 = 1 \Rightarrow r^5\,\mathrm{cis}\,5\alpha = 1\,\mathrm{cis}\,0$

$r^5 = 1 \Rightarrow r = 1$

$5\alpha = 0 \pm 360k,\ k \in \mathbb{Z} \Rightarrow \alpha = 72k$

$\alpha = 72^\circ$

Note: Award ***M1A0*** if final answer is given in radians.

(c) use of $\sin(5\times 72) = 0$ **OR** the imaginary part of 1 is 0

$0 = 5\cos^4\alpha\sin\alpha - 10\cos^2\alpha\sin^3\alpha + \sin^5\alpha$

$\sin\alpha \neq 0 \Rightarrow 0 = 5\left(1-\sin^2\alpha\right)^2 - 10\left(1-\sin^2\alpha\right)\sin^2\alpha + \sin^4\alpha$

Note: Award ***M1*** for replacing $\cos^2\alpha$.

$0 = 5\left(1 - 2\sin^2\alpha + \sin^4\alpha\right) - 10\sin^2\alpha + 10\sin^4\alpha + \sin^4\alpha$

Note: Award ***A1*** for any correct simplification.

so $16\sin^4\alpha - 20\sin^2\alpha + 5 = 0$

(d) $\sin^2\alpha = \dfrac{20 \pm \sqrt{400-320}}{32}$

$\sin\alpha = \pm\sqrt{\dfrac{20 \pm \sqrt{80}}{32}}$

$\sin\alpha = \dfrac{\pm\sqrt{10 \pm 2\sqrt{5}}}{4}$

Note: Award ***A1*** regardless of signs. Accept equivalent forms with integral denominator, simplification may be seen later.

as $72 > 60$, $\sin 72 > \dfrac{\sqrt{3}}{2} = 0.866\ldots$ so we have to take both positive signs (or equivalent argument)

Note: Allow verification of correct signs with calculator if clearly stated

$\sin 72 = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$

7 - (MAT/21_HL_Summer_2016_Q9) - *Complex Numbers*

(a) other two roots are $c - \mathrm{i}$ and $2 - \mathrm{i}d$

(b) **METHOD 1**

use of sum of roots

$2c + 4 = 10$

$c = 3$

use of product of roots

product is $(c + \mathrm{i})(c - \mathrm{i})(2 + \mathrm{i}d)(2 - \mathrm{i}d)$

$(c^2 + 1)(4 + d^2)\left[= 10\left(4 + d^2\right)\right] = 50$

Note: The line above can be awarded if they have used their value of c.

$d = 1$

METHOD 2

$z^4 - 10z^3 + az^2 + bz + 50 = \left(z^2 - 2cz + c^2 + 1\right)\left(z^2 - 4z + 4 + d^2\right)$

compare constant terms or coefficients of z^3

$4 + 2c = 10$

$\left(c^2 + 1\right)\left(4 + d^2\right) = 50$

$c = 3,\ d = 1$

8 – (MAT/22_HL_Summer_2018_Q1) – *Complex Numbers*

(a) $z = \dfrac{(2+7\mathrm{i})}{(6+2\mathrm{i})} \times \dfrac{(6-2\mathrm{i})}{(6-2\mathrm{i})}$ ***(M1)***

$= \dfrac{26+38\mathrm{i}}{40} \left(= \dfrac{13+19\mathrm{i}}{20} = 0.65 + 0.95\mathrm{i} \right)$ ***A1***

[2 marks]

(b) attempt to use $|z| = \sqrt{a^2 + b^2}$ ***(M1)***

$|z| = \sqrt{\dfrac{53}{40}} \left(= \dfrac{\sqrt{530}}{20} \right)$ or equivalent ***A1***

Note: ***A1*** is only awarded for the correct exact value.

[2 marks]

(c) **EITHER**

$\arg z = \arg(2+7\mathrm{i}) - \arg(6+2\mathrm{i})$ ***(M1)***

OR

$\arg z = \arctan\left(\dfrac{19}{13}\right)$ ***(M1)***

THEN

$\arg z = 0.9707$ (radians) (= 55.6197 degrees) ***A1***

Note: Only award the last ***A1*** if 4 decimal places are given.

[2 marks]

9 – (MAT/21_HL_Summer_2019_Q2) – *Complex Numbers*

(a) $z = 2e^{\frac{\pi}{4}i} \left(= 2e^{0.785i}\right)$ ***A1***

Note: Accept all answers in the form $2e^{\left(\frac{\pi}{4}+2\pi n\right)i}$.

$z = 2e^{\frac{5\pi}{4}i} \left(= 2e^{3.93i}\right)$ **OR** $z = 2e^{-\frac{3\pi}{4}i} \left(= 2e^{-2.36i}\right)$ ***(M1)A1***

Note: Accept all answers in the form $2e^{\left(-\frac{3\pi}{4}+2\pi n\right)i}$.

Note: Award ***M1A0*** for correct answers in the incorrect form, *eg* $-2e^{\frac{\pi}{4}i}$.

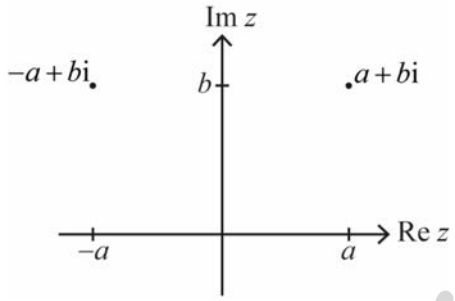
[3 marks]

(b) $z = 1.41 + 1.41i$, $z = -1.41 - 1.41i$ ***A1A1***
$\left(z = \sqrt{2} + \sqrt{2}i, z = -\sqrt{2} - \sqrt{2}i\right)$

[2 marks]

10 – (MAT/21_HL_Summer_2019_Q6) – *Complex Numbers*

(a)



A1

Note: Award ***A1*** for z in first quadrant and z-$2a$ its reflection in the y-axis.

[1 mark]

(b) (i) $\pi - \theta$ (or any equivalent) ***A1***

(ii) $\arg\left(\frac{z}{z-2a}\right) = \arg(z) - \arg(z-2a)$ ***(M1)***

$= 2\theta - \pi$ (or any equivalent) ***A1***

[3 marks]

(c) **METHOD 1**

if $\mathrm{Re}\left(\frac{z}{z-2a}\right) = 0$ then $2\theta - \pi = \frac{n\pi}{2}$, ($n$ odd) ***(M1)***

$-\pi < 2\theta - \pi < 0 \to n = -1$

$2\theta - \pi = -\frac{\pi}{2}$ ***(A1)***

$\theta = \frac{\pi}{4}$ ***A1***

METHOD 2

$\frac{a+bi}{-a+bi} = \frac{b^2 - a^2 - 2abi}{a^2 + b^2}$ ***M1***

$\mathrm{Re}\left(\frac{z}{z-2a}\right) = 0 \Rightarrow b^2 - a^2 = 0$

$b = a$ ***A1***

$\theta = \frac{\pi}{4}$ ***A1***

Note: Accept any equivalent, *eg* $\theta = -\frac{7\pi}{4}$.

[3 marks]

11 – (MAT/22_HL_Summer_2019_Q8) – *Complex Numbers*

(a) **METHOD 1**

$w^3 = 8\mathrm{i}$

writing $8\mathrm{i} = 8\left(\cos\left(\frac{\pi}{2} + 2\pi k\right) + \mathrm{i}\sin\left(\frac{\pi}{2} + 2\pi k\right)\right)$ ***(M1)***

Note: Award ***M1*** for an attempt to find cube roots of w using modulus-argument form.

cube roots $w = 2\left(\cos\left(\frac{\frac{\pi}{2} + 2\pi k}{3}\right) + \mathrm{i}\sin\left(\frac{\frac{\pi}{2} + 2\pi k}{3}\right)\right)$ ***(M1)***

ie. $w = \sqrt{3} + \mathrm{i}, -\sqrt{3} + \mathrm{i}, -2\mathrm{i}$ ***A2***

Note: Award ***A2*** for all 3 correct, ***A1*** for 2 correct.

Note: Accept $w = 1.73 + \mathrm{i}$ and $w = -1.73 + \mathrm{i}$.

[4 marks]

METHOD 2

$w^3 + (2\mathrm{i})^3 = 0$

$(w + 2\mathrm{i})\left(w^2 - 2w\mathrm{i} - 4\right) = 0$ ***M1***

$w = \frac{2\mathrm{i} \pm \sqrt{12}}{2}$ ***M1***

$w = \sqrt{3} + \mathrm{i}, -\sqrt{3} + \mathrm{i}, -2\mathrm{i}$ ***A2***

Note: Award ***A2*** for all 3 correct, ***A1*** for 2 correct.

Note: Accept $w = 1.73 + \mathrm{i}$ and $w = -1.73 + \mathrm{i}$.

[4 marks]

(b) $w_1 = -2\mathrm{i}$

$\frac{z}{z - \mathrm{i}} = -2\mathrm{i}$ ***M1***

$z = -2\mathrm{i}(z - \mathrm{i})$

$z(1 + 2\mathrm{i}) = -2$

$z = \frac{-2}{1 + 2\mathrm{i}}$ ***A1***

$z = -\frac{2}{5} + \frac{4}{5}\mathrm{i}$ ***A1***

Note: Accept $a = -\frac{2}{5}, b = \frac{4}{5}$.

[3 marks]

12 – (MAT/20_HL_Winter_2019_Q6) – *Complex Numbers*

METHOD 1

one other root is $3-\mathrm{i}$ ***A1***
let third root be α ***(M1)***
considering sum or product of roots ***(M1)***
sum of roots $=6+\alpha=\frac{37}{a}$ ***A1***
product of roots $=10\alpha=\frac{10}{a}$ ***A1***
hence $a=6$ ***A1***
[6 marks]

METHOD 2

one other root is $3-\mathrm{i}$ ***A1***
quadratic factor will be $z^2-6z+10$ ***(M1)A1***
$P(z)=az^3-37z^2+66z-10=\left(z^2-6z+10\right)(az-1)$ ***M1***
comparing coefficients ***(M1)***
hence $a=6$ ***A1***
[6 marks]

METHOD 3

substitute $3+\mathrm{i}$ into $P(z)$ ***(M1)***
$a(18+26\mathrm{i})-37(8+6\mathrm{i})+66(3+\mathrm{i})-10=0$ ***(M1)A1***
equating real or imaginary parts or dividing ***M1***
$18a-296+198-10=0$ or $26a-222+66=0$ or $\frac{10-66(3+\mathrm{i})+37(8+6\mathrm{i})}{18+26\mathrm{i}}$ ***A1***
hence $a=6$ ***A1***
[6 marks]

13 – (MathAA/21_HL_Summer_2021_Q8) – *Complex Numbers*

(a) $\left(|zw|=\right)16$ ***A1***
[1 mark]

(b) attempt to find $\arg(z)+\arg(w)$ ***(M1)***
$\arg(zw)=\arg(z)+\arg(w)$
$=\frac{\pi}{5}-\frac{2k\pi}{5}\left(=\frac{(1-2k)\pi}{5}\right)$ ***A1***
[2 marks]

(c) (i) $zw \in \mathrm{Z} \Rightarrow \arg(zw)$ is a multiple of π ***(M1)***

$\Rightarrow 1-2k$ is a multiple of 5 ***(M1)***

$k=3$ ***A1***

(ii) $zw = 16(\cos(-\pi)+\mathrm{i}\sin(-\pi))$

-16 ***A1***

[4 marks]

14 – (MathAA/22_HL_Summer_2021_Q8) – *Complex Numbers*

$$\frac{1+z}{1-z} = \frac{1+\cos\theta+\mathrm{i}\sin\theta}{1-\cos\theta-\mathrm{i}\sin\theta}$$

attempt to use the complex conjugate of their denominator ***M1***

$$= \frac{(1+\cos\theta+\mathrm{i}\sin\theta)(1-\cos\theta+\mathrm{i}\sin\theta)}{(1-\cos\theta-\mathrm{i}\sin\theta)(1-\cos\theta+\mathrm{i}\sin\theta)}$$ ***A1***

$$\mathrm{Re}\left(\frac{1+z}{1-z}\right) = \frac{1-\cos^2\theta-\sin^2\theta}{(1-\cos\theta)^2+\sin^2\theta} \quad \left(= \frac{1-\cos^2\theta-\sin^2\theta}{2-2\cos\theta}\right)$$ ***M1A1***

Note: Award ***M1*** for expanding the numerator and ***A1*** for a correct numerator. Condone either an incorrect denominator or the absence of a denominator.

using $\cos^2\theta+\sin^2\theta=1$ to simplify the numerator ***(M1)***

$$\mathrm{Re}\left(\frac{1+z}{1-z}\right) = 0$$ ***AG***

[5 marks]

15 – (MathAA/20_HL_Winter_2022_Q7) – *Complex Numbers*

(a) **METHOD 1**

attempt to use scalar product or formula for angle between two vectors ***(M1)***

$\boldsymbol{u}.\boldsymbol{v} = \cos\frac{1}{n} + \sin\frac{1}{n}$ (seen anywhere) ***(A1)***

$$\cos\theta = \frac{\cos\frac{1}{n} + \sin\frac{1}{n}}{\sqrt{2}\sqrt{\left(\cos^2\frac{1}{n} + \sin^2\frac{1}{n}\right)}} \left(= \frac{\cos\frac{1}{n} + \sin\frac{1}{n}}{\sqrt{2}}\right)$$ ***A1***

METHOD 2

attempt to use an Argand diagram showing two complex numbers in the first quadrant with the angle between them marked as θ ***(M1)***

$\arg(\boldsymbol{u}) = \frac{\pi}{4}$ (accept 45° or $\arctan(1)$) **and** $\arg(\boldsymbol{v}) = \frac{1}{n}$ ***(A1)***

$\cos\theta = \cos\left|\frac{\pi}{4} - \frac{1}{n}\right|$ ***A1***

(b) use of $\frac{1}{n} \to 0$ as $n \to \infty$ ***(M1)***

EITHER

$(\cos\theta \to)\frac{1}{\sqrt{2}}$ ***(A1)***

OR

$(\boldsymbol{v} \to)\boldsymbol{i}$ ***(A1)***

THEN

the limit is $\frac{\pi}{4}$ ***A1***

Note: Accept $45°$. Do not accept rounded values such as 0.785.

[3 marks]

1 – (MAT/21_HL_Summer_2012_Q3) – *Permutation - Combination*

(a) $\binom{10}{6} = 210$

(b) $2 \times \binom{8}{5} = 112$

Note: Accept 210 – 28 – 70 = 112

(c) $\frac{112}{210} \left(= \frac{8}{15} = 0.533 \right)$

2 – (MAT/22_HL_Summer_2012_Q4) – *Permutation - Combination*

(a) number of arrangements of boys is $15!$ and number of arrangements of girls is $10!$
total number of arrangements is $15! \times 10! \times 2 (= 9.49 \times 10^{18})$

Note: If 2 is omitted, award ***(A1)M1A0***.

(b) number of ways of choosing two boys is $\binom{15}{2}$ and the number of ways of choosing three girls is $\binom{10}{3}$
number of ways of choosing two boys and three girls is $\binom{15}{2} \times \binom{10}{3} = 12600$

3 – (MAT/21_HL_Summer_2013_Q8) – *Permutation - Combination*

(a) the three girls can sit together in $3! = 6$ ways
this leaves 4 'objects' to arrange so the number of ways this can be done is $4!$
so the number of arrangements is $6 \times 4! = 144$

(b) Finding more than one position that the girls can sit
Counting exactly four positions
number of ways = $4 \times 3! \times 3! = 144$

4 – (MAT/21_HL_Summer_2014_Q3) – *Permutation - Combination*

the number of ways of allocating presents to the first child is $\binom{7}{3}\left(\text{or } \binom{7}{2}\right)$

multiplying by $\binom{4}{2}\left(\text{or } \binom{5}{3} \text{ or } \binom{5}{2}\right)$

Note: Award ***M1*** for multiplication of combinations.

$$\binom{7}{3}\binom{4}{2} = 210$$

5 – (MAT/22_HL_Summer_2015_Q2) – *Permutation - Combination*

(a) $\binom{11}{4} = \frac{11\times10\times9\times8}{4\times3\times2\times1} = 330$

(b) $\binom{5}{2}\times\binom{6}{2} = \frac{5\times4}{2\times1}\times\frac{6\times5}{2\times1}$

$= 150$

(c) **METHOD1**

number of ways all men $= \binom{5}{4} = 5$

$330 - 5 = 325$

METHOD 2

$$\binom{6}{1}\binom{5}{3} + \binom{6}{2}\binom{5}{2} + \binom{6}{3}\binom{5}{1} + \binom{6}{4}$$

=325

6 – (MAT/22_HL_Summer_2017_Q8) – *Permutation - Combination*

METHOD 1

consideration of all papers
all papers may be sat in $18!$ ways ***A1***
number of ways of positioning "pairs" of science subjects $= 3! \times 17!$ ***A1***
but this includes two copies of each "triple" ***(R1)***
number of ways of positioning "triplets" of science subjects $= 3! \times 16!$ ***A1***
hence number of arrangements is $18! - 3! \times 17! + 3! \times 16!$ ***M1A1***
$\left(= 4.39 \times 10^{15}\right)$

METHOD 2

consideration of all the non-science papers ***(M1)***
hence all non-science papers can be sat in $15!$ ways ***A1***
there are $16 \times 15 \times 14 \ (= 3360)$ ways of positioning the three science papers ***(M1)A1***
hence the number of arrangements is $16 \times 15 \times 14 \times 15! \left(= 4.39 \times 10^{15}\right)$ ***(M1)A1***

METHOD 3

consideration of all papers
all papers may be sat in $18!$ ways ***A1***
number of ways of positioning exactly two science subjects $= 3! \times 15! \times 16 \times 15$ ***M1A1***
number of ways of positioning "triplets" of science subjects $= 3! \times 16!$ ***A1***
hence number of arrangements is $18! - 3! \times 16! - 3! \times 15! \times 16 \times 15$ ***M1A1***
$\left(= 4.39 \times 10^{15}\right)$

[6 marks]

7 – (MAT/20_HL_Winter_2017_Q9) – *Permutation - Combination*

(a) $12!(=479001600)$ ***A1***

[1 mark]

(b) **METHOD 1**

$8 \times 2 = 16$ ways of sitting Helen and Nicky, $10!$ ways of sitting everyone else ***(A1)***

$16 \times 10!$
$= 58060800$ ***A1***

METHOD 2

$8 \times 1 \times 10!(=29030400)$ ways if Helen sits in the front or back row

$4 \times 2 \times 10!(=29030400)$ ways if Helen sits in the middle row ***(A1)***

Note: Award ***A1*** for one correct value.

2×29030400
$= 58060800$ ***A1***

[2 marks]

(c) **METHOD 1**

$9 \times 2 \times 10!(=65318400)$ ways if Helen and Nicky sit next to each other ***(A1)***

attempt to subtract from total number of ways ***(M1)***

$12! - 9 \times 2 \times 10!$
$= 413683200$ ***A1***

METHOD 2

$6 \times 10 \times 10!(=217728000)$ ways if Helen sits in column 1 or 4 ***(A1)***

$6 \times 9 \times 10!(=195955200)$ ways if Helen sits in column 2 or 3 ***(A1)***

$217728000 + 195955200$
$= 413683200$ ***A1***

[3 marks]

8 – (MAT/20_HL_Winter_2019_Q8) – *Permutation - Combination*

(a) **METHOD 1**

$10! - 2 \times 9! \left(= 2903040\right)$ ***(A1)(A1)A1***

Note: Award ***A1*** for $10!$, ***A1*** for $2 \times 9!$, ***A1*** for final answer.

METHOD 2

$9 \times 8 \times 8!$ ***(A1)(A1)A1***

Note: Award ***A1*** for 9×8 or equivalent, ***A1*** for $8!$ and ***A1*** for answer.

[3 marks]

(b) **METHOD 1**

$8 \times 7 \times 8! \left(= 2257920\right)$ ***(A1)A1***

Note: Award ***(A1)*** for 8×7, ***A1*** for final answer.

METHOD 2

$10! - 2 \times 8! - 2 \times 2 \times 7 \times 8!$

Note: Award ***A1*** for $10!$ minus EITHER subtracted terms and ***A1*** for final correct answer.

[2 marks]

(c) **METHOD 1**

$8 \times 7 \times \left(8! - 2 \times 7!\right)\left(= 1693440\right)$ ***(A1)(A1)A1***

Note: Award ***(A1)*** for 8×7, ***(A1)*** for $2 \times 7!$, ***A1*** for final answer.
$\left(8! - 2 \times 7!\right)$ can be replaced by $6 \times 7!$ *or* ${}^7P_2 \times 6!$ which may be awarded the second ***A1***.

METHOD 2

their answer to (a) $-2 \times 8! - 2 \times 2 \times 7 \times 8!$ ***(A1)(A1)A1***

Note: Award ***A1*** for subtracting each of the terms and ***A1*** for final answer.

METHOD 3

their answer to (b) $-2 \times 7 \times 8!$ or equivalent ***(A1)A2***

Note: Award ***A1*** for the subtraction and ***A2*** for final answer.

[3 marks]

9 – (MAT/20_HL_Winter_2020_Q7) – *Permutation - Combination*

(a) identifying two or three possible cases ***(M1)***

total number of possible groups is $\binom{7}{5}+\binom{7}{4}\binom{5}{1}+\binom{7}{3}\binom{5}{2}$ ***(A1)(A1)***

Note: Award ***A1*** for any two correct cases, ***A1*** for the other one.

$=21+(35\times5)+(35\times10)$

$=546$ ***A1***

[4 marks]

(b) **METHOD 1**

identifying at least two of the three possible cases- Gary goes, Gerwyn goes or neither goes ***(M1)***

total number of possible groups is $\binom{10}{5}+\binom{10}{4}+\binom{10}{4}$ ***(A1)***

$=252+210+210$

$=672$ ***A1***

[3 marks]

METHOD 2

identifying the overall number of groups and no. of cases where both Gary and Gerwyn go. ***(M1)***

total number of possible groups is $\binom{12}{5}-\binom{10}{3}$ ***(A1)***

$=792-120$

$=672$ ***A1***

[3 marks]

10 – (MathAA/22_HL_Summer_2021_Q7) – *Permutation - Combination*

(a) Jack and Andrea finish in that order (as a unit) so we are considering the arrangement of 7 objects ***(M1)***

$7!\ (=5040)$ ways ***A1***

[2 marks]

(b) **METHOD 1**

the number of ways that Andrea finishes in front of Jack is equal to the number of ways that Jack finishes in front of Andrea ***(M1)***

total number of ways is $8!$ ***(A1)***

$\frac{8!}{2}\ (=20160)$ ways ***A1***

[3 marks]

METHOD 2

the other six runners can finish in $6!\ (=720)$ ways ***(A1)***

when Andrea finishes first, Jack can finish in 7 different positions

when Andrea finishes second, Jack can finish in 6 different positions etc

$7+6+5+4+3+2+1\ (=28)$ ways ***(A1)***

hence there are $(7+6+5+4+3+2+1)\times 6!$ ways

$28\times 6!\ (=20160)$ ways ***A1***

[3 marks]

11 – (MathAA/21_HL_Summer_2022_Q9) – *Permutation - Combination*

(a) $6\times 5!$ ***(A1)(A1)***

$=720$ (accept $6!$) ***A1***

[3 marks]

(b) **METHOD 1**

(Peter apart from girls, in an end seat) ${}^{8}P_{4}(=1680)$ OR

(Peter apart from girls, not in end seat) ${}^{7}P_{4}(=840)$ ***(A1)***

case 1: Peter at either end

$2\times {}^{8}P_{4}(=3360)$ OR $2\times {}^{8}C_{4}\times 4!(=3360)$ ***(A1)***

case 2: Peter not at the end

$8\times {}^{7}P_{4}(=6720)$ OR $8\times {}^{7}C_{4}\times 4!(=6720)$ ***(A1)***

Total number of ways $=3360+6720$

$=10080$ ***A1***

METHOD 2

(Peter next to girl, in an end seat) $4\times {}^{8}P_{3}(=1344)$ OR

(Peter next to one girl, not in end seat) $2\times 4\times {}^{7}P_{3}(=1680)$ OR

(Peter next to two girls, not in end seat) $4\times 3\times {}^{7}P_{2}(=504)$ ***(A1)***

case 1: Peter at either end

$2\times 4\times {}^{8}P_{3}(=2688)$ ***(A1)***

case 2: Peter not at the end

$8(2\times 4\times {}^{7}P_{3}+4\times 3\times {}^{7}P_{2})(=17472)$ ***(A1)***

Total number of ways $={}^{10}P_{5}\cdot(2688+17472)$

$=10080$ ***A1***

[4 marks]

12 – (MathAA/22_HL_Summer_2022_Q9) – *Permutation - Combination*

(a) $9\times9\times8\times7\times6\times5\ \left(=9\times{}^{9}P_{5}\right)$ ***(M1)***

$=136080\ \left(=9\times\dfrac{9!}{4!}\right)$ ***A1***

Note: Award ***M1A0*** for $10\times9\times8\times7\times6\times5\ \left(={}^{10}P_{6}=151200\ =\dfrac{10!}{4!}\right)$.

Note: Award ***M1A0*** for ${}^{9}P_{6}=60480$

[2 marks]

(b) **METHOD 1**

EITHER

every unordered subset of 6 digits from the set of 9 non-zero digits can be arranged in exactly one way into a 6-digit number with the digits in increasing order. ***A1***

OR

${}^{9}C_{6}\left(\times1\right)$ ***A1***

THEN

$=84$ ***A1***

METHOD 2

EITHER

removes 3 digits from the set of 9 non-zero digits and these 6 remaining digits can be arranged in exactly one way into a 6-digit number with the digits in increasing order. ***A1***

OR

${}^{9}C_{3}\left(\times1\right)$ ***A1***

THEN

$=84$ ***A1***

[2 marks]

1 - (MAT/21_HL_Summer_2012_Q9) - *Binomial Theorem*

$$\left(x-\frac{2}{x}\right)^4 = x^4 - 8x^2 + 24 - \frac{32}{x^2} + \frac{16}{x^4}$$

$$\left(x^2+\frac{2}{x}\right)^3 = x^6 + 6x^3 + 12 + \frac{8}{x^3}$$

Note: Accept unsimplified or uncalculated coefficients in the constant term

$= 24 \times 12$
$= 288$

2 - (MAT/22_HL_Summer_2014_Q5) - *Binomial Theorem*

expanding $(x-1)^3 = x^3 - 3x^2 + 3x - 1$

expanding $\left(\frac{1}{x}+2x\right)^6$ gives

$$64x^6 + 192x^4 + 240x^2 + \frac{60}{x^2} + \frac{12}{x^4} + \frac{1}{x^6} + 160$$

Note: Award ***(M1)*** for an attempt at expanding using binomial.
Award ***A1*** for $\frac{60}{x^2}$.
Award ***A1*** for $\frac{12}{x^4}$.

$$\frac{60}{x^2} \times -1 + \frac{12}{x^4} \times -3x^2$$

Note: Award ***(M1)*** only if both terms are considered.

therefore coefficient x^{-2} is -96

3 – (MAT/21_HL_Summer_2014_Q8) – *Binomial Theorem, Probability*

(a) $\left(A\binom{6}{5}2^5B+3\binom{6}{4}2^4B^2\right)x^5$

$=\left(192AB+720B^2\right)x^5$

(b) **METHOD 1**

$x=\frac{1}{6},\ A=\frac{3}{6}\left(=\frac{1}{2}\right),\ B=\frac{4}{6}\left(=\frac{2}{3}\right)$

probability is $\frac{4}{81}\ (=0.0494)$

METHOD 2

$\text{P}(5\text{ eaten})=\text{P}(\text{M eats }1)\,\text{P}(\text{N eats }4)+\text{P}(\text{M eats }0)\,\text{P}(\text{N eats }5)$

$=\frac{1}{2}\binom{6}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2+\frac{1}{2}\binom{6}{5}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)$

$=\frac{4}{81}\ (=0.0494)$

4 – (MAT/20_HL_Winter_2016_Q4) – *Binomial Theorem*

attempting a valid method to obtain the required term in the expansion ***(M1)***

Note: Valid methods include an attempt to expand, noting the behaviour of the powers of x, use of the general binomial expansion term, use of a ratio etc.

identifying the correct term ***(A1)***

$\binom{12}{8}\times 4^4\times\left(-\frac{3}{2}\right)^8\ \left(=495\times 4^4\times\left(-\frac{3}{2}\right)^8\right)$ ***M1A1***

Note: Accept $\binom{12}{4}$.

Note: Award ***M1*** for the product of a binomial coefficient, a power of 4 and either a power of $-\frac{3}{2}$ or $\frac{3}{2}$.

$=3\,247\,695$ ***A1***

[5 marks]

5 – (MAT/21_HL_Summer_2017_Q3) – *Binomial Theorem*

METHOD 1

$^8C_r\left(\frac{1}{x}\right)^{8-r}(5x)^r = {}^8C_r(5)^r x^{2r-8}$ ***(M1)***

$r = 5$ ***(A1)***

$^8C_5 \times 5^5 = {}^7C_4 a^3 \times 5^4$ ***M1A1***

$56 \times 5 = 35a^3$

$a^3 = 8$ ***(A1)***

$a = 2$ ***A1***

METHOD 2

attempt to expand both binomials ***M1***

$175000x^2$ ***A1***

$21875a^3x^4$ ***A1***

$175000 = 21875a^3$ ***M1***

$a^3 = 8$ ***(A1)***

$a = 2$ ***A1***

[6 marks]

6 – (MAT/22_HL_Summer_2018_Q5) – *Binomial Theorem*

(a) $\binom{3n+1}{3n-2} = \frac{(3n+1)!}{(3n-2)!3!}$ ***(M1)***

$= \frac{(3n+1)3n(3n-1)}{3!}$ ***A1***

$= \frac{9}{2}n^3 - \frac{1}{2}n$ or equivalent ***A1***

[3 marks]

(b) attempt to solve $\frac{9}{2}n^3 - \frac{1}{2}n > 10^6$ ***(M1)***

$n > 60.57\ldots$ ***(A1)***

Note: Allow equality.

$\Rightarrow n = 61$ ***A1***

[3 marks]

7 – (MAT/20_HL_Winter_2018_Q4) – *Binomial Theorem*

METHOD 1

$\left(x+\frac{3}{x^2}\right)^5 = \ldots + \binom{5}{2}x^2\left(\frac{3}{x^2}\right)^3 + \ldots$ ***(M1)(A1)(A1)***

Note: Award ***M1*** for a product of a binomial coefficient, a power of x, and a power of $\frac{3}{x^2}$, ***A1*** for correct binomial coefficient, ***A1*** for correct powers.

$= \ldots + 10 \times \frac{27}{x^4} + \ldots \left(= \ldots + \frac{270}{x^4} + \ldots\right)$ ***(A1)***

constant term is $x^4\left(\frac{270}{x^4}\right)$

$= 270$ ***A1***

METHOD 2

EITHER

the general term is $x^4\binom{5}{r}x^r\left(\frac{3}{x^2}\right)^{5-r}$ ***(M1)(A1)***

Note: Award ***M1*** for a product of a binomial coefficient, power(s) of x, and a power of $\frac{3}{x^2}$.

$= \binom{5}{r} \times 3^{5-r} \times \frac{x^{r+4}}{x^{10-2r}} \left(= \binom{5}{r} \times 3^{5-r}x^{3r-6}\right)$

constant term occurs when $r = 2$ ***(A1)***

OR

the general term is $x^4\binom{5}{5-r}x^{5-r}\left(\frac{3}{x^2}\right)^r$ ***(M1)(A1)***

Note: Award ***M1*** for a product of a binomial coefficient, power(s) of x, and a power of $\frac{3}{x^2}$.

$= \binom{5}{5-r} \times 3^r \times \frac{x^{9-r}}{x^{2r}} \left(= \binom{5}{5-r} \times 3^r x^{9-3r}\right)$

constant term occurs when $r = 3$ ***(A1)***

THEN

$\binom{5}{2}(3)^3$ ***(A1)***

$= 270$ ***A1***

[5 marks]

8 – (MAT/20_HL_Winter_2019_Q5) – *Binomial Theorem*

attempt to find coefficients in binomial expansion ***(M1)***

coefficient of x^2 : $\binom{n}{2}\times 2^{n-2}$; coefficient of x^3 : $\binom{n}{3}\times 2^{n-3}$ ***A1A1***

Note: Condone terms given rather than coefficients.
Terms may be seen in an equation such as that below.

$\binom{n}{3}\times 2^{n-3} = 4\binom{n}{2}\times 2^{n-2}$ ***(A1)***

attempt to solve equation using GDC or algebraically ***(M1)***

$\binom{n}{3} = 8\binom{n}{2}$

$\frac{n!}{3!(n-3)!} = \frac{8n!}{2!(n-2)!}$

$\frac{1}{3} = \frac{8}{n-2}$

$n = 26$ ***A1***

[6 marks]

9 – (MAT/20_HL_Winter_2020_Q4) – *Binomial Theorem*

use of Binomial expansion to find a term in either $\left(\frac{1}{3x^2}-\frac{x}{2}\right)^9$, $\left(\frac{1}{3x^{7/3}}-\frac{x^{2/3}}{2}\right)^9$,

$\left(\frac{1}{3}-\frac{x^3}{2}\right)^9$, $\left(\frac{1}{3x^3}-\frac{1}{2}\right)^9$ or $\left(2-3x^3\right)^9$ ***(M1)(A1)***

Note: Award ***M1*** for a product of three terms including a binomial coefficient and powers of the two terms, and ***A1*** for a correct expression of a term in the expansion.

finding the powers required to be 2 and 7 ***(M1)(A1)***

constant term is ${}^9C_2\times\left(\frac{1}{3}\right)^2\times\left(-\frac{1}{2}\right)^7$ ***(M1)***

Note: Ignore all x's in student's expression.

therefore term independent of x is $-\frac{1}{32}$ $(= -0.03125)$ ***A1***

10 – (MathAA/21_HL_Summer_2021_Q5) – *Binomial Theorem*

METHOD 1

product of a binomial coefficient, a power of 3 (and a power of x^2) seen ***(M1)***
evidence of correct term chosen ***(A1)***

$^{n+1}C_2 \times 3^{n+1-2} \times \left(x^2\right)^2 \left(= \frac{n(n+1)}{2} \times 3^{n-1} \times x^4\right)$ OR $n - r = 1$

equating their coefficient to 20412 or their term to $20412x^4$ ***(M1)***

EITHER

$^{n+1}C_2 \times 3^{n-1} = 20412$ ***(A1)***

OR

$^{r+2}C_r \times 3^r = 20412 \Rightarrow r = 6$ ***(A1)***

THEN

$n = 7$ ***A1***

METHOD 2

$3^{n+1}\left(1 + \frac{x^2}{3}\right)^{n+1}$

product of a binomial coefficient and a power of $\frac{x^2}{3}$ OR $\frac{1}{3}$ seen ***(M1)***

evidence of correct term chosen ***(A1)***

$3^{n+1} \times {}^{n+1}C_2 \times \left(\frac{x^2}{3}\right)^2 \left(= 3^{n-1}\frac{n(n+1)}{2}x^4\right)$

equating their coefficient to 20412 or their term to $20412x^4$ ***(M1)***

$3^{n-1} \times \frac{n(n+1)}{2} = 20412$ ***(A1)***

$n = 7$ ***A1***

11 – (MathAA/20_HL_Winter_2022_Q5) – *Binomial Theorem*

Note: Do not award any marks if there is clear evidence of adding instead of multiplying, for example ${}^9C_r + (ax)^{9-r} + (1)^r$.

valid approach for expansion (must be the product of a binomial coefficient with $n = 9$ and a power of ax) ***(M1)***

${}^9C_r(ax)^{9-r}(1)^r$ OR ${}^9C_{9-r}(ax)^r(1)^{9-r}$ OR ${}^9C_{11}(ax)^{11}(1)^9 + {}^9C_1(ax)^1(1)^8 + \ldots$

recognizing that the term in x^6 is needed ***(M1)***

$\frac{\text{Term in } x^6}{21x^2} = kx^4$ OR $r = 6$ OR $r = 3$ OR $9 - r = 6$

correct term or coefficient in binomial expansion (seen anywhere) ***(A1)***

${}^9C_6(ax)^6(1)^3$ OR ${}^9C_3a^6x^6$ OR $84\left(a^6x^6\right)(1)$ OR $84a^6$

EITHER

correct term in x^4 or coefficient (may be seen in equation) ***(A1)***

$\frac{{}^9C_6}{21}a^6x^4$ OR $4a^6x^4$ OR $4a^6$

Set their term in x^4 or coefficient of x^4 equal to $\frac{8}{7}a^5x^4$ or $\frac{8}{7}a^5$ (do not accept other powers of *x*) ***(M1)***

$\frac{{}^9C_3}{21}a^6x^4 = \frac{8}{7}a^5x^4$ OR $4a^6 = \frac{8}{7}a^5$

OR

correct term in x^6 or coefficient of x^6 (may be seen in equation) ***(A1)***

$84a^6x^6$ OR $84a^6$

Set their term in x^6 or coefficient of x^6 equal to $24a^5x^6$ or $24a^5$
(do not accept other powers of *x*) ***(M1)***

$84a^6x^6 = 24a^5x^6$ OR $84a = 24$

THEN

$a = \frac{2}{7} \approx 0.286(0.285714...)$ ***A1***

Note: Award ***A0*** for the final mark for $a = \frac{2}{7}$ and $a = 0$.

1 - (MAT/20_HL_Winter_2014_Q6) - *Remainder & Factor Theorem*

using $p(a) = -7$ to obtain $3a^3 + a^2 + 5a + 7 = 0$

$(a+1)(3a^2 - 2a + 7) = 0$

Note: Award ***M1*** for a cubic graph with correct shape and ***A1*** for clearly showing that the above cubic crosses the horizontal axis at $(-1, 0)$ only.

$a = -1$

EITHER

showing that $3a^2 - 2a + 7 = 0$ has no real (two complex) solutions for a

OR

showing that $3a^3 + a^2 + 5a + 7 = 0$ has one real (and two complex) solutions for a

2 - (MAT/21_HL_Summer_2016_Q8) - *Remainder & Factor Theorem*

$a^2 + 4a - b = 2$

EITHER

$a^2 + 4a - (b + 2) = 0$

as a is real $\Rightarrow$ $16 + 4(b + 2) \geq 0$

OR

$b = a^2 + 4a - 2$

$= (a + 2)^2 - 6$

THEN

$b \geq -6$

hence smallest possible value for b is -6

3 – (MAT/22_HL_Summer_2018_Q2) – *Remainder & Factor Theorem*

METHOD 1

substitute each of $x = 1,\ 2$ and 3 into the quartic and equate to zero *(M1)*

$p + q + r = -7$

$4p + 2q + r = -11$ or equivalent *(A2)*

$9p + 3q + r = -29$

Note: Award ***A2*** for all three equations correct, ***A1*** for two correct.

attempting to solve the system of equations *(M1)*

$p = -7,\ q = 17,\ r = -17$ *A1*

Note: Only award ***M1*** when some numerical values are found when solving algebraically or using GDC.

METHOD 2

attempt to find fourth factor *(M1)*

$(x - 1)$ *A1*

attempt to expand $(x-1)^2(x-2)(x-3)$ *M1*

$= x^4 - 7x^3 + 17x^2 - 17x + 6\ (p = -7,\ q = 17,\ r = -17)$ *A2*

Note: Award ***A2*** for all three values correct, ***A1*** for two correct.

Note: Accept long / synthetic division.

[5 marks]

4 – (MAT/20_HL_Winter_2018_Q6) – *Remainder & Factor Theorem*

(a) attempt to substitute $x = 5$ and set equal to zero, or use of long / synthetic division *(M1)*

$2 \times 5^4 - 15 \times 5^3 + a \times 5^2 + 5b + c = 0$ *A1*

$(\Rightarrow 25a + 5b + c = 625)$

[2 marks]

(b) 0 *A1*

[1 mark]

(c) **EITHER**

attempt to solve $P'(5) = 0$ *(M1)*

$\Rightarrow 8 \times 5^3 - 45 \times 5^2 + 4 \times 5 + b = 0$

OR

$\left(x^2 - 10x + 25\right)\left(2x^2 + \alpha x + \beta\right) = 2x^4 - 15x^3 + 2x^2 + bx + c$ *(M1)*

comparing coefficients gives $\alpha = 5, \beta = 2$

THEN

$b = 105$ *A1*

$\therefore c = 625 - 25 \times 2 - 525$

$c = 50$ *A1*

[3 marks]

1 - (MAT/22_HL_Summer_2013_Q8) - *Mathematical Induction*

$\mathrm{P}(n): f(n) = 5^{2n} - 24n - 1$ is divisible by 576 for $n \in \mathbb{Z}^+$

for $n = 1$, $f(1) = 5^2 - 24 - 1 = 0$

Zero is divisible by 576, (as every non-zero number divides zero), and so $\mathrm{P}(1)$ is true.

Note: Award ***R0*** for $\mathrm{P}(1) = 0$ shown and zero is divisible by 576 not specified.

Note: Ignore $\mathrm{P}(2) = 576$ if $\mathrm{P}(1) = 0$ is shown and zero is divisible by 576 is specified.

Assume $\mathrm{P}(k)$ is true for some $k (\Rightarrow f(k) = N \times 576)$.

Note: Do not award ***M1*** for statements such as "let $n = k$".

consider $\mathrm{P}(k+1): f(k+1) = 5^{2(k+1)} - 24(k+1) - 1$

$$= 25 \times 5^{2k} - 24k - 25$$

EITHER

$$= 25 \times (24k + 1 + N \times 576) - 24k - 25$$

$$= 576k + 25 \times 576N \text{ which is a multiple of 576}$$

OR

$$= 25 \times 5^{2k} - 600k - 25 + 600k - 24k$$

$$= 25\left(5^{2k} - 24k - 1\right) + 576k \text{ (or equivalent) which is a multiple of 576}$$

THEN

$\mathrm{P}(1)$ is true and $\mathrm{P}(k)$ true $\Rightarrow \mathrm{P}(k+1)$ true, so $\mathrm{P}(n)$ is true for all $n \in \mathbb{Z}^+$

2 – (MAT/21_HL_Summer_2014_Q7) – *Mathematical Induction*

if $n=0$

$7^3+2=345$ which is divisible by 5, hence true for $n=0$

Note: Award ***A0*** for using $n=1$ but do not penalize further in question.

assume true for $n=k$

Note: Only award the ***M1*** if truth is assumed.

so $7^{8k+3}+2=5p,\ p\in\bullet$

if $n=k+1$

$7^{8(k+1)+3}+2$

$=7^8 7^{8k+3}+2$

$=7^8(5p-2)+2$

$=7^8.5p-2.7^8+2$

$=7^8.5p-11529600$

$=5\left(7^8 p-2305920\right)$

hence if true for $n=k$, then also true for $n=k+1$. Since true for $n=0$, then true for all $n\in\bullet$

3 – (MAT/22_HL_Summer_2018_Q6) – *Mathematical Induction*

let P_n be the statement: $(1-a)^n > 1-na$ for some $n \in \mathbb{Z}^+, n \geq 2,$ where $0 < a < 1$

consider the case $n = 2$: $(1-a)^2 = 1 - 2a + a^2$ ***M1***

$> 1 - 2a$ because $a^2 > 0$. Therefore P_2 is true ***R1***

assume P_n is true for some $n = k$

$(1-a)^k > 1-ka$ ***M1***

Note: Assumption of truth must be present. Following marks are not dependent on this ***M1***.

EITHER

consider $(1-a)^{k+1} = (1-a)(1-a)^k$ ***M1***

$> 1-(k+1)a+ka^2$ ***A1***

$> 1-(k+1)a \Rightarrow P_{k+1}$ is true (as $ka^2 > 0$) ***R1***

OR

multiply both sides by $(1-a)$ (which is positive) ***M1***

$(1-a)^{k+1} > (1-ka)(1-a)$

$(1-a)^{k+1} > 1-(k+1)a+ka^2$ ***A1***

$(1-a)^{k+1} > 1-(k+1)a \Rightarrow P_{k+1}$ is true (as $ka^2 > 0$) ***R1***

THEN

P_2 is true and P_k is true $\Rightarrow P_{k+1}$ is true so P_n true for all $n \geq 2$ (or equivalent) ***R1***

Note: Only award the last ***R1*** if at least four of the previous marks are gained including the ***A1***.

[7 marks]

4 – (MAT/21_HL_Summer_2019_Q8) – *Mathematical Induction*

(a) $x < -0.414,\ x > 2.41$ ***A1A1***

$\left(x < 1-\sqrt{2},\ x > 1+\sqrt{2}\right)$

Note: Award ***A1*** for $-0.414,\ 2.41$ and ***A1*** for correct inequalities.

[2 marks]

(b) check for $n = 3$,

$16 > 9$ so true when $n = 3$ ***A1***

assume true for $n = k$

$2^{k+1} > k^2$ ***M1***

Note: Award ***M0*** for statements such as "let $n = k$".

Note: Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

prove true for $n = k+1$

$2^{k+2} = 2 \times 2^{k+1}$

$> 2k^2$ ***M1***

$= k^2 + k^2$ ***(M1)***

$> k^2 + 2k + 1$ (from part (a)) ***A1***

which is true for $k \geq 3$ ***R1***

Note: Only award the **A1** or the **R1** if it is clear why. Alternate methods are possible.

$= (k+1)^2$

hence if true for $n = k$ true for $n = k+1$, true for $n = 3$ so true for all $n \geq 3$ ***R1***

Note: Only award the final ***R1*** provided at least three of the previous marks are awarded.

[7 marks]

5 - (MAT/20_HL_Winter_2020_Q6) - *Mathematical Induction*

$n=1$: $\text{LHS}=\dfrac{\mathrm{d}\left(x\mathrm{e}^{px}\right)}{\mathrm{d}x}=xp\mathrm{e}^{px}+\mathrm{e}^{px}=\left(px+1\right)\mathrm{e}^{px}$, $\text{RHS}=p^{0}\left(px+1\right)\mathrm{e}^{px}$

$\text{LHS}=\text{RHS}$ so true for $n=1$: ***A1***

Note: Award ***A1*** if $n=0$ is proved.

assume proposition true for $n=k$, i.e. $\dfrac{\mathrm{d}^{k}}{\mathrm{d}x^{k}}\left(x\mathrm{e}^{px}\right)=p^{k-1}\left(px+k\right)\mathrm{e}^{px}$ ***M1***

Notes: Do not award ***M1*** if using n instead of k.
Assumption of truth must be present.
Subsequent marks are not dependent on this ***M1*** mark.

$\dfrac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}\left(x\mathrm{e}^{px}\right)=\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\mathrm{d}^{k}}{\mathrm{d}x^{k}}\left(x\mathrm{e}^{px}\right)\right)$ ***(M1)***

$=\dfrac{\mathrm{d}}{\mathrm{d}x}\left(p^{k-1}\left(px+k\right)\mathrm{e}^{px}\right)$ ***M1***

$=p^{k-1}\left(px+k\right)p\mathrm{e}^{px}+\mathrm{e}^{px}\left(p^{k}\right)$

$=p^{k}\left(px+k\right)\mathrm{e}^{px}+\mathrm{e}^{px}\left(p^{k}\right)$ ***A1***

Note: Award ***A1*** for correct derivative.

$=p^{k}\left(px+k+1\right)\mathrm{e}^{px}$ ***A1***

$=p^{\left((k+1)-1\right)}\left(px+\left(k+1\right)\right)\mathrm{e}^{px}$

Note: The final ***A1*** can be awarded for either of the two lines above.

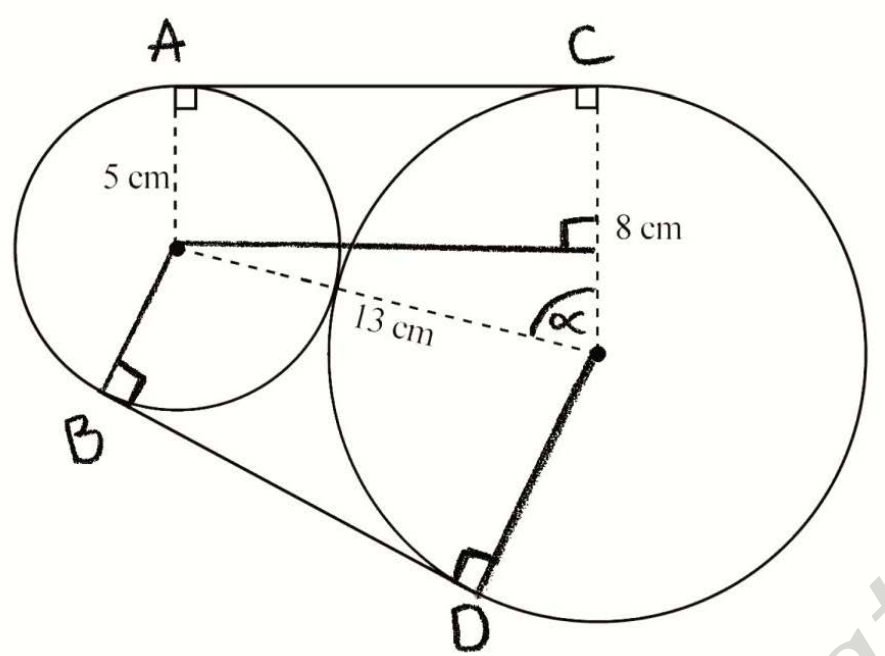
hence true for $n=1$ and $n=k$ true $\Rightarrow n=k+1$ true ***R1***

therefore true for all $n\in\mathbb{Z}^{+}$

Note: Only award the final ***R1*** if the three method marks have been awarded.

[7 marks]

1 - (MAT/22_HL_Summer_2012_Q9) - *Radian*



$\text{AC} = \text{BD} = \sqrt{13^2 - 3^2} = 12.64\ldots$

$\cos\alpha = \frac{3}{13} \Rightarrow \alpha = 1.337\ldots\left(76.65\ldots^\circ\right)$

attempt to find either arc length AB or arc length CD

arc length AB $= 5(\pi - 2\times 0.232\ldots)$ $(=13.37\ldots)$

arc length CD $= 8(\pi + 2\times 0.232\ldots)$ $(=28.85\ldots)$

$$\text{length of string} = 13.37\ldots + 28.85\ldots + 2(12.64\ldots)$$
$$= 67.5\,(\text{cm})$$

2 - (MAT/21_HL_Summer_2013_Q5) - *Radian*

$\frac{1}{2}r^2 \times 1 = 7$

$r = 3.7\ldots\left(=\sqrt{14}\right)$ (or 37… mm)

$\text{height} = 2r\cos\left(\frac{\pi - 1}{2}\right)$ $\left(\text{or } 2r\sin\frac{1}{2}\right)$

3.59 or anything that rounds to 3.6

so the dimensions are 3.7 by 3.6 (cm or 37 by 36 mm)

3 – (MAT/22_HL_Summer_2014_Q4) – *Trigonometry, Radian*

(a) **METHOD 1**

$2\arcsin\left(\frac{1.5}{4}\right)$

$\alpha = 0.769^c \ (44.0^\circ)$

METHOD 2

using the cosine rule:

$3^2 = 4^2 + 4^2 - 2(4)(4)\cos\alpha$

$\alpha = 0.769^c \ (44.0^\circ)$

(b) one segment

$A_1 = \frac{1}{2} \times 4^2 \times 0.76879 - \frac{1}{2} \times 4^2 \times \sin(0.76879)$

$= 0.58819K$

$2A_1 = 1.18 \ (cm^2)$

4 – (MAT/21_HL_Summer_2016_Q4) – *Radian*

attempting to use the area of sector formula (including for a semicircle)

semi-circle $\frac{1}{2}\pi \times 5^2 = \frac{25\pi}{2} = 39.26990817\ldots$

angle in smaller sector is $\pi - \theta$

area of sector $= \frac{1}{2} \times 2^2 \times (\pi - \theta)$

attempt to total a sum of areas of regions to 44

$2(\pi - \theta) = 44 - 39.26990817\ldots$

$\theta = 0.777 \left(= \frac{29\pi}{4} - 22\right)$

5 - (MAT/20_HL_Winter_2016_Q9) - *Radian, Trigonometry*

(a) $A = 2(\alpha - \sin\alpha)r^2 + \frac{1}{2}(\theta - \sin\theta)r^2$ ***M1A1A1***

Note: Award ***M1A1A1*** for alternative correct expressions *eg.* $A = 4\left(\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)r^2 + \frac{1}{2}\theta r^2$.

[3 marks]

(b) **METHOD 1**

consider for example triangle ADM where M is the midpoint of BD ***M1***

$\sin\frac{\alpha}{4} = \frac{1}{4}$ ***A1***

$\frac{\alpha}{4} = \arcsin\frac{1}{4}$

$\alpha = 4\arcsin\frac{1}{4}$ ***AG***

METHOD 2

attempting to use the cosine rule (to obtain $1 - \cos\frac{\alpha}{2} = \frac{1}{8}$) ***M1***

$\sin\frac{\alpha}{4} = \frac{1}{4}$ (obtained from $\sin\frac{\alpha}{4} = \sqrt{\frac{1-\cos\frac{\alpha}{2}}{2}}$) ***A1***

$\frac{\alpha}{4} = \arcsin\frac{1}{4}$

$\alpha = 4\arcsin\frac{1}{4}$ ***AG***

METHOD 3

$\sin\left(\frac{\pi}{2} - \frac{\alpha}{4}\right) = 2\sin\frac{\alpha}{2}$ where $\frac{\theta}{2} = \frac{\pi}{2} - \frac{\alpha}{4}$

$\cos\frac{\alpha}{4} = 4\sin\frac{\alpha}{4}\cos\frac{\alpha}{4}$ ***M1***

Note: Award ***M1*** either for use of the double angle formula or the conversion from sine to cosine.

$\frac{1}{4} = \sin\frac{\alpha}{4}$ ***A1***

$\frac{\alpha}{4} = \arcsin\frac{1}{4}$

$\alpha = 4\arcsin\frac{1}{4}$ ***AG***

[2 marks]

(c) (from triangle ADM), $\theta = \pi - \frac{\alpha}{2}\ \left(= \pi - 2\arcsin\frac{1}{4} = 2\arccos\frac{1}{4} = 2.6362\ldots\right)$ ***A1***

attempting to solve $2(\alpha - \sin\alpha)r^2 + \frac{1}{2}(\theta - \sin\theta)r^2 = 4$

with $\alpha = 4\arcsin\frac{1}{4}$ and $\theta = \pi - \frac{\alpha}{2}\ \left(= 2\arccos\frac{1}{4}\right)$ for r ***(M1)***

$r = 1.69$ ***A1***

[3 marks]

6 – (MAT/20_HL_Winter_2017_Q3) – *Radian*

METHOD 1

area = (four sector areas radius 9) + (four sector areas radius 3) ***(M1)***

$= 4\left(\frac{1}{2}9^2\frac{\pi}{9}\right) + 4\left(\frac{1}{2}3^2\frac{7\pi}{18}\right)$ ***(A1)(A1)***

$= 18\pi + 7\pi$

$= 25\pi\left(= 78.5\text{cm}^2\right)$ ***A1***

METHOD 2

area =
(area of circle radius 3) + (four sector areas radius 9) – (four sector areas radius 3) ***(M1)***

$\pi 3^2 + 4\left(\frac{1}{2}9^2\frac{\pi}{9}\right) - 4\left(\frac{1}{2}3^2\frac{\pi}{9}\right)$ ***(A1)(A1)***

Note: Award ***A1*** for the second term and ***A1*** for the third term.

$= 9\pi + 18\pi - 2\pi$

$= 25\pi\left(= 78.5\text{cm}^2\right)$ ***A1***

Note: Accept working in degrees.

[4 marks]

7 – (MAT/22_HL_Summer_2018_Q4) – *Radian*

(a) (i) **METHOD 1**

$\mathrm{PC} = \frac{\sqrt{3}}{2}$ or 0.8660 ***(M1)***

$\mathrm{PM} = \frac{1}{2}\mathrm{PC} = \frac{\sqrt{3}}{4}$ or 0.4330 ***(A1)***

$\mathrm{AM} = \sqrt{\frac{1}{4} + \frac{3}{16}}$

$= \frac{\sqrt{7}}{4}$ or 0.661 (m) ***A1***

Note: Award ***M1*** for attempting to solve triangle AMP.

METHOD 2

using the cosine rule

$\mathrm{AM}^2 = 1^2 + \left(\frac{\sqrt{3}}{4}\right)^2 - 2 \times \frac{\sqrt{3}}{4} \times \cos(30°)$ ***M1A1***

$\mathrm{AM} = \frac{\sqrt{7}}{4}$ or 0.661 (m) ***A1***

(ii) $\tan\left(\mathrm{A\hat{M}P}\right) = \frac{2}{\sqrt{3}}$ or equivalent ***(M1)***

$= 0.857$ ***A1***

[5 marks]

(b) **EITHER**

$\frac{1}{2}\mathrm{AM}^2\left(2\mathrm{A\hat{M}P} - \sin(2\mathrm{A\hat{M}P})\right)$ ***(M1)A1***

OR

$\frac{1}{2}\mathrm{AM}^2 \times 2\mathrm{A\hat{M}P} - \frac{\sqrt{3}}{8}$ ***(M1)A1***

$= 0.158(\mathrm{m}^2)$ ***A1***

Note: Award ***M1*** for attempting to calculate area of a sector minus area of a triangle.

[3 marks]

8 – (MAT/20_HL_Winter_2019_Q4) – *Radian*

(a) each arc has length $r\theta = 6\times\frac{\pi}{3} = 2\pi\left(=6.283...\right)$ ***(M1)***

perimeter is therefore $6\pi\ \left(=18.8\right)$ (cm) ***A1***

[2 marks]

(b) area of sector, s, is $\frac{1}{2}r^2\theta = 18\times\frac{\pi}{3} = 6\pi\left(=18.84...\right)$ ***(A1)***

area of triangle, t, is $\frac{1}{2}\times 6\times 3\sqrt{3} = 9\sqrt{3}\left(=15.58...\right)$ ***(M1)(A1)***

Note: area of segment, k, is 3.261… implies area of triangle

finding $3s-2t$ or $3k+t$ or similar

area $=3s-2t=18\pi-18\sqrt{3}\ \left(=25.4\right)$ (cm^2) ***(M1)A1***

[5 marks]

9 – (MAT/20_HL_Winter_2020_Q1) – *Radian, Trigonometry*

attempt to find $\mathrm{A\hat{O}B}$ by right-angled trigonometry or the cosine rule (***M1***)

EITHER

$\mathrm{A\hat{O}B} = 2\arcsin\left(\frac{5.5}{15}\right)$ ***A1***

OR

$\mathrm{A\hat{O}B} = \arccos\left(\frac{15^2+15^2-11^2}{2\times 15\times 15}\right)$ ***A1***

THEN

$=0.750847...\ \left(=43.0204...^\circ\right)$

Note: Award ***(M1)A1*** for correct calculation of $\mathrm{A\hat{O}B}$ or $\frac{1}{2}\mathrm{A\hat{O}B}$

shaded area = area of sector- area of triangle $\left(=\frac{1}{2}r^2\left(\theta-\sin\theta\right)\right)$ (***M1***)

$=\frac{1}{2}\times 15^2\times\left(0.750847...-\sin 0.750847...\right)$ (***A1***)

$=7.72$ (cm^2) ***A1***

[5 marks]

10 – (MathAA/20_HL_Winter_2021_Q4) – *Radian*

(a) attempt to find the area of either shaded region in terms of r and θ ***(M1)***

Note: Do not award ***M1*** if they have only copied from the booklet and not applied to the shaded area.

Area of segment $= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$ ***A1***

Area of triangle $= \frac{1}{2}r^2\sin(\pi - \theta)$ ***A1***

correct equation in terms of θ only ***(A1)***

$\theta - \sin\theta = \sin(\pi - \theta)$

$\theta - \sin\theta = \sin\theta$ ***A1***

$\theta = 2\sin\theta$ ***AG***

Note: Award a maximum of ***M1A1A0A0A0*** if a candidate uses degrees (i.e., $\frac{1}{2}r^2\sin(180° - \theta)$), even if later work is correct.

Note: If a candidate directly states that the area of the triangle is $\frac{1}{2}r^2\sin\theta$, award a maximum of ***M1A1A0A1A1***.

[5 marks]

(b) $\theta = 1.89549...$

$\theta = 1.90$ ***A1***

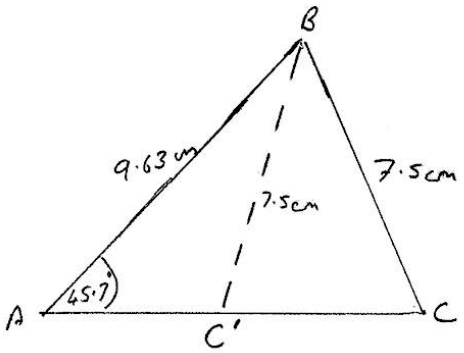
Note: Award ***A0*** if there is more than one solution. Award ***A0*** for an answer in degrees.

[1 mark]

Total [6 marks]

1 – (MAT/22_HL_Summer_2012_Q3) – *Trigonometry*

(a)



Note:	Accept 2 separate triangles. The diagram(s) should show that one triangle has an acute angle and the other triangle has an obtuse angle. The values 9.63, 7.5 and 45.7 and/or the letters, A, B C′ and C should be correctly marked on the diagram(s).

(b) **METHOD 1**

$$\frac{\sin 45.7}{7.5} = \frac{\sin C}{9.63}$$

$$\Rightarrow \hat{C} = 66.77\ldots^\circ,\ 113.2\ldots^\circ$$

$$\Rightarrow \hat{B} = 67.52\ldots^\circ,\ 21.07\ldots^\circ$$

$$\frac{b}{\sin B} = \frac{7.5}{\sin 45.7} \Rightarrow b = 9.68(\text{cm}),\ b = 3.77(\text{cm})$$

Note:	If only the acute value of $\hat{C}$ is found, award ***M1(A1)(A0)(A0)A1A0***.

METHOD 2

$$7.5^2 = 9.63^2 + b^2 - 2 \times 9.63 \times b\cos 45.7^\circ$$

$$b^2 - 13.45\ldots b + 36.48\ldots = 0$$

$$b = \frac{13.45\ldots \pm \sqrt{13.45\ldots^2 - 4 \times 36.48\ldots}}{2}$$

$$\text{AC} = 9.68(\text{cm}),\ \text{AC} = 3.77(\text{cm})$$

2 – (MAT/22_HL_Summer_2013_Q1) – *Trigonometry*

(a) **EITHER**

$\hat{AOB} = 2\arcsin\left(\frac{3}{4}\right)$ or equivalent (*eg* $\hat{AOB} = 2\arctan\left(\frac{3}{\sqrt{7}}\right)$, $\hat{AOB} = 2\arccos\left(\frac{\sqrt{7}}{4}\right)$)

OR

$\cos\hat{AOB} = \frac{4^2 + 4^2 - 6^2}{2 \times 4 \times 4} \quad \left(= -\frac{1}{8}\right)$

THEN

$= 1.696$ (correct to 4sf)

(b) use of area of segment = area of sector – area of triangle

$= \frac{1}{2} \times 4^2 \times 1.696 - \frac{1}{2} \times 4^2 \times \sin 1.696$

$= 5.63\,(\text{cm}^2)$

3 – (MAT/22_HL_Summer_2013_Q6) – *Trigonometry*

(a) attempting to solve for $\cos x$ or for u where $u = \cos x$ or for x graphically.

EITHER

$\cos x = \frac{2}{3}$ (and 2)

OR

$x = 48.1897\ldots^\circ$

THEN

$x = 48^\circ$

Note: Award ***(M1)(A1)A0*** for $x = 48^\circ, 132^\circ$.

Note: Award ***(M1)(A1)A0*** for 0.841 radians.

(b) attempting to solve for $\sec x$ or for v where $v = \sec x$.

$\sec x = \pm\sqrt{2} \left(\text{and } \pm\sqrt{\frac{2}{3}}\right)$

$\sec x = \pm\sqrt{2}$

4 – (MAT/21_HL_Summer_2013_Q7) – *Trigonometry*

(a) let the distance the cable is laid along the seabed be y

$y^2 = x^2 + 200^2 - 2 \times x \times 200 \cos 60^0$

(or equivalent method)

$y^2 = x^2 - 200x + 40000$

$\text{cost} = C = 80y + 20x$

$C = 80(x^2 - 200x + 40000)^{\frac{1}{2}} + 20x$

(b) $x = 55.2786\ldots = 55$ (m to the nearest metre)

$\left(x = 100 - \sqrt{2000}\right)$

5 – (MAT/20_HL_Winter_2013_Q7) – *Vectors - Lines - Planes, Trigonometry*

(a) attempting to form $(3\cos\theta + 6)(\cos\theta - 2) + 7(1 + \sin\theta) = 0$

$3\cos^2\theta - 12 + 7\sin\theta + 7 = 0$

$3\left(1 - \sin^2\theta\right) + 7\sin\theta - 5 = 0$

$3\sin^2\theta - 7\sin\theta + 2 = 0$

(b) attempting to solve algebraically (including substitution) or graphically for $\sin\theta$

$\sin\theta = \frac{1}{3}$

$\theta = 0.340 \quad (= 19.5^\circ)$

6 – (MAT/20_HL_Winter_2013_Q8) – *Trigonometry*

(a) $A = \frac{1}{2} \times 10^2 \times \theta - \frac{1}{2} \times 10^2 \times \sin\theta$ ***M1A1***

Note: Award ***M1*** for use of area of segment = area of sector − area of triangle.

$= 50\theta - 50\sin\theta$ ***AG***

(b) **METHOD 1**

unshaded area $= \frac{\pi \times 10^2}{2} - 50(\theta - \sin\theta)$

(or equivalent *eg* $50\pi - 50\theta + 50\sin\theta$) ***(M1)***

$50\theta - 50\sin\theta = \frac{1}{2}(50\pi - 50\theta + 50\sin\theta)$ ***(A1)***

$3\theta - 3\sin\theta - \pi = 0$

$\Rightarrow \theta = 1.969$ (rad) ***A1***

METHOD 2

$50\theta - 50\sin\theta = \frac{1}{3}\left(\frac{\pi \times 10^2}{2}\right)$ ***(M1)(A1)***

$3\theta - 3\sin\theta - \pi = 0$

$\Rightarrow \theta = 1.969$ (rad) ***A1***

7 – (MAT/22_HL_Summer_2014_Q4) – *Trigonometry, Radian*

(a) **METHOD 1**

$$2\arcsin\left(\frac{1.5}{4}\right)$$

$$\alpha = 0.769^c \ \ (44.0^\circ)$$

METHOD 2

using the cosine rule:

$$3^2 = 4^2 + 4^2 - 2(4)(4)\cos\alpha$$

$$\alpha = 0.769^c \ \ (44.0^\circ)$$

(b) one segment

$$A_1 = \frac{1}{2}\times 4^2 \times 0.76879 - \frac{1}{2}\times 4^2 \times \sin(0.76879)$$

$$= 0.58819\text{K}$$

$$2A_1 = 1.18 \ (\text{cm}^2)$$

8 – (MAT/20_HL_Winter_2014_Q9) – *Trigonometry*

(a) each triangle has area $\frac{1}{8}x^2 \sin\frac{2\pi}{n}$ (use of $\frac{1}{2}ab\sin C$)

there are n triangles so $A = \frac{1}{8}nx^2 \sin\frac{2\pi}{n}$

$$C = \frac{4\left(\frac{1}{8}nx^2 \sin\frac{2\pi}{n}\right)}{\pi x^2}$$

so $C = \frac{n}{2\pi}\sin\frac{2\pi}{n}$

(b) attempting to find the least value of n such that $\frac{n}{2\pi}\sin\frac{2\pi}{n} > 0.99$

$n = 26$

attempting to find the least value of n such that $\frac{n\sin\frac{2\pi}{n}}{\pi\left(1+\cos\frac{\pi}{n}\right)} > 0.99$

$n = 21$ (and so a regular polygon with 21 sides)

Note: Award ***(M0)A0(M1)A1*** if $\frac{n}{2\pi}\sin\frac{2\pi}{n} > 0.99$ is not considered and $\frac{n\sin\frac{2\pi}{n}}{\pi\left(1+\cos\frac{\pi}{n}\right)} > 0.99$ is correctly considered.

Award ***(M1)A1(M0)A0*** for $n = 26$.

(c) **EITHER**

for even and odd values of n, the value of C seems to increase towards the limiting value of the circle $(C = 1)$ *ie* as n increases, the polygonal regions get closer and closer to the enclosing circular region

OR

the differences between the odd and even values of n illustrate that this measure of compactness is not a good one.

9 – (MAT/20_HL_Winter_2014_Q14) – *Trigonometry*

(a) **METHOD 1**

squaring both equations

$9\sin^2 B + 24\sin B\cos C + 16\cos^2 C = 36$

$9\cos^2 B + 24\cos B\sin C + 16\sin^2 C = 1$

adding the equations and using $\cos^2\theta + \sin^2\theta = 1$ to obtain

$9 + 24\sin(B+C) + 16 = 37$

$24(\sin B\cos C + \cos B\sin C) = 12$

$24\sin(B+C) = 12$

$\sin(B+C) = \frac{1}{2}$

METHOD 2

substituting for $\sin B$ and $\cos B$ to obtain

$\sin(B+C) = \left(\frac{6-4\cos C}{3}\right)\cos C + \left(\frac{1-4\sin C}{3}\right)\sin C$

$= \frac{6\cos C + \sin C - 4}{3}$ (or equivalent)

substituting for $\sin C$ and $\cos C$ to obtain

$\sin(B+C) = \sin B\left(\frac{6-3\sin B}{4}\right) + \cos B\left(\frac{1-3\cos B}{4}\right)$

$= \frac{\cos B + 6\sin B - 3}{4}$ (or equivalent)

Adding the two equations for $\sin(B+C)$:

$2\sin(B+C) = \frac{(18\sin B + 24\cos C) + (4\sin C + 3\cos B) - 25}{12}$

$\sin(B+C) = \frac{36+1-25}{24}$

$\sin(B+C) = \frac{1}{2}$

METHOD 3

substituting for $\sin B$ and $\sin C$ to obtain

$\sin(B+C) = \left(\frac{6-4\cos C}{3}\right)\cos C + \cos B\left(\frac{1-3\cos B}{4}\right)$

substituting for $\cos B$ and $\cos C$ to obtain

$\sin(B+C) = \sin B\left(\frac{6-3\sin B}{4}\right) + \left(\frac{1-4\sin C}{3}\right)\sin C$

Adding the two equations for $\sin(B+C)$:

$2\sin(B+C) = \frac{6\cos C + \sin C - 4}{3} + \frac{6\sin B + \cos B - 3}{4}$ (or equivalent)

$$2\sin\left(B+C\right)=\frac{(18\sin B+24\cos C)+(4\sin C+3\cos B)-25}{12}$$

$$\sin\left(B+C\right)=\frac{36+1-25}{24}$$

$$\sin\left(B+C\right)=\frac{1}{2}$$

(b) $\sin A=\sin\left(180^\circ-(B+C)\right)$ so $\sin A=\sin(B+C)$

$\sin(B+C)=\frac{1}{2}\Rightarrow\sin A=\frac{1}{2}$

$\Rightarrow A=30^\circ$ or $A=150^\circ$

if $A=150^\circ$, then $B<30^\circ$

for example, $3\sin B+4\cos C<\frac{3}{2}+4<6$, *ie* a contradiction

only one possible value ($A=30^\circ$)

10 – (MAT/22_HL_Summer_2015_Q1) – *Trigonometry*

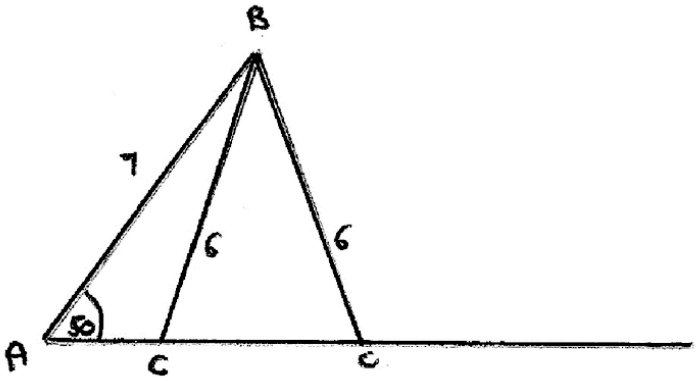
(a) $A=\frac{1}{2}\times5\times12\times\sin100^\circ$

$=29.5\left(\text{cm}^2\right)$

(b) $\text{AC}^2=5^2+12^2-2\times5\times12\times\cos100^\circ$

therefore $\text{AC}=13.8$ (cm)

11 - (MAT/21_HL_Summer_2015_Q4) - *Trigonometry*



METHOD 1

$$\frac{6}{\sin 50} = \frac{7}{\sin C} \Rightarrow \sin C = \frac{7\sin 50}{6}$$

$C = 63.344...$

or $C = 116.655...$

$B = 13.344...$ (or $B = 66.656...$)

area $= \frac{1}{2} \times 6 \times 7 \times \sin 13.344...$ (or $\frac{1}{2} \times 6 \times 7 \times \sin 66.656...$)

$4.846...$ (or $= 19.281...$)

so answer is 4.85 (cm^2)

METHOD 2

$6^2 = 7^2 + b^2 - 2 \times 7b\cos 50$

$b^2 - 14\,b\cos 50 + 13 = 0$ or equivalent method to solve the above equation

$b = 7.1912821...$ or $b = 1.807744...$

area $= \frac{1}{2} \times 7 \times 1.8077...\sin 50 = 4.846...$

(or $\frac{1}{2} \times 7 \times 7.1912821...\sin 50 = 19.281...$)

so answer is 4.85 (cm^2)

METHOD 3

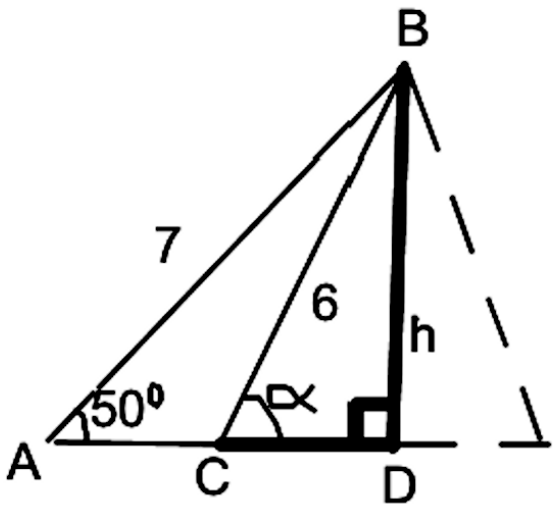


Diagram showing triangles ACB and ADB

$h = 7\sin(50) = 5.3623...$ (cm)

$\alpha = \arcsin\frac{h}{6} = 63.3442...$

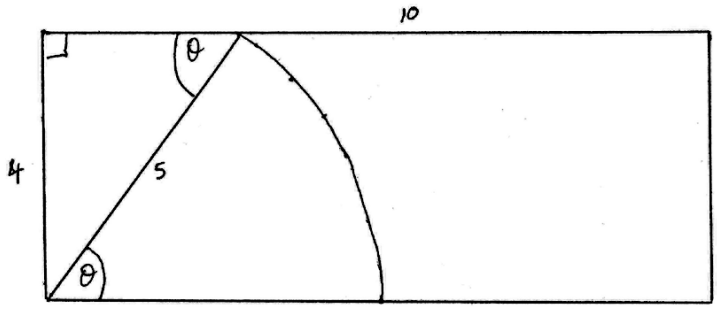
$AC = AD - CD = 7\cos 50 - 6\cos\alpha = 1.8077...$ (cm)

area $= \frac{1}{2} \times 1.8077... \times 5.3623...$

$= 4.85$ (cm^2)

12 – (MAT/22_HL_Summer_2015_Q8) – *Trigonometry*

(a)



EITHER

area of triangle $= \frac{1}{2} \times 3 \times 4 \ (= 6)$

area of sector $= \frac{1}{2} \arcsin\left(\frac{4}{5}\right) \times 5^2 \ (= 11.5911\ldots)$

OR

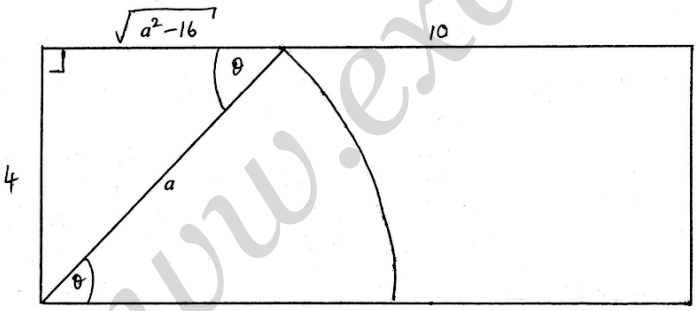
$\int_0^4 \sqrt{25 - x^2}\,\mathrm{d}x$

THEN

total area $= 17.5911\ldots \mathrm{m}^2$

percentage $= \frac{17.5911\ldots}{40} \times 100 = 44\%$

(b) **METHOD 1**



area of triangle $= \frac{1}{2} \times 4 \times \sqrt{a^2 - 16}$

$\theta = \arcsin\left(\frac{4}{a}\right)$

area of sector $= \frac{1}{2} r^2 \theta = \frac{1}{2} a^2 \arcsin\left(\frac{4}{a}\right)$

therefore total area $= 2\sqrt{a^2 - 16} + \frac{1}{2} a^2 \arcsin\left(\frac{4}{a}\right) = 20$

rearrange to give: $a^2 \arcsin\left(\frac{4}{a}\right) + 4\sqrt{a^2 - 16} = 40$

METHOD 2

$$\int_0^4 \sqrt{a^2 - x^2}\,dx = 20$$

use substitution $x = a\sin\theta,\ \frac{\mathrm{d}x}{\mathrm{d}\theta} = a\cos\theta$

$$\int_0^{\arcsin\left(\frac{4}{a}\right)} a^2\cos^2\theta\,\mathrm{d}\theta = 20$$

$$\frac{a^2}{2}\int_0^{\arcsin\left(\frac{4}{a}\right)} \left(\cos 2\theta + 1\right)\mathrm{d}\theta = 20$$

$$a^2\left[\left(\frac{\sin 2\theta}{2} + \theta\right)\right]_0^{\arcsin\left(\frac{4}{a}\right)} = 40$$

$$a^2\left[\left(\sin\theta\cos\theta + \theta\right)\right]_0^{\arcsin\left(\frac{4}{a}\right)} = 40$$

$$a^2\arcsin\left(\frac{4}{a}\right) + a^2\left(\frac{4}{a}\right)\sqrt{\left(1 - \left(\frac{4}{a}\right)^2\right)} = 40$$

$$a^2\arcsin\left(\frac{4}{a}\right) + 4\sqrt{a^2 - 16} = 40$$

(c) solving using GDC $\Rightarrow a = 5.53\,\text{cm}$

13 – (MAT/20_HL_Winter_2015_Q4) – *Trigonometry*

(a) (i) $A = -3$

(ii) period $= \frac{2\pi}{B}$

$B = 2$

Note: Award as above for $A = 3$ and $B = -2$.

(iii) $C = 2$

(b) $x = 1.74,\ 2.97\ \left(x = \frac{1}{2}\left(\pi + \arcsin\frac{1}{3}\right), \frac{1}{2}\left(2\pi - \arcsin\frac{1}{3}\right)\right)$

14 - (MAT/20_HL_Winter_2015_Q7) - *Trigonometry*

$$21 = \frac{1}{2} \cdot 6 \cdot 11 \cdot \sin A$$

$$\sin A = \frac{7}{11}$$

EITHER

$$\hat{A} = 0.6897..., 2.452\ldots \left(\hat{A} = \arcsin\frac{7}{11}, \pi - \arcsin\frac{7}{11} = 39.521...^\circ, 140.478...^\circ \right)$$

OR

$$\cos A = \pm\frac{6\sqrt{2}}{11} (= \pm 0.771...)$$

THEN

$$\mathrm{BC}^2 = 6^2 + 11^2 - 2 \cdot 6 \cdot 11 \cos A$$

$$\mathrm{BC} = 16.1 \text{ or } 7.43$$

15 - (MAT/22_HL_Summer_2016_Q1) - *Trigonometry*

$$\mathrm{AC}^2 = 7.8^2 + 10.4^2$$

$$\mathrm{AC} = 13$$

use of cosine rule *eg*, $\cos\left(\mathrm{A\hat{B}C}\right) = \dfrac{6.5^2 + 9.1^2 - 13^2}{2(6.5)(9.1)}$

$$\mathrm{A\hat{B}C} = 111.804\ldots^\circ \ (= 1.95134\ldots)$$

$$= 112^\circ$$

16 – (MAT/20_HL_Winter_2016_Q7) – *Trigonometry*

(a) **METHOD 1**

let $\mathrm{AC} = x$

$3^2 = x^2 + 4^2 - 8x\cos\frac{\pi}{9}$ ***M1A1***

attempting to solve for x ***(M1)***

$x = 1.09,\ 6.43$ ***A1A1***

METHOD 2

let $\mathrm{AC} = x$

using the sine rule to find a value of C ***M1***

$4^2 = x^2 + 3^2 - 6x\cos(152.869\ldots^\circ) \Rightarrow x = 1.09$ ***(M1)A1***

$4^2 = x^2 + 3^2 - 6x\cos(27.131\ldots^\circ) \Rightarrow x = 6.43$ ***(M1)A1***

METHOD 3

let $\mathrm{AC} = x$

using the sine rule to find a value of B and a value of C ***M1***

obtaining $B = 132.869\ldots^\circ,\ 7.131\ldots^\circ$ and $C = 27.131\ldots^\circ,\ 152.869\ldots^\circ$ ***A1***

$(B = 2.319\ldots,\ 0.124\ldots$ and $C = 0.473\ldots,\ 2.668\ldots)$

attempting to find a value of x using the cosine rule ***(M1)***

$x = 1.09,\ 6.43$ ***A1A1***

Note: Award ***M1A0(M1)A1A0*** for one correct value of x

[5 marks]

(b) $\frac{1}{2}\times 4\times 6.428\ldots\times\sin\frac{\pi}{9}$ and $\frac{1}{2}\times 4\times 1.088\ldots\times\sin\frac{\pi}{9}$ ***(A1)***

$(4.39747\ldots$ and $0.744833\ldots)$

let D be the difference between the two areas

$D = \frac{1}{2}\times 4\times 6.428\ldots\times\sin\frac{\pi}{9} - \frac{1}{2}\times 4\times 1.088\ldots\times\sin\frac{\pi}{9}$ ***(M1)***

$(D = 4.39747\ldots - 0.744833\ldots)$

$= 3.65\left(\mathrm{cm}^2\right)$ ***A1***

[3 marks]

17 – (MAT/20_HL_Winter_2016_Q9) – *Radian, Trigonometry*

(a) $A = 2(\alpha - \sin\alpha)r^2 + \frac{1}{2}(\theta - \sin\theta)r^2$ ***M1A1A1***

Note: Award ***M1A1A1*** for alternative correct expressions *eg.* $A = 4\left(\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)r^2 + \frac{1}{2}\theta r^2$.

[3 marks]

(b) **METHOD 1**

consider for example triangle ADM where M is the midpoint of BD ***M1***

$\sin\frac{\alpha}{4} = \frac{1}{4}$ ***A1***

$\frac{\alpha}{4} = \arcsin\frac{1}{4}$

$\alpha = 4\arcsin\frac{1}{4}$ ***AG***

METHOD 2

attempting to use the cosine rule (to obtain $1 - \cos\frac{\alpha}{2} = \frac{1}{8}$) ***M1***

$\sin\frac{\alpha}{4} = \frac{1}{4}$ (obtained from $\sin\frac{\alpha}{4} = \sqrt{\frac{1-\cos\frac{\alpha}{2}}{2}}$) ***A1***

$\frac{\alpha}{4} = \arcsin\frac{1}{4}$

$\alpha = 4\arcsin\frac{1}{4}$ ***AG***

METHOD 3

$\sin\left(\frac{\pi}{2} - \frac{\alpha}{4}\right) = 2\sin\frac{\alpha}{2}$ where $\frac{\theta}{2} = \frac{\pi}{2} - \frac{\alpha}{4}$

$\cos\frac{\alpha}{4} = 4\sin\frac{\alpha}{4}\cos\frac{\alpha}{4}$ ***M1***

Note: Award ***M1*** either for use of the double angle formula or the conversion from sine to cosine.

$\frac{1}{4} = \sin\frac{\alpha}{4}$ ***A1***

$\frac{\alpha}{4} = \arcsin\frac{1}{4}$

$\alpha = 4\arcsin\frac{1}{4}$ ***AG***

[2 marks]

(c) (from triangle ADM), $\theta = \pi - \frac{\alpha}{2}$ $\left(= \pi - 2\arcsin\frac{1}{4} = 2\arccos\frac{1}{4} = 2.6362\ldots\right)$ ***A1***

attempting to solve $2(\alpha - \sin\alpha)r^2 + \frac{1}{2}(\theta - \sin\theta)r^2 = 4$

with $\alpha = 4\arcsin\frac{1}{4}$ and $\theta = \pi - \frac{\alpha}{2}$ $\left(= 2\arccos\frac{1}{4}\right)$ for r ***(M1)***

$r = 1.69$ ***A1***

[3 marks]

18 - (MAT/22_HL_Summer_2017_Q4) - *Quadratics, Trigonometry*

(a) $k^2 - k - 12 < 0$

$(k-4)(k+3) < 0$ ***(M1)***

$-3 < k < 4$ ***A1***

[2 marks]

(b) $\cos B = \frac{2^2 + c^2 - 4^2}{4c}$ $\left(\text{or } 16 = 2^2 + c^2 - 4c\cos B\right)$ ***M1***

$\Rightarrow \frac{c^2 - 12}{4c} < \frac{1}{4}$ ***A1***

$\Rightarrow c^2 - c - 12 < 0$

from result in (a)

$0 < \text{AB} < 4$ or $-3 < \text{AB} < 4$ ***(A1)***

but AB must be at least 2

$\Rightarrow 2 < AB < 4$ ***A1***

Note: Allow $\leq AB$ for either of the final two ***A*** marks.

[4 marks]

19 – (MAT/21_HL_Summer_2017_Q10) – *Trigonometry*

(a) $p^2 = 12^2 + r^2 - 2 \times 12 \times r \times \cos(30^\circ)$ ***M1A1***

$r^2 - 12\sqrt{3}\,r + 144 - p^2 = 0$ ***AG***

[2 marks]

(b) **EITHER**

$r^2 - 12\sqrt{3}r + 80 = 0$ ***(M1)***

OR

using the sine rule ***(M1)***

THEN

$PQ = 5.10$ (cm) or ***A1***

$PQ = 15.7$ (cm) ***A1***

[3 marks]

(c) area $= \frac{1}{2} \times 12 \times 5.1008\ldots \times \sin(30^\circ)$ ***M1A1***

$= 15.3\,(\text{cm}^2)$ ***A1***

[3 marks]

(d) **METHOD 1**

EITHER

$r^2 - 12\sqrt{3}\,r + 144 - p^2 = 0$

discriminant $= \left(12\sqrt{3}\right)^2 - 4 \times \left(144 - p^2\right)$ ***M1***

$= 4\left(p^2 - 36\right)$ ***A1***

$\left(p^2 - 36\right) > 0$ ***M1***

$p > 6$ ***A1***

OR

construction of a right angle triangle ***(M1)***

$12 \sin 30^\circ = 6$ ***M1(A1)***

hence for two triangles $p > 6$ ***R1***

THEN

$p < 12$ ***A1***

$144 - p^2 > 0$ to ensure two positive solutions or valid geometric argument ***R1***

$\therefore\ 6 < \mathrm{p} < 12$ ***A1***

METHOD 2

diagram showing two triangles ***(M1)***

$12 \sin 30^\circ = 6$ ***M1A1***

one right angled triangle when $p = 6$ ***(A1)***

$\therefore p > 6$ for two triangles ***R1***

$p < 12$ for two triangles ***A1***

$6 < p < 12$ ***A1***

[7 marks]

20 – (MAT/20_HL_Winter_2017_Q5) – *Trigonometry*

attempt to use tan, or sine rule, in triangle BXN or BXS ***(M1)***

$\mathrm{NX} = 80 \tan 55^\circ \left(= \dfrac{80}{\tan 35^\circ} = 114.25 \right)$ ***(A1)***

$\mathrm{SX} = 80 \tan 65^\circ \left(= \dfrac{80}{\tan 25^\circ} = 171.56 \right)$ ***(A1)***

Attempt to use cosine rule ***M1***

$\mathrm{SN}^2 = 171.56^2 + 114.25^2 - 2 \times 171.56 \times 114.25 \cos 70^\circ$ ***(A1)***

$\mathrm{SN} = 171(\mathrm{m})$ ***A1***

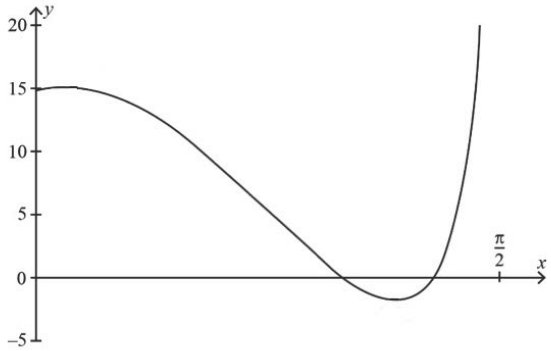
Note: Award final ***A1*** only if the correct answer has been given to 3 significant figures.

[6 marks]

21 – (MAT/20_HL_Winter_2017_Q11) – *Trigonometry, Differentiation*

(a) (i) $f'(x) = 4\sin x\cos x + 14\cos 2x + \sec^2 x$ (or equivalent) ***(M1)A1***

(ii)



A1A1A1A1

Note: Award ***A1*** for correct behaviour at $x = 0$, ***A1*** for correct domain and correct behaviour for $x \to \frac{\pi}{2}$, ***A1*** for two clear intersections with x-axis and minimum point, ***A1*** for clear maximum point.

(iii) $x = 0.0736$ ***A1***

$x = 1.13$ ***A1***

[8 marks]

(b) (i) attempt to write $\sin x$ in terms of u only ***(M1)***

$\sin x = \dfrac{u}{\sqrt{1+u^2}}$ ***A1***

(ii) $\cos x = \dfrac{1}{\sqrt{1+u^2}}$ ***(A1)***

attempt to use $\sin 2x = 2\sin x\cos x \left(= 2\dfrac{u}{\sqrt{1+u^2}}\dfrac{1}{\sqrt{1+u^2}} \right)$ ***(M1)***

$\sin 2x = \dfrac{2u}{1+u^2}$ ***A1***

(iii) $2\sin^2 x + 7\sin 2x + \tan x - 9 = 0$

$\dfrac{2u^2}{1+u^2} + \dfrac{14u}{1+u^2} + u - 9\ (= 0)$ ***M1***

$\dfrac{2u^2 + 14u + u\left(1+u^2\right) - 9\left(1+u^2\right)}{1+u^2} = 0$ (or equivalent) ***A1***

$u^3 - 7u^2 + 15u - 9 = 0$ ***AG***

[7 marks]

(c) $u = 1$ or $u = 3$ ***(M1)***

$x = \arctan(1)$ ***A1***

$x = \arctan(3)$ ***A1***

Note: Only accept answers given the required form.

[3 marks]

22 – (MAT/21_HL_Summer_2018_Q3) – *Trigonometry*

$\tan(x+\pi) = \tan x \left(= \dfrac{\sin x}{\cos x} \right)$ ***(M1)A1***

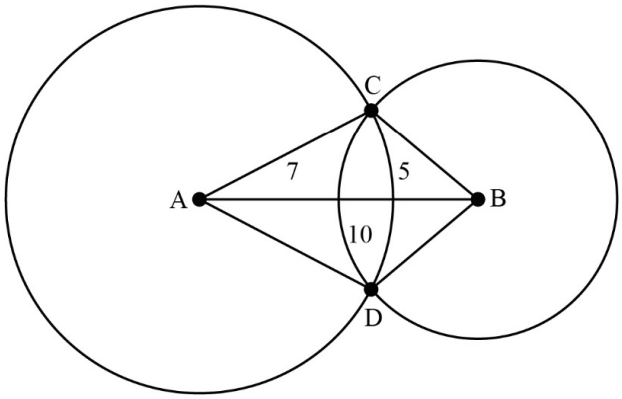
$\cos\left(x - \dfrac{\pi}{2}\right) = \sin x$ ***(M1)A1***

Note: The two ***M1***'s can be awarded for observation or for expanding.

$\tan\left(x+\pi\right)\cos\left(x - \dfrac{\pi}{2}\right) = \dfrac{\sin^2 x}{\cos x}$ ***A1***

[5 marks]

23 – (MAT/20_HL_Winter_2018_Q7) – *Trigonometry*



use of cosine rule ***(M1)***

$\mathrm{C\hat{A}B} = \arccos\left(\dfrac{49+100-25}{2\times 7\times 10}\right) = 0.48276\ldots(=27.660...^{\circ})$ ***(A1)***

$\mathrm{C\hat{B}A} = \arccos\left(\dfrac{25+100-49}{2\times 5\times 10}\right) = 0.70748\ldots(=40.535...^{\circ})$ ***(A1)***

attempt to subtract triangle area from sector area ***(M1)***

area $= \dfrac{1}{2}\times 49\left(2\mathrm{C\hat{A}B} - \sin 2\mathrm{C\hat{A}B}\right) + \dfrac{1}{2}\times 25\left(2\mathrm{C\hat{B}A} - \sin 2\mathrm{C\hat{B}A}\right)$

$= 3.5079... + 5.3385...$ ***(A1)***

Note: Award this ***A1*** for either of these two values.

$= 8.85\left(\mathrm{km}^2\right)$ ***A1***

Note: Accept all answers that round to 8.8 or 8.9.

[6 marks]

24 – (MAT/20_HL_Winter_2018_Q11) – *Trigonometry*

(a) **METHOD 1**

use of tan ***(M1)***

$\tan\theta_p = \dfrac{1}{p}$ ***(A1)***

$\theta_p = \arctan\left(\dfrac{1}{p}\right)$ ***A1***

METHOD 2

$\mathrm{AP} = \sqrt{p^2+1}$ ***(A1)***

use of sin, cos, sine rule or cosine rule using the correct length of AP ***(M1)***

$\theta_p = \arcsin\left(\dfrac{1}{\sqrt{p^2+1}}\right)$ or $\theta_p = \arccos\left(\dfrac{p}{\sqrt{p^2+1}}\right)$ ***A1***

[3 marks]

(b) $\mathrm{QR} = 1 \Rightarrow r = q + 1$ ***(A1)***

Note: This may be seen anywhere.

$\tan\theta_p = \tan\left(\theta_q + \theta_r\right)$

attempt to use compound angle formula for tan ***M1***

$\tan\theta_p = \dfrac{\tan\theta_q + \tan\theta_r}{1 - \tan\theta_q \tan\theta_r}$ ***(A1)***

$\dfrac{1}{p} = \dfrac{\dfrac{1}{q} + \dfrac{1}{r}}{1 - \left(\dfrac{1}{q}\right)\left(\dfrac{1}{r}\right)}$ ***(M1)***

$\dfrac{1}{p} = \dfrac{\dfrac{1}{q} + \dfrac{1}{q+1}}{1 - \left(\dfrac{1}{q}\right)\left(\dfrac{1}{q+1}\right)}$ or $p = \dfrac{1 - \left(\dfrac{1}{q}\right)\left(\dfrac{1}{q+1}\right)}{\dfrac{1}{q} + \dfrac{1}{q+1}}$ ***A1***

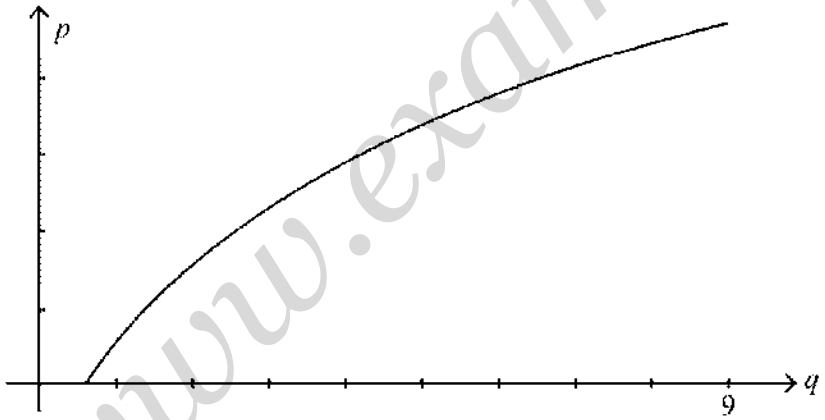
$\dfrac{1}{p} = \dfrac{q + q + 1}{q(q+1) - 1}$ ***M1***

Note: Award ***M1*** for multiplying top and bottom by $q(q+1)$.

$p = \dfrac{q^2 + q - 1}{2q + 1}$ ***AG***

[6 marks]

(c)



increasing function with positive q-intercept ***A1***

Note: Accept curves which extend beyond the domain shown above.

$(0.618 <)\ q < 9$ ***(A1)***

$\Rightarrow$ range is $(0 <)\ p < 4.68$ ***(A1)***

$0 < p < 4.68$ ***A1***

[4 marks]

25 – (MAT/22_HL_Summer_2019_Q1) – *Trigonometry*

attempt to apply cosine rule ***M1***

$$\cos A = \frac{5^2 + 11^2 - 14^2}{2 \times 5 \times 11} = -0.4545...$$

$\Rightarrow A = 117.03569...$

$\Rightarrow A = 117.0$ ***A1***

attempt to apply sine rule or cosine rule: ***M1***

$$\frac{\sin 117.03569...}{14} = \frac{\sin B}{11}$$

$\Rightarrow B = 44.4153...$

$\Rightarrow B = 44.4$ ***A1***

$C = 180 - A - B$

$C = 18.5$ ***A1***

Note: Candidates may attempt to find angles in any order of their choosing.

[5 marks]

26 – (MAT/21_HL_Summer_2019_Q4) – *Trigonometry*

(a) $f(x) \geq 3$ ***A1***

[1 mark]

(b) $x = \sec y + 2$ ***(M1)***

Note: Exchange of variables can take place at any point.

$\cos y = \frac{1}{x-2}$ ***(A1)***

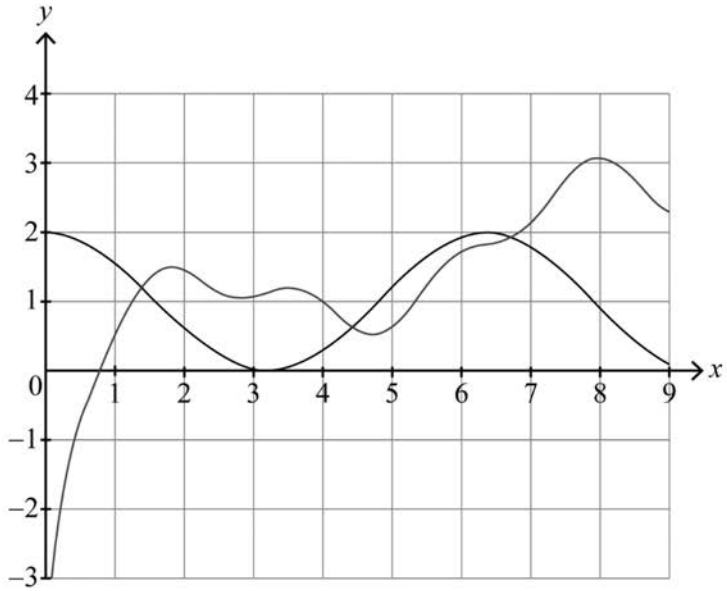
$f^{-1}(x) = \arccos\left(\frac{1}{x-2}\right),\ x \geq 3$ ***A1A1***

Note: Allow follow through from (a) for last ***A1*** mark which is independent of earlier marks in (b).

[4 marks]

27 – (MAT/22_HL_Summer_2019_Q4) – *Trigonometry*

(a)



A1A1

Note: Award ***A1*** for each correct curve, showing all local max & mins.

Note: Award ***A0A0*** for the curves drawn in degrees.

[2 marks]

(b) $x = 1.35, 4.35, 6.64$ ***(M1)***

Note: Award ***M1*** for attempt to find points of intersections between two curves.

$0 < x < 1.35$ ***A1***

Note: Accept $x < 1.35$.

$4.35 < x < 6.64$ ***A1A1***

Note: Award ***A1*** for correct endpoints, ***A1*** for correct inequalities.

Note: Award ***M1FTA1FTA0FTA0FT*** for $0 < x < 7.31$.

Note: Accept $x < 7.31$.

[4 marks]

28 – (MAT/22_HL_Summer_2019_Q5) – *Trigonometry*

(a) **METHOD 1**

$\text{LHS} = \dfrac{1+\sin 2x}{\cos 2x} = \dfrac{1+2\sin x\cos x}{\cos^2 x-\sin^2 x}$ ***M1***

$= \dfrac{\left(\cos^2 x+\sin^2 x\right)+2\sin x\cos x}{\cos^2 x-\sin^2 x}$ ***M1***

$= \dfrac{(\cos x+\sin x)^2}{(\cos x+\sin x)(\cos x-\sin x)}$ ***A1***

$= \dfrac{\cos x+\sin x}{\cos x-\sin x}$

$= \dfrac{\dfrac{\cos x}{\cos x}+\dfrac{\sin x}{\cos x}}{\dfrac{\cos x}{\cos x}-\dfrac{\sin x}{\cos x}}$ ***A1***

$= \dfrac{1+\tan x}{1-\tan x}$ ***AG***

Note: Candidates may start with RHS, apply MS in reverse.

[4 marks]

METHOD 2

$\text{LHS} = \dfrac{1+\sin 2x}{\cos 2x} = \dfrac{1+2\sin x\cos x}{\cos^2 x-\sin^2 x}$ ***M1***

dividing numerator and denominator by $\cos^2 x$ ***M1***

$= \dfrac{\sec^2 x+2\tan x}{1-\tan^2 x}$

$= \dfrac{1+\tan^2 x+2\tan x}{1-\tan^2 x}$ ***A1***

$= \dfrac{(\tan x+1)^2}{(1-\tan x)(1+\tan x)}$ ***A1***

$= \dfrac{1+\tan x}{1-\tan x}$ ***AG***

Note: Candidates may start with RHS; apply MS in reverse.

[4 marks]

(b) valid attempt to solve $\dfrac{1+\tan x}{1-\tan x} = \sqrt{3}$ ***(M1)***

$\tan x = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

$x = 0.262\left(=\dfrac{\pi}{12}\right),\ x = 3.40\left(=\dfrac{13\pi}{12}\right)$ ***A1***

Note: Award ***M1A0*** if only one correct solution is given.

[2 marks]

29 – (MAT/21_HL_Summer_2019_Q10) – *Trigonometry*

(a) $3, -3$ ***A1A1***

[2 marks]

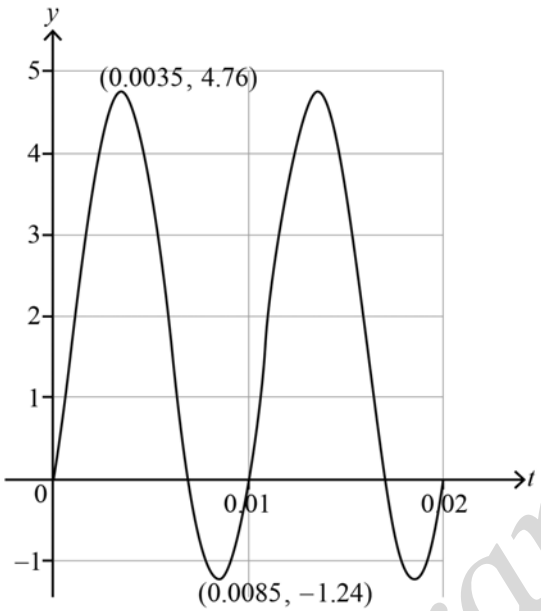
(b) stretch parallel to the y-axis (with x-axis invariant), scale factor $\frac{2}{3}$ ***A1***

translation of $\begin{pmatrix} -0.003 \\ 0 \end{pmatrix}$ (shift to the left by 0.003) ***A1***

Note: Can be done in either order.

[2 marks]

(c)



correct shape over correct domain with correct endpoints ***A1***
first maximum at $(0.0035, 4.76)$ ***A1***
first minimum at $(0.0085, -1.24)$ ***A1***

[3 marks]

(d) $p \geq 3$ between $t = 0.0016762$ and 0.0053238 and $t = 0.011676$ and 0.015324 ***(M1)(A1)***

Note: Award ***M1A1*** for either interval.

$= 0.00730$ ***A1***

[3 marks]

(e) $p_{av} = \frac{1}{0.007}\int_0^{0.007} 6\sin(100\pi t)\sin\left(100\pi(t+0.003)\right)\mathrm{d}t$ ***(M1)***

$= 2.87$ ***A1***

[2 marks]

(f) in each cycle the area under the t axis is smaller than area above the t axis ***R1***
the curve begins with the positive part of the cycle ***R1***
[2 marks]

(g) $a = \frac{4.76-(-1.24)}{2}$ ***(M1)***
$a = 3.00$ ***A1***

$d = \frac{4.76+(-1.24)}{2}$
$d = 1.76$ ***A1***
$b = \frac{2\pi}{0.01}$
$b = 628 (= 200\pi)$ ***A1***
$c = 0.0035 - \frac{0.01}{4}$ ***(M1)***
$c = 0.00100$ ***A1***
[6 marks]

(b) **METHOD 1**

let the geometric series be u_1, $u_1 r$, $u_1 r^2$
$(u_1 r)^3 = 8$ ***M1***
$u_1 r = 2$ ***A1***
hence one of the roots is $\log_2 2 = 1$ ***R1***

METHOD 2

$\frac{b}{a} = \frac{c}{b}$
$b^2 = ac \Rightarrow b^3 = abc = 8$ ***M1***
$b = 2$ ***A1***
hence one of the roots is $\log_2 2 = 1$ ***R1***
[3 marks]

(c) **METHOD 1**
product of the roots is $r_1 \times r_2 \times 1 \times d\mathrm{i} \times -d\mathrm{i} = -8d^2$ ***(M1)(A1)***
$r_1 \times r_2 = -8$ ***A1***
sum of the roots is $r_1 + r_2 + 1 + d\mathrm{i} + -d\mathrm{i} = 3$ ***(M1)(A1)***
$r_1 + r_2 = 2$ ***A1***
solving simultaneously ***(M1)***
$r_1 = -2$, $r_2 = 4$ ***A1A1***

30 – (MAT/20_HL_Winter_2019_Q9) – *Trigonometry*

(a) (i) $A(7.47, 2.28)$ and $B(43.4, -2.45)$ ***A1A1A1A1***

(ii) maximum speed is $2.45\ (\text{m s}^{-1})$ ***A1***

[5 marks]

(b) (i) $v = 0 \Rightarrow t_1 = 25.1\,(\text{s})$ ***(M1)A1***

(ii) $\int_0^{t_1} v\,\mathrm{d}t$ ***(M1)***

$= 41.0\,(\text{m})$ ***A1***

(iii) $a = \frac{\mathrm{d}v}{\mathrm{d}t}$ at $t = t_1 = 25.1$ ***(M1)***

$a = -0.200\ (\text{m s}^{-2})$ ***A1***

Note: Accept $a = -0.2$.

[6 marks]

(c) attempt to integrate between 0 and 30 ***(M1)***

Note: An unsupported answer of 38.6 can imply integrating from 0 to 30.

EITHER

$\int_0^{30} |v|\,\mathrm{d}t$ ***(A1)***

OR

$41.0 - \int_{t_1}^{30} v\,\mathrm{d}t$ ***(A1)***

THEN

$= 43.3\,(\text{m})$ ***A1***

[3 marks]

31 – (MAT/20_HL_Winter_2020_Q1) – *Radian, Trigonometry*

attempt to find $\mathrm{A\hat{O}B}$ by right-angled trigonometry or the cosine rule (***M1***)

EITHER

$\mathrm{A\hat{O}B} = 2\arcsin\left(\frac{5.5}{15}\right)$ ***A1***

OR

$\mathrm{A\hat{O}B} = \arccos\left(\frac{15^2 + 15^2 - 11^2}{2\times 15\times 15}\right)$ ***A1***

THEN

$= 0.750847...\ (= 43.0204...^\circ)$

Note: Award ***(M1)A1*** for correct calculation of $\mathrm{A\hat{O}B}$ or $\frac{1}{2}\mathrm{A\hat{O}B}$

shaded area = area of sector- area of triangle $\left(= \frac{1}{2}r^2(\theta - \sin\theta)\right)$ (***M1***)

$= \frac{1}{2}\times 15^2 \times (0.750847... - \sin 0.750847...)$ (***A1***)

$= 7.72$ (cm^2) ***A1***

[5 marks]

32 – (MAT/20_HL_Winter_2020_Q3) – *Trigonometry, Integration*

(a) the principal axis is $\frac{5+(-1)}{2}(=2)$

so $p = 2$ ***A1***

the amplitude is $\frac{5-(-1)}{2}(=3)$

so $q = 3$ ***A1***

EITHER

one period is $2\left(-\frac{3\pi}{4}-\left(-\frac{9\pi}{4}\right)\right)$ ***(M1)***

$=3\pi$

$\Rightarrow \frac{2\pi}{r}=3\pi$

OR

Substituting a point eg $-1=2+\sin\left(-\frac{3\pi}{4}r\right)$

$\sin\left(-\frac{3\pi}{4}r\right)=-1\Rightarrow -\frac{3\pi}{4}r=\ldots-\frac{5\pi}{2},-\frac{\pi}{2},\frac{3\pi}{2},\ldots$

Choice of correct solution $-\frac{3\pi}{4}r=-\frac{\pi}{2}$ ***(M1)***

THEN

$\Rightarrow r=\frac{2}{3}$ ***A1***

$\left(\Rightarrow y=2+3\sin\left(\frac{2x}{3}\right)\right)$

Note: q and r can be both given as negatives for full marks

[4 marks]

(b) roots are $x=-1.09459\ldots, x=-3.617797\ldots$ ***(A1)***

$\int_{-3.617797\ldots}^{-1.09459\ldots}\left(2+3\sin\left(\frac{2x}{3}\right)\right)\mathrm{d}x$ ***(M1)***

$=-1.66\left(=-1.66179\ldots\right)$ ***(A1)***

so area $=1.66$ (units^2) ***A1***

[4 marks]

33 – (MathAA/21_HL_Summer_2021_Q3) – *Trigonometry*

Amplitude is $\frac{110}{2} = 55$ ***(A1)***

$a = -55$ ***A1***

$c = 65$ ***A1***

$\frac{2\pi}{b} = 20$ OR $-55\cos(20b) + 65 = 10$ ***(M1)***

$b = \frac{\pi}{10} (= 0.314)$ ***A1***

34 – (MathAA/20_HL_Winter_2021_Q2) – *Trigonometry*

EITHER

attempt to use cosine rule ***(M1)***

$12^2 + AB^2 - 2 \times 12 \times \cos 25^\circ \times AB = 7^2$ OR $AB^2 - 21.7513... AB + 95 = 0$ ***(A1)***

at least one correct value for AB ***(A1)***

$AB = 6.05068...$ OR $AB = 15.7007...$

using their smaller value for AB to find minimum perimeter ***(M1)***

$12 + 7 + 6.05068...$

OR

attempt to use sine rule ***(M1)***

$\frac{\sin B}{12} = \frac{\sin 25^\circ}{7}$ OR $\sin B = 0.724488...$ OR $\hat{B} = 133.573...^\circ$ OR $\hat{B} = 46.4263...^\circ$ ***(A1)***

at least one correct value for $\hat{C}$

$\hat{C} = 21.4263...^\circ$ OR $\hat{C} = 108.573...^\circ$ ***(A1)***

using their acute value for $\hat{C}$ to find minimum perimeter ***(M1)***

$12 + 7 + \sqrt{12^2 + 7^2 - 2 \times 12 \times 7 \cos 21.4263...}$ OR $12 + 7 + \frac{7 \sin 21.4263...^\circ}{\sin 25^\circ}$

THEN

$25.0506...$

minimum perimeter = 25.1. ***A1***

Total [5 marks]

35 – (MathAA/20_HL_Winter_2021_Q9) – *Trigonometry*

(a) $12 = \frac{2\pi}{b}$ OR $b = \frac{2\pi}{12}$ ***A1***

$b = \frac{\pi}{6}$ ***AG***

[1 mark]

(b) $a = \frac{6.8 - 2.2}{2}$ OR $a = \frac{\max - \min}{2}$ ***(M1)***

$= 2.3$ (m) ***A1***

[2 marks]

(c) $d = \frac{6.8 + 2.2}{2}$ OR $d = \frac{\max + \min}{2}$ ***(M1)***

$= 4.5$ (m) ***A1***

[2 marks]

(d) **METHOD 1**

substituting $t = 4.5$ and $H = 6.8$ for example into their equation for H ***(A1)***

$$6.8 = 2.3\sin\left(\frac{\pi}{6}(4.5 - c)\right) + 4.5$$

attempt to solve their equation ***(M1)***

$c = 1.5$ ***A1***

METHOD 2

using horizontal translation of $\frac{12}{4}$ ***(M1)***

$4.5 - c = 3$ ***(A1)***

$c = 1.5$ ***A1***

METHOD 3

$$H'(t) = (2.3)\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}(t - c)\right)$$ ***(A1)***

attempts to solve their $H'(4.5) = 0$ for c ***(M1)***

$$(2.3)\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}(4.5 - c)\right) = 0$$

$c = 1.5$ ***A1***

[3 marks]

(e) attempt to find H when $t = 12$ or $t = 0$, graphically or algebraically ***(M1)***

$H = 2.87365...$

$H = 2.87\,(\text{m})$ ***A1***

[2 marks]

(f) attempt to solve $5 = 2.3\sin\left(\frac{\pi}{6}(t-1.5)\right)+4.5$ ***(M1)***

times are $t = 1.91852...$ and $t = 7.08147...$, $(t = 13.9185..., t = 19.0814...)$ ***(A1)***

total time is $2\times(7.081...-1.919...)$

$10.3258...$

$= 10.3$ (hours) ***A1***

Note: Accept 10.

[3 marks]

(g) **METHOD 1**

substitutes $t = \frac{11}{3}$ and $H = 6.8$ into their equation for H and attempts to solve for c ***(M1)***

$6.8 = 2.3\sin\left(\frac{\pi}{6}\left(\frac{11}{3}-c\right)\right)+4.5 \Rightarrow c = \frac{2}{3}$

$H(t) = 2.3\sin\left(\frac{\pi}{6}\left(t-\frac{2}{3}\right)\right)+4.5$ ***A1***

METHOD 2

uses their horizontal translation $\left(\frac{12}{4} = 3\right)$ ***(M1)***

$\frac{11}{3} - c = 3 \Rightarrow c = \frac{2}{3}$

$H(t) = 2.3\sin\left(\frac{\pi}{6}\left(t-\frac{2}{3}\right)\right)+4.5$ ***A1***

[2 marks]
Total [15 marks]

36 – (MathAA/21_HL_Summer_2022_Q2) – *Trigonometry*

(a) valid approach to find area of segment by finding area of sector – area of triangle ***(M1)***

$\frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$

$\frac{1}{2}(2)^2\theta - \frac{1}{2}(2)^2\sin\theta$ ***(A1)***

area = $2\theta - 2\sin\theta$ ***A1***

[3 marks]

(b) EITHER

area of logo = area of rectangle – area of segments ***(M1)***

$5\times 4 - 2\times(2\theta - 2\sin\theta) = 13.4$ ***(A1)***

OR

area of one segment = $\frac{20-13.4}{2}$ $(=3.3)$ ***(M1)***

$2\theta - 2\sin\theta = 3.3$ ***(A1)***

THEN

$\theta = 2.35672...$

$\theta = 2.36$ (do not accept an answer in degrees) ***A1***

Note: Award ***(M1)(A1)A0*** if there is more than one solution.
Award ***(M1)(A1FT)A0*** if the candidate works in degrees and obtains a final answer of $135.030\ldots$

[3 marks]

37 – (MathAA/22_HL_Summer_2022_Q1) – *Trigonometry*

(a) **EITHER**

uses the cosine rule ***(M1)***

$AB^2 = 5^2 + 5^2 - 2\times5\times5\times\cos 1.9$ ***(A1)***

OR

uses right-angled trigonometry ***(M1)***

$\frac{AB}{2} = \sin 0.95$ ***(A1)***

OR

uses the sine rule ***(M1)***

$\alpha = \frac{1}{2}(\pi - 1.9)(= 0.6207...)$

$\frac{AB}{\sin 1.9} = \frac{5}{\sin 0.6207...}$ ***(A1)***

THEN

$AB = 8.1341...$

$AB = 8.13$ (m) ***A1***

[3 marks]

(b) let the shaded area be A

METHOD 1

Attempt at finding reflex angle ***(M1)***

$\mathrm{A\hat{O}B} = 2\pi - 1.9 \ (= 4.3831...)$

substitution into area formula ***(A1)***

$A = \frac{1}{2} \times 5^2 \times 4.3831...$ OR $\left(\frac{2\pi - 1.9}{2\pi}\right) \times \pi\left(5^2\right)$

$= 54.7898...$

$= 54.8$ (m²) ***A1***

METHOD 2

let the area of the circle be A_C and the area of the unshaded sector be A_U

$A = A_C - A_U$ ***(M1)***

$A = \pi \times 5^2 - \frac{1}{2} \times 5^2 \times 1.9 \ (= 78.5398... - 23.75)$ ***(A1)***

$= 54.7898...$

$= 54.8$ (m²) ***A1***

[3 marks]

38 – (MathAA/22_HL_Summer_2022_Q10) – *Trigonometry, Differentiation*

(a) (i) 32 (cm) ***A1***

(ii) $h_A(0) = \sin(6) + 27$ ***(M1)***

$= 26.7205...$

$= 26.7$ (cm) ***A1***

[3 marks]

(b) attempts to solve $h_A(t) = h_B(t)$ for t ***(M1)***

$t = 4.0074..., 4.7034..., 5.88332...$

$t = 4.01, 4.70, 5.88$ (weeks) ***A2***

[3 marks]

(c) $h_A(t) - h_B(t) = \sin(2t+6) + t - 5$ ***A1***

EITHER

for $t > 6$, $t - 5 > 1$ ***A1***

and as $\sin(2t+6) \geq -1 \Rightarrow h_A(t) - h_B(t) > 0$ ***R1***

OR

the minimum value of $\sin(2t+6) = -1$ ***R1***

so for $t > 6$, $h_A(t) - h_B(t) = t - 6 > 0$ ***A1***

THEN

hence for $t > 6$, Plant A was always taller than Plant B ***AG***

[3 marks]

(d) recognises that $h_A'(t)$ and $h_B'(t)$ are required ***(M1)***

attempts to solve $h_A'(t) = h_B'(t)$ for t ***(M1)***

$t = 1.18879...$ and $2.23598...$ OR $4.33038...$ and $5.37758...$ OR $7.47197...$ and $8.51917...$ ***(A1)***

Note: Award full marks for $t = \frac{4\pi}{3} - 3, \frac{5\pi}{3} - 3, \left(\frac{7\pi}{3} - 3, \frac{8\pi}{3} - 3, \frac{10\pi}{3} - 3, \frac{11\pi}{3} - 3\right)$.
Award subsequent marks for correct use of these exact values.

$1.18879... < t < 2.23598...$ OR $4.33038... < t < 5.37758...$ OR
$7.47197... < t < 8.51917...$ ***(A1)***

attempts to calculate the total amount of time ***(M1)***

$3(2.2359... - 1.1887...)$ $\left(= 3\left(\left(\frac{5\pi}{3} - 3\right) - \left(\frac{4\pi}{3} - 3\right)\right)\right)$

$= 3.14$ $(= \pi)$ (weeks) ***A1***

[6 marks]

39 – (MathAA/20_HL_Winter_2022_Q9) – *Trigonometry*

(a) (i) attempt to use the cosine rule ***(M1)***

$\mathrm{AC} = \sqrt{2^2 + 4^2 - 2(2)(4)\cos\alpha}\left(= \sqrt{20 - 16\cos\alpha} = 2\sqrt{5 - 4\cos\alpha}\right)$ ***A1***

(ii) $\mathrm{AC} = \sqrt{6^2 + 8^2 - 2(6)(8)\cos\beta}\left(= \sqrt{100 - 96\cos\beta} = 2\sqrt{25 - 24\cos\beta}\right)$ ***A1***

(iii) $5 - 4\cos\alpha = 25 - 24\cos\beta$

$\alpha = \arccos(6\cos\beta - 5)$ ***A1***

[4 marks]

(b) attempt to find the sum of two triangle areas using $A = \frac{1}{2}ab\sin C$ ***(M1)***

Note: Do not award this ***M1*** if the triangle is assumed to be right angled.

$\text{Area} = \frac{1}{2}(8)\sin\alpha + \frac{1}{2}(48)\sin\beta$ ***(A1)***

attempt to express the area in terms of one variable only ***(M1)***

$= 4\sqrt{1 - (6\cos\beta - 5)^2} + 24\sin\beta$ or $4\sin(\arccos(6\cos\beta - 5)) + 24\sin\beta$ OR

$4\sin\alpha + 24\sqrt{1 - \left(\frac{5 + \cos\alpha}{6}\right)^2}$ or $4\sin\alpha + 24\sin\left(\arccos\left(\frac{5 + \cos\alpha}{6}\right)\right)$

Max area = 19.5959…

= 19.6 ***A1***

[4 marks]

1 - (MAT/20_HL_Winter_2012_Q3) - *Matrices*

$\det A = 3\ln x - 2\ln(5-x)$

A singular $\Rightarrow \det A = 0$

attempt to solve $3\ln x - 2\ln(5-x) = 0$ (*eg* graph sketch)

$x = 2.0547...$

$x = 2.05$ (3sf)

2 - (MAT/21_HL_Summer_2013_Q2) - *Matrices*

METHOD 1

determinant $= 0$

$k(-2-16)-(0-12)+2(0+3)=0$

$-18k+18=0$

$k=1$

METHOD 2

writes in the form

$$\begin{pmatrix} k & 1 & 2 & 4 \\ 0 & -1 & 4 & 5 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$ (or attempts to solve simultaneous equations)

Having two 0's in first column (obtaining two equations in the same two variables)

$$\begin{pmatrix} k & 1 & 2 & 4 \\ 0 & -1 & 4 & 5 \\ 0 & 0 & 18k-18 & 21k-27 \end{pmatrix}$$ (or isolating one variable)

Note: The *A1* is to be awarded for the $18k-18$. The final column may not be seen.

$k=1$

3 - (MAT/22_HL_Summer_2013_Q2) - *Matrices*

(a) attempting to express the system in matrix form

$$\begin{pmatrix} 0.1 & -1.7 & 0.9 \\ -2.4 & 0.3 & 3.2 \\ 2.5 & 0.6 & -3.7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4.4 \\ 1.2 \\ 0.8 \end{pmatrix}$$

Note: Award ***M1A1*** for a correct augmented matrix.

(b) either direct GDC use, attempting elimination or using an inverse matrix.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2.4 \\ 1.6 \\ -1.6 \end{pmatrix} \text{ (correct to 2sf) or } \begin{pmatrix} -2.40 \\ 1.61 \\ -1.57 \end{pmatrix} \text{ (correct to 3sf) or } \begin{pmatrix} -\frac{932}{389} \\ \frac{628}{389} \\ -\frac{612}{389} \end{pmatrix} \text{ (exact)}$$

4 - (MAT/20_HL_Winter_2013_Q1) - *Matrices*

$\boldsymbol{AX} = \boldsymbol{B}$

EITHER

$\Rightarrow \boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{B}$

OR

attempting row reduction:

$$eg \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & -1 & -6 \\ 0 & -2 & 0 & -1 \end{array}\right)$$

THEN

$$\Rightarrow \boldsymbol{X} = \begin{pmatrix} -\frac{7}{2} \\ \frac{1}{2} \\ 5 \end{pmatrix}$$

5 - (MAT/21_HL_Summer_2014_Q4) - *Matrices*

(a) $\begin{cases} x+2y-z=2 \\ 2x+y+z=1 \\ -x+4y+az=4 \end{cases}$

$\rightarrow \begin{cases} x+2y-z=2 \\ -3y+3z=-3 \\ 6y+(a-1)z=6 \end{cases}$

$\rightarrow \begin{cases} x+2y-z=2 \\ -3y+3z=-3 \\ (a+5)z=0 \end{cases}$

(or equivalent)
if not a unique solution then $a=-5$

Note: The first ***M1*** is for attempting to eliminate a variable, the first ***A1*** for obtaining two expression in just two variables (plus a), and the second ***A1*** for obtaining an expression in just a and one other variable

(b) if $a=-5$ there are an infinite number of solutions as last equation always true
and if $a \neq -5$ there is a unique solution
hence always a solution

6 – (MAT/22_HL_Summer_2015_Q7) – *Matrices*

(a)
$$\begin{aligned} 2x+y+6z&=0\\ 4x+3y+14z&=4\\ 2x-2y+(\alpha-2)z&=\beta-12 \end{aligned}$$

attempt at row reduction

eg R_2-2R_1 and R_3-R_1

$$\begin{aligned} 2x+y+6z&=0\\ y+2z&=4\\ -3y+(\alpha-8)z&=\beta-12 \end{aligned}$$

eg R_3+3R_2

$$\begin{aligned} 2x+y+6z&=0\\ y+2z&=4\\ (\alpha-2)z&=\beta \end{aligned}$$

(i) no solutions if $\alpha=2$, $\beta\neq 0$

(ii) one solution if $\alpha\neq 2$

(iii) infinite solutions if $\alpha=2$, $\beta=0$

Note: Accept alternative methods e.g. determinant of a matrix

Note: Award ***A1A1A0*** if all three consistent with their reduced form, ***A1A0A0*** if two or one answer consistent with their reduced form.

(b) $y+2z=4\Rightarrow y=4-2z$
$2x=-y-6z=2z-4-6z=-4z-4\Rightarrow x=-2z-2$

therefore Cartesian equation is $\dfrac{x+2}{-2}=\dfrac{y-4}{-2}=\dfrac{z}{1}$ or equivalent

1 – (MAT/21_HL_Summer_2012_Q4) – *Vectors - Lines - Planes*

point on line is $x=\frac{-1-5\lambda}{5}$, $y=\frac{9+5\lambda}{5}$, $z=\lambda$ or similar

Note:	Accept use of point on the line or elimination of one of the variables using the equations of the planes

$$\frac{-1-5\lambda}{5}-\frac{9+5\lambda}{5}+2\lambda=k$$

Note:	Award ***M1A1*** if coordinates of point and equation of a plane is used to obtain linear equation in k or equations of the line are used in combination with equation obtained by elimination to get linear equation in k.

$k=-2$

2 – (MAT/22_HL_Summer_2012_Q11) – *Vectors - Lines - Planes*

(a) in augmented matrix form $\begin{vmatrix} 1 & -3 & 1 & 3 \\ 1 & 5 & -2 & 1 \\ 0 & 16 & -6 & k \end{vmatrix}$

attempt to find a line of zeros

$$r_2 - r_1 \begin{vmatrix} 1 & -3 & 1 & 3 \\ 0 & 8 & -3 & -2 \\ 0 & 16 & -6 & k \end{vmatrix}$$

$$r_3 - 2r_2 \begin{vmatrix} 1 & -3 & 1 & 3 \\ 0 & 8 & -3 & -2 \\ 0 & 0 & 0 & k+4 \end{vmatrix}$$

there is an infinite number of solutions when $k = -4$

there is no solution when

$k \neq -4, (k \in \mathbb{R})$

Note: Approaches other than using the augmented matrix are acceptable.

(b) using $\begin{vmatrix} 1 & -3 & 1 & 3 \\ 0 & 8 & -3 & -2 \\ 0 & 0 & 0 & k+4 \end{vmatrix}$ and letting $z = \lambda$

$8y - 3\lambda = -2$

$\Rightarrow y = \dfrac{3\lambda - 2}{8}$

$x - 3y + z = 3$

$\Rightarrow x - \left(\dfrac{9\lambda - 6}{8}\right) + \lambda = 3$

$\Rightarrow 8x - 9\lambda + 6 + 8\lambda = 24$

$\Rightarrow x = \dfrac{18 + \lambda}{8}$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{18}{8} \\ -\frac{2}{8} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{8} \\ \frac{3}{8} \\ 1 \end{pmatrix}$$

$$\boldsymbol{r} = \begin{pmatrix} \frac{9}{4} \\ -\frac{1}{4} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix}$$

(c) recognition that $\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$ is parallel to the plane

direction normal of the plane is given by $\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 3 & 8 \\ 3 & -2 & 0 \end{vmatrix}$

$= 16\boldsymbol{i} + 24\boldsymbol{j} - 11\boldsymbol{k}$

Cartesian equation of the plane is given by $16x + 24y - 11z = d$ and a point which fits this equation is $(1, 2, 0)$

$\Rightarrow 16 + 48 = d$

$d = 64$

hence Cartesian equation of plane is $16x + 24y - 11z = 64$

Note: Accept alternative methods using dot product.

(d) the plane crosses the z-axis when $x = y = 0$

coordinates of P are $\left(0, 0, -\frac{64}{11}\right)$

(e) recognition that the angle between the line and the direction normal is given by:

$\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \sqcap \begin{pmatrix} 16 \\ 24 \\ -11 \end{pmatrix} = \sqrt{29}\sqrt{953}\cos\theta$ where θ is the angle between the line and the normal vector

$\Rightarrow 122 = \sqrt{29}\sqrt{953}\cos\theta$

$\Rightarrow \theta = 42.8^\circ$ (0.747 radians)

hence the angle between the line and the plane is $90^\circ - 42.8^\circ = 47.2^\circ$ (0.824 radians)

3 – (MAT/21_HL_Summer_2012_Q13) – *Vectors - Lines - Planes*

(a) $\overrightarrow{AB} = \begin{pmatrix} 0 \\ 6 \\ -6 \end{pmatrix} \Rightarrow AB = \sqrt{72}$ ***A1***

$\overrightarrow{AC} = \begin{pmatrix} -6 \\ 0 \\ -6 \end{pmatrix} \Rightarrow AC = \sqrt{72}$ ***A1***

so they are the same ***AG***

$\overrightarrow{AB} \cdot \overrightarrow{AC} = 36 = \left(\sqrt{72}\right)\left(\sqrt{72}\right)\cos\theta$ ***(M1)***

$\cos\theta = \dfrac{36}{\left(\sqrt{72}\right)\left(\sqrt{72}\right)} = \dfrac{1}{2} \Rightarrow \theta = 60^\circ$ ***A1AG***

Note: Award ***M1A1*** if candidates find BC and claim that triangle ABC is equilateral.

(b) **METHOD 1**

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 6 & -6 \\ -6 & 0 & -6 \end{vmatrix} = -36\boldsymbol{i} + 36\boldsymbol{j} + 36\boldsymbol{k}$ ***(M1)A1***

equation of plane is $x - y - z = k$ ***(M1)***

goes through A, B or C $\Rightarrow x - y - z = 2$ ***A1***

METHOD 2

$x + by + cz = d$ (or similar) ***M1***

$5 - 2b + 5c = d$

$5 + 4b - c = d$ ***A1***

$-1 - 2b - c = d$

solving simultaneously ***M1***

$b = -1,\ c = -1,\ d = 2$

so $x - y - z = 2$ ***A1***

(c) (i) midpoint is $(5, 1, 2)$, so equation of Π_1 is $y - z = -1$ ***A1A1***

(ii) midpoint is $(2, -2, 2)$, so equation of Π_2 is $x + z = 4$ ***A1A1***

Note: In each part, award ***A1*** for midpoint and ***A1*** for the equation of the plane.

(d) **EITHER**

solving the two equations above

$$L: \boldsymbol{r} = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

OR

L has the direction of the vector product of the normal vectors to the planes Π_1 and Π_2

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{vmatrix} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$$

(or its opposite)

THEN

direction is $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ as required

(e) D is of the form $(4-\lambda, -1+\lambda, \lambda)$

$(1+\lambda)^2 + (-1-\lambda)^2 + (5-\lambda)^2 = 72$

$3\lambda^2 - 6\lambda - 45 = 0$

$\lambda = 5$ or $\lambda = -3$

D(–1, 4, 5)

(f) **EITHER**

G is of the form $(4-\lambda, -1+\lambda, \lambda)$ and $\mathrm{DG} = \mathrm{AG}$, BG or CG

e.g. $(1+\lambda)^2 + (-1-\lambda)^2 + (5-\lambda)^2 = (5-\lambda)^2 + (5-\lambda)^2 + (5-\lambda)^2$

$(1+\lambda)^2 = (5-\lambda)^2$

$\lambda = 2$

$\mathrm{G}(2, 1, 2)$

OR

G is the centre of mass (barycentre) of the regular tetrahedron ABCD

$$\mathrm{G}\left(\frac{5+5+(-1)+(-1)}{4}, \frac{-2+4+(-2)+4}{4}, \frac{5+(-1)+(-1)+5}{4}\right)$$

THEN

Note: the following part is independent of previous work and candidates may use ***AG*** to answer it (here it is possible to award ***M0M0A0A1M1A1)***

$$\overrightarrow{\mathrm{GD}} = \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix} \text{ and } \overrightarrow{\mathrm{GA}} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$$

$$\cos\theta = \frac{-9}{\left(3\sqrt{3}\right)\left(3\sqrt{3}\right)} = -\frac{1}{3} \Rightarrow \theta = 109^\circ \text{ (or 1.91 radians)}$$

4 – (MAT/20_HL_Winter_2012_Q13) – *Vectors - Lines - Planes*

(a) $\boldsymbol{n} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$ and $\boldsymbol{m} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$

$\cos\theta = \dfrac{\boldsymbol{n} \bullet \boldsymbol{m}}{|\boldsymbol{n}||\boldsymbol{m}|}$

$\cos\theta = \dfrac{2+2+3}{\sqrt{1+4+9}\sqrt{4+1+1}} = \dfrac{7}{\sqrt{14}\sqrt{6}}$

$\theta = 40.2^{\circ}$ (0.702 rad)

(b) **METHOD 1**

eliminate z from $x - 2y - 3z = 2$ and $2x - y - z = k$

$5x - y = 3k - 2 \Rightarrow x = \dfrac{y - (2 - 3k)}{5}$

eliminate y from $x - 2y - 3z = 2$ and $2x - y - z = k$

$3x + z = 2k - 2 \Rightarrow x = \dfrac{z - (2k - 2)}{-3}$

$x = t$, $y = (2 - 3k) + 5t$ and $z = (2k - 2) - 3t$

$\boldsymbol{r} = \begin{pmatrix} 0 \\ 2 - 3k \\ 2k - 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$

METHOD 2

$\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \\ 3 \end{pmatrix} \Rightarrow$ direction is $\begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$

Let $x = 0$

$0 - 2y - 3z = 2$ and $2 \times 0 - y - z = k$

solve simultaneously

$y = 2 - 3k$ and $z = 2k - 2$

therefore $\mathbf{r} = \begin{pmatrix} 0 \\ 2 - 3k \\ 2k - 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$

METHOD 3

substitute $x = t$, $y = (2-3k)+5t$ and $z = (2k-2)-3t$ into π_1 and π_2

for $\pi_1 : t - 2(2-3k+5t) - 3(2k-2-3t) = 2$

for $\pi_2 : 2t - (2-3k+5t) - (2k-2-3t) = k$

the planes have a unique line of intersection

therefore the line is $\boldsymbol{r} = \begin{pmatrix} 0 \\ 2-3k \\ 2k-2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$

(c) $5 - t = (2-3k+5t)+3 = 2-2(2k-2-3t)$

Note: Award ***M1A1*** if candidates use vector or parametric equations of L_2

$$eg\ \begin{pmatrix} 0 \\ 2-3k \\ 2k-2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix} + s\begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} \text{ or } \Rightarrow \begin{cases} t = 5-2s \\ 2-3k+5t = -3+2s \\ 2k-2-3t = 1+s \end{cases}$$

solve simultaneously

$k = 2,\ t = 1\ (s = 2)$

intersection point $(1, 1, -1)$

(d) $\vec{l_2} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

$$\vec{l_1} \times \vec{l_2} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 5 & -3 \\ 2 & -2 & 1 \end{vmatrix} = \begin{pmatrix} -1 \\ -7 \\ -12 \end{pmatrix}$$

$$\boldsymbol{r} \cdot \begin{pmatrix} 1 \\ 7 \\ 12 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 7 \\ 12 \end{pmatrix}$$

$x + 7y + 12z = -4$

(e) Let θ be the angle between the lines $\vec{l_1} = \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$ and $\vec{l_2} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

$\cos\theta = \dfrac{|2-10-3|}{\sqrt{35}\sqrt{9}} \Rightarrow \theta = 0.902334\ldots\,(51.699\ldots^{\circ})$

as the triangle XYZ has a right angle at Y,

$\text{XZ} = a \Rightarrow \text{YZ} = a\sin\theta$ and $\text{XY} = a\cos\theta$

$\text{area} = 3 \Rightarrow \dfrac{a^2\sin\theta\cos\theta}{2} = 3$

$a = 3.5122\ldots$

$\text{perimeter} = a + a\sin\theta + a\cos\theta = 8.44537\ldots = 8.45$

5 – (MAT/21_HL_Summer_2013_Q11) – *Vectors - Lines - Planes*

(a) $\overrightarrow{PQ} = \begin{pmatrix} 8 \\ 6 \\ 4 \end{pmatrix}$

equation of line: $\boldsymbol{r} = \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 8 \\ 6 \\ 4 \end{pmatrix}$ (or equivalent)

Note: Award ***M1A0*** if $\boldsymbol{r} =$ is omitted.

(b) **METHOD 1**

$x: \quad -4 + 5s = -3 + 8t$
$y: \quad 2s = -1 + 6t$
$z: \quad 4 = 2 + 4t$

solving any two simultaneously
$t = 0.5,\ s = 1$ (or equivalent)
verification that these values give R when substituted into **both** equations
(or that the three equations are consistent and that one gives R)

METHOD 2

$(1, 2, 4)$ is given by $t = 0.5$ for L_1 and $s = 1$ for L_2
because $(1, 2, 4)$ is on both lines it is the point of intersection of the two lines

(c) $\begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = 26 = \sqrt{29} \times \sqrt{29}\cos\theta$

$\cos\theta = \dfrac{26}{29}$

$\theta = 0.459$ or 26.3°

(d) $\overrightarrow{RP} = \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \\ -2 \end{pmatrix}, \left|\overrightarrow{RP}\right| = \sqrt{29}$

Note: This could also be obtained from $\left| 0.5 \begin{pmatrix} 8 \\ 6 \\ 4 \end{pmatrix} \right|$

EITHER

$\overrightarrow{RS_1} = \begin{pmatrix} -4 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ -2 \\ 0 \end{pmatrix}, \quad \left|\overrightarrow{RS_1}\right| = \sqrt{29}$

$\therefore \overrightarrow{OS_2} = \overrightarrow{OS_1} + 2\overrightarrow{S_1R} = \begin{pmatrix} -4 \\ 0 \\ 4 \end{pmatrix} + 2\begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}$

$\left(\text{or } \overrightarrow{OS_2} = \overrightarrow{OR} + \overrightarrow{S_1R} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} \right)$

$= \begin{pmatrix} 6 \\ 4 \\ 4 \end{pmatrix}$

S_2 is $(6, 4, 4)$

OR

$\begin{pmatrix} -4+5s \\ 2s \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 5s-5 \\ 2s-2 \\ 0 \end{pmatrix}$

$(5s-5)^2 + (2s-2)^2 = 29$

$29s^2 - 58s + 29 = 29$

$s(s-2) = 0, \; s = 0, 2$

$(6, 4, 4)$ (and $(-4, 0, 4)$)

(e) **EITHER**

midpoint of $[PS_1]$ is $M(-3.5, -0.5, 3)$

$$\overrightarrow{RM} = \begin{pmatrix} -4.5 \\ -2.5 \\ -1 \end{pmatrix}$$

OR

$$\overrightarrow{RS_1} = \begin{pmatrix} -5 \\ -2 \\ 0 \end{pmatrix}$$

the direction of the line is $\overrightarrow{RS_1} + \overrightarrow{RP}$

$$\begin{pmatrix} -5 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} -4 \\ -3 \\ -2 \end{pmatrix} = \begin{pmatrix} -9 \\ -5 \\ -2 \end{pmatrix}$$

THEN

the equation of the line is:

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 9 \\ 5 \\ 2 \end{pmatrix} \text{ or equivalent}$$

6 – (MAT/20_HL_Winter_2013_Q7) – *Vectors - Lines - Planes, Trigonometry*

(a) attempting to form $(3\cos\theta + 6)(\cos\theta - 2) + 7(1 + \sin\theta) = 0$

$3\cos^2\theta - 12 + 7\sin\theta + 7 = 0$

$3\left(1 - \sin^2\theta\right) + 7\sin\theta - 5 = 0$

$3\sin^2\theta - 7\sin\theta + 2 = 0$

(b) attempting to solve algebraically (including substitution) or graphically for $\sin\theta$

$\sin\theta = \frac{1}{3}$

$\theta = 0.340 \quad (= 19.5^\circ)$

7 – (MAT/20_HL_Winter_2013_Q9) – *Vectors - Lines - Planes*

(a) **METHOD 1**

for P on L_1, $\overrightarrow{OP} = \begin{pmatrix} -5-\lambda \\ -3+2\lambda \\ 2+2\lambda \end{pmatrix}$

require $\begin{pmatrix} -5-\lambda \\ -3+2\lambda \\ 2+2\lambda \end{pmatrix} . \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = 0$

$5+\lambda-6+4\lambda+4+4\lambda = 0$ (or equivalent)

$\lambda = -\frac{1}{3}$

$\therefore \overrightarrow{OP} = \begin{pmatrix} -\frac{14}{3} \\ -\frac{11}{3} \\ \frac{4}{3} \end{pmatrix}$

$L_2 : \boldsymbol{r} = \mu \begin{pmatrix} -14 \\ -11 \\ 4 \end{pmatrix}$

Note: Do not award the final ***A1*** if $\boldsymbol{r} =$ is not seen.

METHOD 2

Calculating either $\left|\overrightarrow{OP}\right|$ or $\left|\overrightarrow{OP}\right|^2$ *eg*

$\left|\overrightarrow{OP}\right| = \sqrt{(-5-\lambda)^2 + (-3+2\lambda)^2 + (2+2\lambda)^2}$

$= \sqrt{9\lambda^2 + 6\lambda + 38}$

Solving either $\frac{d}{d\lambda}\left(\left|\overrightarrow{OP}\right|\right) = 0$ or $\frac{d}{d\lambda}\left(\left|\overrightarrow{OP}\right|^2\right) = 0$ for λ.

$\lambda = -\frac{1}{3}$

$\overrightarrow{OP} = \begin{pmatrix} -\frac{14}{3} \\ -\frac{11}{3} \\ \frac{4}{3} \end{pmatrix}$

$L_2 : \boldsymbol{r} = \mu \begin{pmatrix} -14 \\ -11 \\ 4 \end{pmatrix}$

(b) **METHOD 1**

$$\left|\overrightarrow{OP}\right| = \sqrt{\left(-\frac{14}{3}\right)^2 + \left(-\frac{11}{3}\right)^2 + \left(\frac{4}{3}\right)^2}$$

$$= 6.08 \quad \left(= \sqrt{37}\right)$$

METHOD 2

$$\text{shortest distance} = \frac{\left|\begin{pmatrix}-1\\2\\2\end{pmatrix} \times \begin{pmatrix}-5\\-3\\2\end{pmatrix}\right|}{\left|\begin{pmatrix}-1\\2\\2\end{pmatrix}\right|}$$

$$= \frac{\left|10\boldsymbol{i} + 8\boldsymbol{j} + 13\boldsymbol{k}\right|}{\sqrt{1+4+4}}$$

$$= 6.08 \left(= \sqrt{37}\right)$$

8 – (MAT/20_HL_Winter_2014_Q1) – *Vectors - Lines - Planes*

$$\boldsymbol{n}_1 = \begin{pmatrix}4\\2\\-1\end{pmatrix} \text{ and } \boldsymbol{n}_2 = \begin{pmatrix}1\\3\\3\end{pmatrix}$$

$$\text{use of } \cos\theta = \frac{\boldsymbol{n}_1 \bullet \boldsymbol{n}_2}{\left|\boldsymbol{n}_1\right|\left|\boldsymbol{n}_2\right|}$$

$$\cos\theta = \frac{7}{\sqrt{21}\sqrt{19}} \quad \left(= \frac{7}{\sqrt{399}}\right)$$

Note: Award *A1* for a correct numerator and *A1* for a correct denominator.

$$\theta = 69^{\circ}$$

9 – (MAT/20_HL_Winter_2014_Q5) – *Vectors - Lines - Planes*

(a) attempting to find a normal to π eg $\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 8 \\ 11 \\ 6 \end{pmatrix}$

$$\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 8 \\ 11 \\ 6 \end{pmatrix} = 17 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

$$\boldsymbol{r} \bullet \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 12 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

$2x - 2y + z = 4$ (or equivalent)

(b) $l_3 : \boldsymbol{r} = \begin{pmatrix} 4 \\ 0 \\ 8 \end{pmatrix} + t \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, t \in \mathbb{R}$

attempting to solve $\begin{pmatrix} 4+2t \\ -2t \\ 8+t \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 4$ for t ie $9t + 16 = 4$ for t

$t = -\dfrac{4}{3}$

$\left(\dfrac{4}{3}, \dfrac{8}{3}, \dfrac{20}{3}\right)$

10 – (MAT/22_HL_Summer_2015_Q5) – *Vectors - Lines - Planes*

(a) $\boldsymbol{u}\times\boldsymbol{v}=\begin{pmatrix}2(0)+2b\\-2a-1(0)\\b-2a\end{pmatrix}=\begin{pmatrix}2b\\-2a\\b-2a\end{pmatrix}$

$\begin{pmatrix}2b\\-2a\\b-2a\end{pmatrix}=\begin{pmatrix}4\\b\\c\end{pmatrix}$

$\Rightarrow a=-1,\ b=2,\ c=4$

Note: Award ***A1*** for two correct.

(b) $\boldsymbol{n}=\begin{pmatrix}4\\2\\4\end{pmatrix}$

$\Rightarrow 4x+2y+4z=0\ (2x+y+2z=0)$

11 – (MAT/21_HL_Summer_2015_Q8) – *Vectors - Lines - Planes*

(a) require $\begin{pmatrix}4\\\lambda\\10\end{pmatrix}=s\begin{pmatrix}2\\3\\5\end{pmatrix}$

$\Rightarrow 4=2s\Rightarrow s=2\Rightarrow\lambda=6$

Note: Accept cross product solution.

(b) require $\boldsymbol{v}\bullet\boldsymbol{w}=2\times4+3\times\lambda+5\times10=0\Rightarrow 3\lambda=-58\Rightarrow\lambda=\frac{-58}{3}(-19.3)$

(c) $\boldsymbol{v}\bullet\boldsymbol{w}=2\times4+3\times\lambda+5\times10=\sqrt{2^2+3^2+5^2}\times\sqrt{4^2+\lambda^2+10^2}\times\cos10^\circ$

$58+3\lambda=\sqrt{38}\times\sqrt{116+\lambda^2}\times\cos10^\circ$

$\lambda=3.73$ or 8.76

12 – (MAT/22_HL_Summer_2015_Q13) – *Vectors - Lines - Planes*

(a) L_1 and L_2 are not parallel, since $\begin{pmatrix}-1\\1\\2\end{pmatrix} \neq k\begin{pmatrix}2\\1\\6\end{pmatrix}$ ***R1***

if they meet, then $1-\lambda = 1+2\mu$ and $2+\lambda = 2+\mu$ ***M1***
solving simultaneously $\Rightarrow \lambda = \mu = 0$ ***A1***
$2+2\lambda = 4+6\mu \Rightarrow 2 \neq 4$ contradiction, ***R1***
so lines are skew ***AG***

Note: Do not award the second ***R1*** if their values of parameters are incorrect.

(b) $\begin{pmatrix}-1\\1\\2\end{pmatrix} \bullet \begin{pmatrix}2\\1\\6\end{pmatrix} (=11) = \sqrt{6}\sqrt{41}\cos\theta$ ***M1A1***

$\cos\theta = \dfrac{11}{\sqrt{246}}$ ***(A1)***

$\theta = 45.5^\circ$ (0.794 radians) ***A1***

(c) (i) $\begin{pmatrix}-1\\1\\2\end{pmatrix} \times \begin{pmatrix}2\\1\\6\end{pmatrix} = \begin{pmatrix}6-2\\4+6\\-1-2\end{pmatrix}$ ***(M1)***

$= \begin{pmatrix}4\\10\\-3\end{pmatrix} = 4\boldsymbol{i} + 10\boldsymbol{j} - 3\boldsymbol{k}$ ***A1***

(ii) **METHOD 1**

let P be the intersection of L_1 and L_3

let Q be the intersection of L_2 and L_3

$$\overrightarrow{\mathrm{OP}} = \begin{pmatrix} 1-\lambda \\ 2+\lambda \\ 2+2\lambda \end{pmatrix} \quad \overrightarrow{\mathrm{OQ}} = \begin{pmatrix} 1+2\mu \\ 2+\mu \\ 4+6\mu \end{pmatrix}$$

therefore $\overrightarrow{\mathrm{PQ}} = \overrightarrow{\mathrm{OQ}} - \overrightarrow{\mathrm{OP}} = \begin{pmatrix} 2\mu+\lambda \\ \mu-\lambda \\ 2+6\mu-2\lambda \end{pmatrix}$

$$\begin{pmatrix} 2\mu+\lambda \\ \mu-\lambda \\ 2+6\mu-2\lambda \end{pmatrix} = t\begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix}$$

$$2\mu+\lambda-4t=0$$
$$\mu-\lambda-10t=0$$
$$6\mu-2\lambda+3t=-2$$

solving simultaneously

$$\lambda = \frac{32}{125}(0.256),\ \mu = -\frac{28}{125}(-0.224)$$

Note: Award ***A1*** for either correct λ or μ.

EITHER

therefore $\overrightarrow{\mathrm{OP}} = \begin{pmatrix} 1-\lambda \\ 2+\lambda \\ 2+2\lambda \end{pmatrix} = \begin{pmatrix} \frac{93}{125} \\ \frac{282}{125} \\ \frac{314}{125} \end{pmatrix} = \begin{pmatrix} 0.744 \\ 2.256 \\ 2.512 \end{pmatrix}$

therefore $L_3 : \boldsymbol{r}_3 = \begin{pmatrix} 0.744 \\ 2.256 \\ 2.512 \end{pmatrix} + \alpha\begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix}$

OR

therefore $\overrightarrow{OQ} = \begin{pmatrix} 1+2\mu \\ 2+\mu \\ 4+6\mu \end{pmatrix} = \begin{pmatrix} \frac{69}{125} \\ \frac{222}{125} \\ \frac{332}{125} \end{pmatrix} = \begin{pmatrix} 0.552 \\ 1.776 \\ 2.656 \end{pmatrix}$ ***A1***

therefore $L_3 : \boldsymbol{r}_3 = \begin{pmatrix} 0.552 \\ 1.776 \\ 2.656 \end{pmatrix} + \alpha \begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix}$ ***A1***

Note: Allow position vector(s) to be expressed in decimal or fractional form.

METHOD 2

$$L_3 : r_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + t \begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix}$$

forming two equations as intersections with L_1 and L_2

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + t_1 \begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + t_2 \begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}$$ ***M1A1A1***

Note: Only award ***M1A1A1*** if two different parameters t_1, t_2 used.

attempting to solve simultaneously ***M1***

$\lambda = \frac{32}{125}(0.256),\ \mu = -\frac{28}{125}(-0.224)$ ***A1***

EITHER

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0.552 \\ 1.776 \\ 2.656 \end{pmatrix}$$

therefore $L_3 : \boldsymbol{r}_3 = \begin{pmatrix} 0.552 \\ 1.776 \\ 2.656 \end{pmatrix} + t \begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix}$

OR

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0.744 \\ 2.256 \\ 2.512 \end{pmatrix}$$

therefore $L_3 : \boldsymbol{r}_3 = \begin{pmatrix} 0.744 \\ 2.256 \\ 2.512 \end{pmatrix} + t \begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix}$

13 – (MAT/20_HL_Winter_2015_Q2) – *Vectors - Lines - Planes*

using technology and/or by elimination (*eg* ref on GDC)

$x = 1.89\left(= \frac{17}{9}\right),\ y = 1.67\left(= \frac{5}{3}\right),\ z = -2.22\left(= \frac{-20}{9}\right)$

14 – (MAT/20_HL_Winter_2015_Q10) – *Vectors - Lines - Planes*

(a) $\boldsymbol{a} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ ***A1***

$\boldsymbol{b} = \frac{1}{3}\left(\begin{pmatrix} 4 \\ 16 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \end{pmatrix}\right) = \begin{pmatrix} \frac{5}{3} \\ 4 \end{pmatrix}$ ***(M1)A1***

(b) **METHOD 1**

Roderick must signal in a direction vector perpendicular to Ed's path. ***(M1)***

the equation of the signal is $\boldsymbol{s} = \begin{pmatrix} 11 \\ 9 \end{pmatrix} + \lambda\begin{pmatrix} -12 \\ 5 \end{pmatrix}$ (or equivalent) ***A1***

$\begin{pmatrix} -1 \\ 4 \end{pmatrix} + \frac{t}{3}\begin{pmatrix} 5 \\ 12 \end{pmatrix} = \begin{pmatrix} 11 \\ 9 \end{pmatrix} + \lambda\begin{pmatrix} -12 \\ 5 \end{pmatrix}$ ***M1***

$\frac{5}{3}t + 12\lambda = 12$ and $4t - 5\lambda = 5$ ***M1***

$t = 2.13\left(= \frac{360}{169}\right)$ ***A1***

METHOD 2

$\begin{pmatrix} 5 \\ 12 \end{pmatrix} \cdot \left(\begin{pmatrix} 11 \\ 9 \end{pmatrix} - \begin{pmatrix} -1 + \frac{5}{3}t \\ 4 + 4t \end{pmatrix}\right) = 0$ (or equivalent) ***M1A1A1***

Note: Award the ***M1*** for an attempt at a scalar product equated to zero, ***A1*** for the first factor and ***A1*** for the complete second factor.

attempting to solve for t ***(M1)***

$t = 2.13\left(= \frac{360}{169}\right)$ ***A1***

METHOD 3

$x=\sqrt{\left(12-\frac{5t}{3}\right)^2+(5-4t)^2}$ (or equivalent) $\left(x^2=\left(12-\frac{5t}{3}\right)^2+(5-4t)^2\right)$ ***M1A1A1***

Note: Award ***M1*** for use of Pythagoras' theorem, ***A1*** for $\left(12-\frac{5t}{3}\right)^2$ and ***A1*** for $(5-4t)^2$.

attempting (graphically or analytically) to find *t* such that $\frac{\mathrm{d}x}{\mathrm{d}t}=0\left(\frac{\mathrm{d}\left(x^2\right)}{\mathrm{d}t}=0\right)$ ***(M1)***

$t=2.13\left(=\frac{360}{169}\right)$ ***A1***

METHOD 4

$\cos\theta=\frac{\begin{pmatrix}12\\5\end{pmatrix}\cdot\begin{pmatrix}5\\12\end{pmatrix}}{\left|\begin{pmatrix}12\\5\end{pmatrix}\right|\left|\begin{pmatrix}5\\12\end{pmatrix}\right|}=\frac{120}{169}$ ***M1A1***

Note: Award ***M1*** for attempting to calculate the scalar product.

$\frac{120}{13}=\frac{t}{3}\left|\begin{pmatrix}5\\12\end{pmatrix}\right|$ (or equivalent) ***(A1)***

attempting to solve for *t* ***(M1)***

$t=2.13\left(=\frac{360}{169}\right)$ ***A1***

15 – (MAT/21_HL_Summer_2016_Q1) – *Vectors - Lines - Planes*

(a) $\overrightarrow{OA} \times \overrightarrow{OB} = \begin{pmatrix} 4 \\ -4 \\ -2 \end{pmatrix}$ ***(M1)A1***

Note: ***M1A0*** can be awarded for attempt at a correct method **shown**, or correct method implied by the digits 4, 4, 2 found in the correct order.

[2 marks]

(b) area $= \frac{1}{2}\sqrt{4^2 + 4^2 + 2^2} = 3$ ***M1A1***

16 – (MAT/22_HL_Summer_2016_Q9) – *Vectors - Lines - Planes*

(a) (i) **METHOD 1**

$$\begin{aligned} \left|\overrightarrow{OC}\right|^2 &= \overrightarrow{OC} \cdot \overrightarrow{OC} \\ &= (\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{b}) \\ &= \boldsymbol{a} \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{a} + \boldsymbol{b} \cdot \boldsymbol{b} \\ &= |\boldsymbol{a}|^2 + 2\boldsymbol{a} \cdot \boldsymbol{b} + |\boldsymbol{b}|^2 \end{aligned}$$

METHOD 2

$|\overrightarrow{OC}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2 - 2|\overrightarrow{OA}||\overrightarrow{OB}|\cos(O\hat{A}C)$ ***A1***

$|\overrightarrow{OA}||\overrightarrow{OB}|\cos(O\hat{A}C) = -(\boldsymbol{a}\cdot\boldsymbol{b})$ ***A1***

$|\overrightarrow{OC}|^2 = |\boldsymbol{a}|^2 + 2\boldsymbol{a}\cdot\boldsymbol{b} + |\boldsymbol{b}|^2$ ***AG***

(ii) **METHOD 1**

$|\overrightarrow{AB}|^2 = \overrightarrow{AB}\cdot\overrightarrow{AB}$

$= (\boldsymbol{b}-\boldsymbol{a})\cdot(\boldsymbol{b}-\boldsymbol{a})$ ***A1***

$= \boldsymbol{b}\cdot\boldsymbol{b} - \boldsymbol{b}\cdot\boldsymbol{a} - \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\boldsymbol{a}$ ***A1***

$= |\boldsymbol{a}|^2 - 2\boldsymbol{a}\cdot\boldsymbol{b} + |\boldsymbol{b}|^2$ ***AG***

METHOD 2

$|\overrightarrow{AB}|^2 = |\overrightarrow{AC}|^2 + |\overrightarrow{BC}|^2 - 2|\overrightarrow{AC}||\overrightarrow{BC}|\cos(A\hat{C}B)$ ***A1***

$|\overrightarrow{AC}||\overrightarrow{BC}|\cos(A\hat{C}B) = \boldsymbol{a}\cdot\boldsymbol{b}$ ***A1***

$|\overrightarrow{AB}|^2 = |\boldsymbol{a}|^2 - 2\boldsymbol{a}\cdot\boldsymbol{b} + |\boldsymbol{b}|^2$ ***AG***

[4 marks]

(b) $|\overrightarrow{OC}| = |\overrightarrow{AB}| \Rightarrow |\overrightarrow{OC}|^2 = |\overrightarrow{AB}|^2 \Rightarrow |\boldsymbol{a}|^2 + 2\boldsymbol{a}\cdot\boldsymbol{b} + |\boldsymbol{b}|^2 = |\boldsymbol{a}|^2 - 2\boldsymbol{a}\cdot\boldsymbol{b} + |\boldsymbol{b}|^2$ ***R1(M1)***

Note: Award ***R1*** for $|\overrightarrow{OC}| = |\overrightarrow{AB}| \Rightarrow |\overrightarrow{OC}|^2 = |\overrightarrow{AB}|^2$ and ***(M1)*** for $|\boldsymbol{a}|^2 + 2\boldsymbol{a}\cdot\boldsymbol{b} + |\boldsymbol{b}|^2 = |\boldsymbol{a}|^2 - 2\boldsymbol{a}\cdot\boldsymbol{b} + |\boldsymbol{b}|^2$.

$\boldsymbol{a}\cdot\boldsymbol{b} = 0$ ***A1***

hence OACB is a rectangle ($\boldsymbol{a}$ and $\boldsymbol{b}$ both non-zero)
with adjacent sides at right angles ***R1***

Note: Award ***R1(M1)A0R1*** if the dot product has not been used.

17 – (MAT/20_HL_Winter_2016_Q2) – *Vectors - Lines - Planes*

$\boldsymbol{n}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\boldsymbol{n}_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ ***(A1)(A1)***

EITHER

$\theta = \arccos\left(\frac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}\right) \left(\cos\theta = \frac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}\right)$ ***(M1)***

$= \arccos\left(\frac{2+0-1}{\sqrt{3}\sqrt{5}}\right) \left(\cos\theta = \frac{2+0-1}{\sqrt{3}\sqrt{5}}\right)$ ***(A1)***

$= \arccos\left(\frac{1}{\sqrt{15}}\right) \left(\cos\theta = \frac{1}{\sqrt{15}}\right)$

OR

$\theta = \arcsin\left(\frac{|\boldsymbol{n}_1 \times \boldsymbol{n}_2|}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}\right) \left(\sin\theta = \frac{|\boldsymbol{n}_1 \times \boldsymbol{n}_2|}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}\right)$ ***(M1)***

$= \arcsin\left(\frac{\sqrt{14}}{\sqrt{3}\sqrt{5}}\right) \left(\sin\theta = \frac{\sqrt{14}}{\sqrt{3}\sqrt{5}}\right)$ ***(A1)***

$= \arcsin\left(\frac{\sqrt{14}}{\sqrt{15}}\right) \left(\sin\theta = \frac{\sqrt{14}}{\sqrt{15}}\right)$

THEN

$= 75.0^\circ$ (or 1.31) ***A1***

[5 marks]

18 – (MAT/21_HL_Summer_2017_Q7) – *Vectors - Lines - Planes*

METHOD 1

$\overrightarrow{AB} = \begin{pmatrix} -3 \\ -3 \\ 0 \end{pmatrix}$ ***(A1)***

$\begin{pmatrix} -3 \\ -3 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ ***M1A1***

$= \begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix}$ ***A1***

$x - y - z = k$ ***M1***

$k = 3$

equation of plane Π is $x - y - z = 3$ or equivalent ***A1***

METHOD 2

let plane Π be $ax + by + cz = d$

attempt to form one or more simultaneous equations: ***M1***

$a + 2b - c = 0$ (1) ***A1***

$6a + 2b + c = d$ (2)

$3a - b + c = d$ (3) ***A1***

Note: Award second ***A1*** for equations (2) and (3).

attempt to solve ***M1***

EITHER

using GDC gives $a = \frac{d}{3},\ b = -\frac{d}{3},\ c = -\frac{d}{3}$ ***(A1)***

equation of plane Π is $x - y - z = 3$ or equivalent ***A1***

OR

row reduction ***M1***

equation of plane Π is $x - y - z = 3$ or equivalent ***A1***

[6 marks]

19 – (MAT/22_HL_Summer_2017_Q7) – *Vectors - Lines - Planes*

METHOD 1

$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c}$

$(\boldsymbol{a} \times \boldsymbol{b}) - (\boldsymbol{b} \times \boldsymbol{c}) = 0$

$(\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{c} \times \boldsymbol{b}) = 0$ ***M1A1***

$(\boldsymbol{a} + \boldsymbol{c}) \times \boldsymbol{b} = 0$ ***A1***

$(\boldsymbol{a} + \boldsymbol{c})$ is parallel to $\boldsymbol{b} \Rightarrow \boldsymbol{a} + \boldsymbol{c} = s\boldsymbol{b}$ ***R1AG***

Note: Condone absence of arrows, underlining, or other otherwise "correct" vector notation throughout this question.

Note: Allow "is in the same direction to", for the final ***R*** mark.

METHOD 2

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c} \Rightarrow \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} = \begin{pmatrix} b_2c_3 - b_3c_2 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - b_2c_1 \end{pmatrix}$$ ***M1A1***

$$a_2b_3 - a_3b_2 = b_2c_3 - b_3c_2 \Rightarrow b_3(a_2 + c_2) = b_2(a_3 + c_3)$$
$$a_3b_1 - a_1b_3 = b_3c_1 - b_1c_3 \Rightarrow b_1(a_3 + c_3) = b_3(a_1 + c_1)$$
$$a_1b_2 - a_2b_1 = b_1c_2 - b_2c_1 \Rightarrow b_2(a_1 + c_1) = b_1(a_2 + c_2)$$

$$\frac{(a_1 + c_1)}{b_1} = \frac{(a_2 + c_2)}{b_2} = \frac{(a_3 + c_3)}{b_3} = s$$ ***A1***

$$\Rightarrow a_1 + c_1 = sb_1$$
$$\Rightarrow a_2 + c_2 = sb_2$$
$$\Rightarrow a_3 + c_3 = sb_3$$

$$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = s\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$ ***A1***

$\Rightarrow \boldsymbol{a} + \boldsymbol{c} = s\boldsymbol{b}$ ***AG***

[4 marks]

20 – (MAT/22_HL_Summer_2017_Q9) – *Vectors - Lines - Planes*

(a) $\overrightarrow{BC} = (\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}) - (2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) = -\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}$ ***(A1)***

$\boldsymbol{r} = (2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) + \lambda(-\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k})$

(or $\boldsymbol{r} = (\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}) + \lambda(-\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k})$ ***(M1)A1***

Note: Do not award ***A1*** unless $\boldsymbol{r} =$ or equivalent correct notation seen.

[3 marks]

(b) attempt to write in parametric form using two different parameters **AND** equate ***M1***

$2\mu = 2 - \lambda$

$\mu = -1 + 4\lambda$

$-2\mu = 2 + \lambda$ ***A1***

attempt to solve first pair of simultaneous equations for two parameters ***M1***

solving first two equations gives $\lambda = \frac{4}{9},\ \mu = \frac{7}{9}$ ***(A1)***

substitution of these two values in third equation ***(M1)***

since the values do not fit, the lines do not intersect ***R1***

Note: Candidates may note that adding the first and third equations immediately leads to a contradiction and hence they can immediately deduce that the lines do not intersect.

[6 marks]

(c) **METHOD 1**

plane is of the form $\boldsymbol{r} \bullet (2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) = d$ ***(A1)***

$d = (\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}) \bullet (2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) = -1$ ***(M1)***

hence Cartesian form of plane is $2x + y - 2z = -1$ ***A1***

METHOD 2

plane is of the form $2x + y - 2z = d$ ***(A1)***

substituting $(1, 3, 3)$ (to find gives $2 + 3 - 6 = -1$) ***(M1)***

hence Cartesian form of plane is $2x + y - 2z = -1$ ***A1***

[3 marks]

(d) **METHOD 1**

attempt scalar product of direction vector BC with normal to plane *M1*

$(-\boldsymbol{i}+4\boldsymbol{j}+\boldsymbol{k})\bullet(2\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k})=-2+4-2$

$=0$ *A1*

hence BC lies in Π_1 *AG*

METHOD 2

substitute eqn of line into plane *M1*

line $r=\begin{pmatrix}2\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}-1\\4\\1\end{pmatrix}$. Plane $\pi_1: 2x+y-2z=-1$

$2(2-\lambda)+(-1+4\lambda)-2(2+\lambda)$

$=-1$ *A1*

hence BC lies in Π_1 *AG*

Note: Candidates may also just substitute $2i-j+2k$ into the plane since they are told C lies on π_1.

Note: Do not award ***A1FT***.

[2 marks]

(e) **METHOD 1**

applying scalar product to $\overrightarrow{OA}$ and $\overrightarrow{OB}$ *M1*

$(2\boldsymbol{j}+\boldsymbol{k})\bullet(2\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k})=0$ *A1*

$(2\boldsymbol{j}+\boldsymbol{k})\bullet(2\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k})=0$ *A1*

METHOD 2

attempt to find cross product of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ *M1*

plane Π_2 has normal $\overrightarrow{OA}\times\overrightarrow{OB}=-8\boldsymbol{j}-4\boldsymbol{k}$ *A1*

since $-8\boldsymbol{j}-4\boldsymbol{k}=-4(2\boldsymbol{j}+\boldsymbol{k})$, $2\boldsymbol{j}+\boldsymbol{k}$ is perpendicular to the plane Π_2 *R1*

[3 marks]

(f) plane Π_3 has normal $\overrightarrow{OA}\times\overrightarrow{OC}=9\boldsymbol{i}-8\boldsymbol{j}+5\boldsymbol{k}$ *A1*

[1 mark]

(g) attempt to use dot product of normal vectors *(M1)*

$\cos\theta=\dfrac{(2\boldsymbol{j}+\boldsymbol{k})\bullet(9\boldsymbol{i}-8\boldsymbol{j}+5\boldsymbol{k})}{|2\boldsymbol{j}+\boldsymbol{k}||9\boldsymbol{i}-8\boldsymbol{j}+5\boldsymbol{k}|}$ *(M1)*

$=\dfrac{-11}{\sqrt{5}\sqrt{170}}(=-0.377...)$ *(A1)*

Note: Accept $\dfrac{11}{\sqrt{5}\sqrt{170}}$.

acute angle between planes $=67.8^\circ\left(=1.18^c\right)$ *A1*

[4 marks]

21 – (MAT/21_HL_Summer_2018_Q11) – *Vectors - Lines - Planes*

(a) $\boldsymbol{r}_A = \boldsymbol{r}_B$ ***(M1)***

$2 - t = -0.5t \Rightarrow t = 4$ ***A1***

checking $t = 4$ satisfies $4 + t = 3.2 + 1.2t$ and $-1 - 0.15t = -2 + 0.1t$ ***R1***

$\mathrm{P}(-2, 8, -1.6)$ ***A1***

Note: Do not award final ***A1*** if answer given as column vector.

[4 marks]

(b) (i) $0.9 \times \begin{pmatrix} -0.5 \\ 1.2 \\ 0.1 \end{pmatrix} = \begin{pmatrix} -0.45 \\ 1.08 \\ 0.09 \end{pmatrix}$ ***A1***

Note: Accept use of cross product equalling zero.

hence in the same direction ***AG***

(ii) $\begin{pmatrix} -0.45t \\ 3.2 + 1.08t \\ -2 + 0.09t \end{pmatrix} = \begin{pmatrix} -2 \\ 8 \\ -1.6 \end{pmatrix}$ ***M1***

Note: The ***M1*** can be awarded for any one of the resultant equations.

$\Rightarrow t = \dfrac{40}{9} = 4.44\ldots$ ***A1***

[3 marks]

(c) (i) $\boldsymbol{r}_A - \boldsymbol{r}_B = \begin{pmatrix} 2 - t \\ 4 + t \\ -1 - 0.15t \end{pmatrix} - \begin{pmatrix} -0.45t \\ 3.2 + 1.08t \\ -2 + 0.09t \end{pmatrix}$ ***(M1)(A1)***

$= \begin{pmatrix} 2 - 0.55t \\ 0.8 - 0.08t \\ 1 - 0.24t \end{pmatrix}$ ***(A1)***

Note: Accept $\boldsymbol{r}_B - \boldsymbol{r}_A$.

distance $D = \sqrt{(2 - 0.55t)^2 + (0.8 - 0.08t)^2 + (1 - 0.24t)^2}$ ***M1A1***

$(= \sqrt{8.64 - 2.688t + 0.317t^2})$

(ii) minimum when $\dfrac{\mathrm{d}D}{\mathrm{d}t} = 0$ ***(M1)***

$t = 3.83$ ***A1***

(iii) 0.511 (km) ***A1***

[8 marks]

22 – (MAT/22_HL_Summer_2019_Q11) – *Vectors - Lines - Planes*

(a) **METHOD 1**

for example

$\overrightarrow{PQ} = \begin{pmatrix} -1 \\ -5 \\ 8 \end{pmatrix}, \overrightarrow{PR} = \begin{pmatrix} 1 \\ -6 \\ 3 \end{pmatrix}$ ***A1A1***

$\overrightarrow{PQ} \times \overrightarrow{PR} = 33\boldsymbol{i} + 11\boldsymbol{j} + 11\boldsymbol{k}$ ***(M1)A1***

$\boldsymbol{rn} = \boldsymbol{an}$

$33x + 11y + 11z = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 33 \\ 11 \\ 11 \end{pmatrix} = 22$ ***(M1)***

$\Rightarrow 3x + y + z = 2$ or equivalent ***A1***

[6 marks]

METHOD 2

assume plane can be written as $ax + by + cz = 1$ ***M1***

substituting each set of coordinates gives the system of equations:

$a + 6b - 7c = 1$

$0a + b + c = 1$

$2a + 0b - 4c = 1$ ***A1***

solving by GDC ***(M1)***

$a = \frac{3}{2}, b = \frac{1}{2}, c = \frac{1}{2}$ ***A1A1A1***

$\Rightarrow \frac{3}{2}x + \frac{1}{2}y + \frac{1}{2}z = 1$ or equivalent

[6 marks]

(b) **METHOD 1**

substitution of equation of line into both equations of planes ***M1***

$3\left(\frac{5}{4} + \frac{\lambda}{2}\right) + \lambda + \left(-\frac{7}{4} - \frac{5\lambda}{2}\right) = 2$ ***A1***

$\left(\frac{5}{4} + \frac{\lambda}{2}\right) - 3\lambda - \left(-\frac{7}{4} - \frac{5\lambda}{2}\right) = 3$ ***A1***

[3 marks]

(c) normal to Π_3 is perpendicular to direction of L

$\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} = 0$ ***A1***

$\Rightarrow a + 2b - 5c = 0$ ***AG***

[1 mark]

(d) (i) substituting $\begin{pmatrix} \frac{5}{4} \\ 0 \\ -\frac{7}{4} \end{pmatrix}$ into Π_3: ***M1***

$\frac{5a}{4} - \frac{7c}{4} = 1$ ***A1***

$5a - 7c = 4$ ***AG***

(ii) attempt to find scalar products for Π_1 and Π_3, Π_2 and Π_3 and equating ***M1***

$\frac{3a+b+c}{\sqrt{11}\sqrt{a^2+b^2+c^2}} = \frac{a-3b-c}{\sqrt{11}\sqrt{a^2+b^2+c^2}}$ ***M1***

Note: Accept $3a+b+c = a-3b-c$.

$\Rightarrow a+2b+c=0$ ***A1***

attempt to solve $a+2b+c=0$, $a+2b-5c=0$, $5a-7c=4$ ***M1***

$\Rightarrow a = \frac{4}{5}, b = -\frac{2}{5}, c = 0$ ***A1***

hence equation is $\frac{4x}{5} \quad \frac{2y}{5} = 1$

for second equation:

$\frac{3a+b+c}{\sqrt{11}\sqrt{a^2+b^2+c^2}} = -\frac{a-3b-c}{\sqrt{11}\sqrt{a^2+b^2+c^2}}$ ***(M1)***

$\Rightarrow 2a-b=0$

attempt to solve $2a-b=0$, $a+2b-5c=0$, $5a-7c=4$

$\Rightarrow a=-2, b=-4, c=-2$ ***A1***

hence equation is $-2x-4y-2z=1$

[9 marks]

23 – (MAT/20_HL_Winter_2020_Q10) – *Vectors - Lines - Planes*

(a) $3(1-3\lambda)-(2-\lambda)+(-2+4\lambda)=-13$ ***(M1)***

$\lambda = 3$ ***(A1)***

$\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} + 3\begin{pmatrix} -3 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -8 \\ -1 \\ 10 \end{pmatrix}$ ***(M1)***

so $\mathrm{P}(-8,-1,10)$ ***A1***

Note: Do not award the final ***A1*** if a vector given instead of coordinates

[4 marks]

(b) **METHOD 1**

$\boldsymbol{r} = \mu\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$

substituting into equation of the plane ***M1***

$9\mu + \mu + \mu = -13$

$\mu = -\frac{13}{11}(= -1.18...)$ ***A1***

distance $= \frac{13\sqrt{3^2+(-1)^2+1^2}}{11}$ ***(M1)***

$= \frac{13}{\sqrt{11}}\left(= \frac{13\sqrt{11}}{11} = 3.92\right)$ ***A1***

METHOD 2

choice of any point on the plane, eg $(-8, -1, 10)$ to use in distance formula ***(M1)***

so distance $= \dfrac{\begin{pmatrix}-8\\-1\\10\end{pmatrix}.\begin{pmatrix}-3\\1\\-1\end{pmatrix}}{\sqrt{(-3)^2+1^2+(-1)^2}}$ ***A1A1***

Note: Award ***A1*** for numerator, ***A1*** for denominator.

$= \dfrac{24-1-10}{\sqrt{11}}$

$= \dfrac{13}{\sqrt{11}}\left(= \dfrac{13\sqrt{11}}{11} = 3.92\right)$ ***A1***

[4 marks]

(c) **EITHER**

identify two vectors ***(A1)***

eg, $\begin{pmatrix}1\\2\\-2\end{pmatrix}$ and $\begin{pmatrix}-3\\-1\\4\end{pmatrix}$

$\boldsymbol{n} = \begin{pmatrix}1\\2\\-2\end{pmatrix}\times\begin{pmatrix}-3\\-1\\4\end{pmatrix} = \begin{pmatrix}6\\2\\5\end{pmatrix}$ ***(M1)***

OR

identify three points in the plane ***(A1)***

eg $\lambda = 0, 1$ gives $\begin{pmatrix}1\\2\\-2\end{pmatrix}$ and $\begin{pmatrix}-2\\1\\2\end{pmatrix}$

solving system of equations ***(M1)***

THEN

$\Pi_2 : \boldsymbol{r}.\begin{pmatrix}6\\2\\5\end{pmatrix} = 0$ ***A1***

Note: Accept $6x+2y+5z=0$.

[3 marks]

(d) vector normal to Π_1 is eg $\boldsymbol{n_1} = \begin{pmatrix}3\\-1\\1\end{pmatrix}$

vector normal to Π_2 is eg $\boldsymbol{n_2} = \begin{pmatrix}6\\2\\5\end{pmatrix}$ ***(A1)***

required angle is θ, where $\cos\theta = \dfrac{\begin{pmatrix}3\\-1\\1\end{pmatrix}.\begin{pmatrix}6\\2\\5\end{pmatrix}}{\sqrt{11}\sqrt{65}}$ ***M1A1***

$\cos\theta = \dfrac{21}{\sqrt{11}\sqrt{65}} = 0.785...$ ***(A1)***

$\theta = 0.667526...$

$\theta = 0.668\ (= 38.2^\circ)$ ***A1***

Note: Award the penultimate ***(A1)*** but not the final ***A1*** for the obtuse angle $2.47406\ldots$ or 142°.

[5 marks]

24 – (MathAA/21_HL_Summer_2021_Q6) – *Vectors - Lines - Planes*

(a) attempt to find a vector perpendicular to Π_1 and Π_2 using a cross product ***(M1)***

$$\begin{pmatrix}3\\2\\1\end{pmatrix}\times\begin{pmatrix}1\\-2\\1\end{pmatrix}=\left(2-(-2)\right)\boldsymbol{i}+(1-3)\boldsymbol{j}+(-6-2)\boldsymbol{k}$$

$$=\begin{pmatrix}4\\-2\\-8\end{pmatrix}\left(=2\begin{pmatrix}2\\-1\\-4\end{pmatrix}\right)$$ ***(A1)***

equation is $4x-2y-8z=0\left(\Rightarrow 2x-y-4z=0\right)$ ***A1***

[3 marks]

(b) attempt to solve 3 simultaneous equations in 3 variables ***(M1)***

$$\left(\frac{41}{21},-\frac{10}{21},\frac{23}{21}\right)\left(=(1.95,-0.476,1.10)\right)$$ ***A1***

[2 marks]

25 – (MathAA/20_HL_Winter_2021_Q11) – *Vectors - Lines - Planes*

(a) (i) attempts to find either $\overrightarrow{AB}$ or $\overrightarrow{AC}$ ***(M1)***

$\overrightarrow{AB} = \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} -2 \\ 1 \\ -7 \end{pmatrix}$ ***A1***

(ii) **METHOD 1**

attempts to find $\overrightarrow{AB} \times \overrightarrow{AC}$ ***(M1)***

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 14 \\ -21 \\ -7 \end{pmatrix}$ ***A1***

EITHER

equation of plane is of the form $14x - 21y - 7z = d$ $(2x - 3y - z = d)$ ***(A1)***

substitutes a valid point e.g $(3, 0, 0)$ to obtain a value of d ***M1***

$d = 42$ $(d = 6)$

OR

attempts to use $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$ ***(M1)***

$\boldsymbol{r} \cdot \begin{pmatrix} 14 \\ -21 \\ -7 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 14 \\ -21 \\ -7 \end{pmatrix}$ $\left(\boldsymbol{r} \cdot \begin{pmatrix} 14 \\ -21 \\ -7 \end{pmatrix} = 42 \right)$ ***A1***

$\boldsymbol{r} \cdot \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$ $\left(\boldsymbol{r} \cdot \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = 6 \right)$

THEN

$14x - 21y - 7z = 42$ $(2x - 3y - z = 6)$ ***A1***

METHOD 2

equation of plane is of the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ -7 \end{pmatrix}$ ***A1***

attempts to form equations for x, y, z in terms of their parameters ***(M1)***

$x = 3 - 3s - 2t$, $y = -2s + t$, $z = -7t$

A1

eliminates at least one of their parameters ***(M1)***

for example, $2x - 3y = 6 - 7t$ $(\Rightarrow 2x - 3y = 6 + z)$

$2x - 3y - z = 6$ ***A1***

[7 marks]

(b) **METHOD 1**

substitutes $\boldsymbol{r}=\begin{pmatrix}0\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\-1\end{pmatrix}$ into their Π_1 and Π_2 (given) ***(M1)***

Π_1: $2\lambda-3(-2+\lambda)-(-\lambda)=6$ and Π_2: $3\lambda-(-2+\lambda)+2(-\lambda)=2$ ***A1***

Note: Award ***(M1)A0*** for correct verification using a specific value of λ.

so the vector equation of L can be written as $\boldsymbol{r}=\begin{pmatrix}0\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\-1\end{pmatrix}$ ***AG***

METHOD 2
EITHER

attempts to find $\begin{pmatrix}2\\-3\\-1\end{pmatrix}\times\begin{pmatrix}3\\-1\\2\end{pmatrix}$ ***M1***

$=\begin{pmatrix}-7\\-7\\7\end{pmatrix}$

OR

$\begin{pmatrix}2\\-3\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix}=(2-3+1)=0$ and $\begin{pmatrix}3\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix}=(3-1-2)=0$ ***M1***

THEN

substitutes $(0,-2,0)$ into Π_1 and Π_2

Π_1: $2(0)-3(-2)-(0)=6$ and Π_2: $3(0)-(-2)+2(0)=2$ ***A1***

so the vector equation of L can be written as $\boldsymbol{r}=\begin{pmatrix}0\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\-1\end{pmatrix}$ ***AG***

METHOD 3

attempts to solve $2x-3y-z=6$ and $3x-y+2z=2$ ***(M1)***

for example, $x=-\lambda, y=-2-\lambda, z=\lambda$ ***A1***

Note: Award ***A1*** for substituting $x=0$ (or $y=-2$ or $z=0$) into Π_1 and Π_2 and solving simultaneously. For example, solving $-3y-z=6$ and $-y+2z=2$ to obtain $y=-2$ and $z=0$.

so the vector equation of L can be written as $\boldsymbol{r}=\begin{pmatrix}0\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\-1\end{pmatrix}$ ***AG***

[2 marks]

(c) (i) substitutes the equation of L into the equation of Π_3 ***(M1)***

$2\lambda+2\lambda=3\Rightarrow 4\lambda=3$ ***A1***

$\lambda=\frac{3}{4}$ ***AG***

(ii) **P** has coordinates $\left(\frac{3}{4},-\frac{5}{4},-\frac{3}{4}\right)$ ***A1***

[3 marks]

(d) (i) normal to Π_3 is $\boldsymbol{n}=\begin{pmatrix}2\\0\\-2\end{pmatrix}$ ***(A1)***

Note: May be seen or used anywhere.

considers the line normal to Π_3 passing through $\mathrm{B}(0,-2,0)$ ***(M1)***

$\boldsymbol{r}=\begin{pmatrix}0\\-2\\0\end{pmatrix}+\mu\begin{pmatrix}2\\0\\-2\end{pmatrix}$ ***A1***

EITHER

finding the point on the normal line that intersects Π_3

attempts to solve simultaneously with plane $2x-2z=3$ ***(M1)***

$4\mu+4\mu=3$

$\mu=\frac{3}{8}$ ***A1***

point is $\left(\frac{3}{4},-2,-\frac{3}{4}\right)$

OR

$$\left(\begin{pmatrix}2\mu\\-2\\-2\mu\end{pmatrix}-\begin{pmatrix}\frac{3}{4}\\-\frac{5}{4}\\-\frac{3}{4}\end{pmatrix}\right)\cdot\begin{pmatrix}2\\0\\-2\end{pmatrix}=0$$ ***(M1)***

$$4\mu-\frac{3}{2}+4\mu-\frac{3}{2}=0$$

$$\mu=\frac{3}{8}$$ ***A1***

OR

attempts to find the equation of the plane parallel to Π_3 containing B' $(x-z=3)$ and solve simultaneously with L ***(M1)***

$$2\mu'+2\mu'=3$$

$$\mu'=\frac{3}{4}$$ ***A1***

THEN

so, another point on the reflected line is given by

$$\boldsymbol{r}=\begin{pmatrix}0\\-2\\0\end{pmatrix}+\frac{3}{4}\begin{pmatrix}2\\0\\-2\end{pmatrix}$$ ***(A1)***

$$\Rightarrow \mathrm{B}'\left(\frac{3}{2},-2,-\frac{3}{2}\right)$$ ***A1***

(ii) **EITHER**

attempts to find the direction vector of the reflected line using their P and B' ***(M1)***

$$\overrightarrow{\mathrm{PB}'}=\begin{pmatrix}\frac{3}{4}\\-\frac{3}{4}\\-\frac{3}{4}\end{pmatrix}$$

OR

attempts to find their direction vector of the reflected line using a vector approach ***(M1)***

$$\overrightarrow{\mathrm{PB}'}=\overrightarrow{\mathrm{PB}}+\overrightarrow{\mathrm{BB}'}=-\frac{3}{4}\begin{pmatrix}1\\1\\-1\end{pmatrix}+\frac{3}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

THEN

$$\boldsymbol{r}=\begin{pmatrix}\frac{3}{2}\\-2\\-\frac{3}{2}\end{pmatrix}+\lambda\begin{pmatrix}\frac{3}{4}\\-\frac{3}{4}\\-\frac{3}{4}\end{pmatrix}$$ (or equivalent) ***A1***

Note: Award ***A0*** for either '$\boldsymbol{r} =$' or '$\begin{pmatrix} x \\ y \\ z \end{pmatrix} =$' not stated. Award ***A0*** for '$L' =$'.

[9 marks]
Total [21 marks]

26 – (MathAA/21_HL_Summer_2022_Q7) – *Vectors - Lines - Planes*

(a) $|\boldsymbol{a}| = \sqrt{12^2 + (-5)^2}\ (= 13)$ ***(A1)***

$2 \le |\boldsymbol{a} + \boldsymbol{b}| \le 28$ (accept min 2 and max 28) ***A1***

Note: Award ***(A1)A0*** for 2 and 28 seen with no indication that they are the endpoints of an interval.

[2 marks]

(b) recognition that $\boldsymbol{p}$ or $\boldsymbol{b}$ is a negative multiple of $\boldsymbol{a}$ ***(M1)***

$\boldsymbol{p} = -2\hat{\boldsymbol{a}}$ OR $\boldsymbol{b} = -\frac{15}{13}\boldsymbol{a}\left(= -\frac{15}{13}\begin{pmatrix}12\\-5\end{pmatrix}\right)$

$\boldsymbol{p} = -\frac{2}{13}\begin{pmatrix}12\\-5\end{pmatrix}\left(= \begin{pmatrix}-1.85\\0.769\end{pmatrix}\right)$ ***A1***

[2 marks]

(c) **METHOD 1**

$\boldsymbol{q}$ is perpendicular to $\begin{pmatrix}12\\-5\end{pmatrix}$

$\Rightarrow \boldsymbol{q}$ is in the direction $\begin{pmatrix}5\\12\end{pmatrix}$ ***(M1)***

$\boldsymbol{q} = k\begin{pmatrix}5\\12\end{pmatrix}$ ***(A1)***

$(|\boldsymbol{q}| =)\sqrt{(5k)^2 + (12k)^2} = 15$ ***(M1)***

$k = \frac{15}{13}$ ***(A1)***

$\boldsymbol{q} = \frac{15}{13}\begin{pmatrix}5\\12\end{pmatrix}\left(= \begin{pmatrix}\frac{75}{13}\\\frac{180}{13}\end{pmatrix} = \begin{pmatrix}5.77\\13.8\end{pmatrix}\right)$ ***A1***

[5 marks]

METHOD 2

$\boldsymbol{q}$ is perpendicular to $\begin{pmatrix} 12 \\ -5 \end{pmatrix}$

attempt to set scalar product $\boldsymbol{q}.\boldsymbol{a} = 0$ OR product of gradients $= -1$ ***(M1)***

$12x - 5y = 0$ ***(A1)***

$(|\boldsymbol{q}| =)\sqrt{x^2 + y^2} = 15$

attempt to solve simultaneously to find a quadratic in x or in y ***(M1)***

$x^2 + \left(\frac{12x}{5}\right)^2 = 15^2$ OR $\left(\frac{5y}{12}\right)^2 + y^2 = 15^2$

$\boldsymbol{q} = \begin{pmatrix} \frac{75}{13} \\ \frac{180}{13} \end{pmatrix} \left(= \begin{pmatrix} 5.77 \\ 13.8 \end{pmatrix} \right)$ ***A1A1***

Note: Award ***A1*** independently for each value. Accept values given as $x = \frac{75}{13}$ and $y = \frac{180}{13}$ or equivalent.

[5 marks]

27 – (MathAA/22_HL_Summer_2022_Q11) – *Vectors - Lines - Planes*

(a) let ϕ be the required angle (bearing)

EITHER

$\phi = 90° - \arctan\frac{1}{2}\ \left(= \arctan 2\right)$ ***(M1)***

Note: Award ***M1*** for a labelled sketch.

OR

$\cos\phi = \frac{\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}4\\2\end{pmatrix}}{\sqrt{1}\times\sqrt{20}}\ \left(= 0.4472...,= \frac{1}{\sqrt{5}}\right)$ ***(M1)***

$\phi = \arccos(0.4472...)$

THEN

$063°$ ***A1***

Note: Do not accept $063.4°$ or $63.4°$ or 1.10^c.

[2 marks]

(b) **Method 1**

let $|\boldsymbol{b}_A|$ be the speed of A and let $|\boldsymbol{b}_B|$ be the speed of B

attempts to find the speed of one of A or B ***(M1)***

$|\boldsymbol{b}_A| = \sqrt{(-6)^2 + 2^2 + 4^2}$ or $|\boldsymbol{b}_B| = \sqrt{4^2 + 2^2 + (-2)^2}$

Note: Award ***M0*** for $|\boldsymbol{b}_A| = \sqrt{19^2 + (-1)^2 + 1^2}$ and $|\boldsymbol{b}_B| = \sqrt{1^2 + 0^2 + 12^2}$.

$|\boldsymbol{b}_A| = 7.48...\ \left(= \sqrt{56}\right)$ (km min^{-1}) and $|\boldsymbol{b}_B| = 4.89...\ \left(= \sqrt{24}\right)$ (km min^{-1}) ***A1***

$|\boldsymbol{b}_A| > |\boldsymbol{b}_B|$ so A travels at a greater speed than B ***AG***

[2 marks]

Method 2

attempts to use $\text{speed} = \dfrac{\text{distance}}{\text{time}}$

$\text{speed}_A = \dfrac{|r_A(t_2) - r_A(t_1)|}{t_2 - t_1}$ and $\text{speed}_B = \dfrac{|r_B(t_2) - r_B(t_1)|}{t_2 - t_1}$ ***(M1)***

for example:

$\text{speed}_A = \dfrac{|r_A(1) - r_A(0)|}{1}$ and $\text{speed}_B = \dfrac{|r_B(1) - r_B(0)|}{1}$

$\text{speed}_A = \dfrac{\sqrt{(-6)^2 + 2^2 + 4^2}}{1}$ and $\text{speed}_B = \dfrac{\sqrt{4^2 + 2^2 + 2^2}}{1}$

$\text{speed}_A = 7.48...\left(2\sqrt{14}\right)$ and $\text{speed}_B = 4.89...\left(\sqrt{24}\right)$ ***A1***

$\text{speed}_A > \text{speed}_B$ so A travels at a greater speed than B ***AG***

[2 marks]

(c) attempts to use the angle between two direction vectors formula ***(M1)***

$\cos\theta = \dfrac{(-6)(4) + (2)(2) + (4)(-2)}{\sqrt{(-6)^2 + 2^2 + 4^2}\sqrt{4^2 + 2^2 + (-2)^2}}$ ***(A1)***

$\cos\theta = -0.7637...\ \left(= -\dfrac{7}{\sqrt{84}}\right)$ or $\theta = \arccos(-0.7637...)\ (= 2.4399...)$

attempts to find the acute angle $180° - \theta$ using their value of θ ***(M1)***

$= 40.2°$ ***A1***

[4 marks]

(d) (i) for example, sets $\boldsymbol{r}_A(t_1) = \boldsymbol{r}_B(t_2)$ and forms at least two equations ***(M1)***

$19 - 6t_1 = 1 + 4t_2$

$-1 + 2t_1 = 2t_2$

$1 + 4t_1 = 12 - 2t_2$

Note: Award ***M0*** for equations involving t only.

EITHER

attempts to solve the system of equations for one of t_1 or t_2 ***(M1)***

$t_1 = 2$ or $t_2 = \frac{3}{2}$ ***A1***

OR

attempts to solve the system of equations for t_1 and t_2 ***(M1)***

$t_1 = 2$ and $t_2 = \frac{3}{2}$ ***A1***

THEN

substitutes their t_1 or t_2 value into the corresponding $\boldsymbol{r}_A$ or $\boldsymbol{r}_B$ ***(M1)***

$\mathrm{P}(7,3,9)$ ***A1***

Note: Accept $\overrightarrow{\mathrm{OP}} = \begin{pmatrix} 7 \\ 3 \\ 9 \end{pmatrix}$. Accept 7 km east of O, 3 km north of O and 9 km above sea level.

(ii) attempts to find the value of $t_1 - t_2$ ***(M1)***

$t_1 - t_2 = 2 - \frac{3}{2}$

0.5 minutes (30 seconds) ***A1***

[7 marks]

(e) **EITHER**

attempts to find $\boldsymbol{r}_B - \boldsymbol{r}_A$ ***(M1)***

$$\boldsymbol{r}_B - \boldsymbol{r}_A = \begin{pmatrix} -18 \\ 1 \\ 11 \end{pmatrix} + t\begin{pmatrix} 10 \\ 0 \\ -6 \end{pmatrix}$$

attempts to find their $D(t)$ ***(M1)***

$$D(t) = \sqrt{(10t-18)^2 + 1 + (11-6t)^2}$$ ***A1***

OR

attempts to find $\boldsymbol{r}_A - \boldsymbol{r}_B$ ***(M1)***

$$\boldsymbol{r}_A - \boldsymbol{r}_B = \begin{pmatrix} 18 \\ -1 \\ -11 \end{pmatrix} + t\begin{pmatrix} -10 \\ 0 \\ 6 \end{pmatrix}$$

attempts to find their $D(t)$ ***(M1)***

$$D(t) = \sqrt{(18-10t)^2 + (-1)^2 + (-11+6t)^2}$$ ***A1***

Note: Award ***M0M0A0*** for expressions using two different time parameters.

THEN

either attempts to find the local minimum point of $D(t)$ or attempts to find the value of t such that $D'(t) = 0$ (or equivalent) ***(M1)***

$$t = 1.8088... \left(= \frac{123}{68}\right)$$

$$D(t) = 1.01459...$$

minimum value of $D(t)$ is $1.01 \left(= \frac{\sqrt{1190}}{34}\right)$ (km) ***A1***

[5 marks]

Note: Award ***M0*** for attempts at the shortest distance between two lines.

28 – (MathAA/20_HL_Winter_2022_Q12) – *Vectors - Lines - Planes*

(a) $\overrightarrow{AB} = \begin{pmatrix} k-1 \\ -4 \\ -2 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix}$ ***A1A1***

[2 marks]

(b) **METHOD 1**

$k - 1 = 2 \times 4$ ***M1***

$k = 9$ ***AG***

METHOD 2

in order by y or z-ordinate, the points are $(k, -2, 1)$, $(5, 0, 2)$ $(1, 2, 3)$

$k - 5 = 5 - 1$ ***M1***

$k = 9$ ***AG***

[1 mark]

(c) (i) attempt to set up a vector equation using a point, a parameter and a direction vector ***(M1)***

$\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix}$ (or equivalent) ***A1***

Note: "$\boldsymbol{r} =$" or equivalent must be seen for ***A1***.

(ii) **METHOD 1**

point on line l_1 has coordinates $(1+4\lambda, 2-2\lambda, 3-\lambda)$

attempt to use a different parameter for l_2 ***(M1)***

$$\frac{x-1}{2}=\frac{y}{3}=1-z=\mu \text{ or } \boldsymbol{r}=\begin{pmatrix}1\\0\\1\end{pmatrix}+\mu\begin{pmatrix}2\\3\\-1\end{pmatrix}$$

point on line l_2 has coordinates $(1+2\mu, 3\mu, 1-\mu)$ ***(A1)***

Note: This ***A1*** may be implied by $\boldsymbol{r}=\begin{pmatrix}1\\0\\1\end{pmatrix}+\mu\begin{pmatrix}2\\3\\-1\end{pmatrix}$.

$1+4\lambda=1+2\mu$

$2-2\lambda=3\mu$

$3-\lambda=1-\mu$

any two of the above equations ***A1***

attempt to solve two simultaneous equations with two parameters ***(M1)***

eg $\lambda=0.25$, $\mu=0.5$ or $\lambda=1.6$, $\mu=-0.4$ or $\lambda=-2$, $\mu=-4$ ***A1***

substitute into third equation or solve a different pair of simultaneous equations ***M1***

obtain contradiction *eg* $3-0.25\neq 1-0.5$ or $1+4(1.6)\neq 1+2(-0.4)$ or $2-2(-2)\neq 3(-4)$ (so the lines do not intersect) ***R1***

Note: Do not award this ***R1*** if it is based on incorrect values.

lines are not parallel ***R1***

so lines are skew ***AG***

METHOD 2

point on line l_1 has coordinates $(1+4\lambda, 2-2\lambda, 3-\lambda)$

attempt to use the equation of l_2 to generate at least two equations in λ ***(M1)***

if the two lines intersect,

$$\frac{(1+4\lambda)-1}{2} = \frac{2-2\lambda}{3}\left(\Rightarrow 2\lambda = \frac{2-2\lambda}{3}\right)$$

$$\frac{(1+4\lambda)-1}{2} = 1-(3-\lambda)(\Rightarrow 2\lambda = \lambda - 2)$$

$$\frac{2-2\lambda}{3} = 1-(3-\lambda) \Rightarrow \left(\frac{2-2\lambda}{3} = \lambda - 2\right)$$

any two of the above equations ***A1A1***

attempt to solve at least one equation in λ ***(M1)***

one of $\lambda = \frac{1}{4}$, $\lambda = -2$, $\lambda = \frac{8}{5}$ seen ***A1***

substitute into second equation or solve second equation ***M1***

obtain contradiction *eg* $\lambda = \frac{1}{4} \neq -2$ or $2\left(\frac{1}{4}\right) \neq \frac{1}{4} - 2$ (so the lines do not intersect) ***R1***

Note: Do not award this ***R1*** if it is based on incorrect values.

lines are not parallel ***R1***

so lines are skew ***AG***

METHOD 3

attempt to use a find Cartesian equation for l_1 ***(M1)***

$\frac{x-1}{4}=\frac{y-2}{-2}=\frac{z-3}{-1}$ ***A1***

attempt to isolate one variable in both equations ***(M1)***

$l_1: z=\frac{1-x}{4}+3=\frac{y-2}{2}+3 \quad l_2: z=\frac{1-x}{2}+1=\frac{-y}{3}+1$ OR

$l_1: y=\frac{1-x}{2}+2=2(z-3)+2 \quad l_2: y=\frac{3(x-1)}{2}=3(1-z)$ OR

$l_1: x=1-2(y-2)=1-4(z-3) \quad l_2: x=\frac{2y}{3}+1=1-2(z-1)$ ***A1***

attempt to solve for each of the other two variables ***(M1)***

e.g. $\frac{1-x}{2}+1=\frac{1-x}{4}+3$ and $\frac{-y}{3}+1=\frac{y-2}{2}+3$

$x=-7,\ y=-1.2$ OR $x=2,\ z=1.4$ OR $y=1.5,\ z=5$ ***A1***

obtain contradiction *eg* $z=5\neq 1.4$ OR $y=1.5\neq -1.2$ OR $x=2\neq -7$

(so the lines do not intersect) ***R1***

Note: Do not award this ***R1*** if it is based on incorrect values.

lines are not parallel ***R1***

so lines are skew ***AG***

[10 marks]

(d) (i) **METHOD 1**

attempt to find cross product of two of $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$ or their opposites ***M1***

eg $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 0 \\ k-9 \\ 18-2k \end{pmatrix} \left(= (k-9)\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right)$ ***A1***

attempt to substitute their cross product and a point into the equation of a plane ***(M1)***

$(k-9)y + 2(9-k)z = 2(k-9) + 6(9-k)$

$(k-9)y + 2(9-k)z = 36 - 4k \ (\Rightarrow y - 2z = -4 \text{ since } k \neq 9)$ ***A1***

METHOD 2

attempt to find vector equation of Π and write x, y and z in parametric form ***M1***

$\left(\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} k-1 \\ -4 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix} \Rightarrow \right) x = 1 + \lambda(k-1) + 4\mu,\ y = 2 - 4\lambda - 2\mu,$

$z = 3 - 2\lambda - \mu$ or equivalent ***A1***

attempt to eliminate both parameters to work towards Cartesian form ***M1***

$(k-9)y + 2(9-k)z = 36 - 4k \ (\Rightarrow y - 2z = -4 \text{ since } k \neq 9)$ ***A1***

(ii) **METHOD 1**

attempt to find the equation of the line through (0, 0, 0) perpendicular to the plane ***(M1)***

EITHER

$$(\boldsymbol{r} =)\, t\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$ ***(A1)***

attempt to find the point where the line and plane intersect ***(M1)***

$t + 4t + 4 = 0$

$t = -\frac{4}{5}$ ***(A1)***

OR

$$(\boldsymbol{r} =)\, t(k-9)\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$ ***(A1)***

attempt to find the point where the line and plane intersect ***(M1)***

$t(k-9)^2 + 4t(k-9)^2 + 4(k-9) = 0$

$t = -\frac{4}{5(k-9)}$ ***(A1)***

THEN

so the point on the plane closest to the origin is $(0,\ -0.8,\ 1.6)$ ***A1***

METHOD 2

choose a point on the plane (p, q, r)

$q - 2r + 4 = 0$ OR $q(k-9) - 2r(k-9) + 4(k-9) = 0 \Rightarrow q = 2r - 4$

distance to the origin is $\sqrt{p^2 + (2r-4)^2 + r^2}$ ***(A1)***

since *p* is independent of *r*, distance is minimised when $p = 0$ ***(R1)***

attempt to find the value of *r* for which their $\sqrt{(2r-4)^2 + r^2}$ is minimised ***(M1)***

$r = 1.6$ ***(A1)***

so the point on the plane closest to the origin is $(0, -0.8, 1.6)$ ***A1***

METHOD 3

attempt to find a vector from the origin to the closest point on the plane ***(M1)***

EITHER

$(\boldsymbol{r} =)\, t\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ ***(A1)***

distance to the origin $= \left(\dfrac{4}{\sqrt{1^2 + (-2)^2}} = \dfrac{4}{\sqrt{5}} \right) = \dfrac{4\sqrt{5}}{5}$ ***(A1)***

$t = \pm\dfrac{4}{5}$

check in equation of plane $y - 2z = -4$ to get $t = -\dfrac{4}{5}$ ***(R1)***

OR

$(\boldsymbol{r} =)\, t(k-9)\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ ***(A1)***

distance to the origin $= \left(\dfrac{4}{\sqrt{1^2 + (-2)^2}} = \dfrac{4}{\sqrt{5}} \right) = \dfrac{4\sqrt{5}}{5}$ ***(A1)***

$t = \pm\dfrac{4}{5(k-9)}$

check in equation of plane $y - 2z = -4$ to get $t = -\dfrac{4}{5(k-9)}$ ***(R1)***

THEN

so the point on the plane closest to the origin is $(0, -0.8, 1.6)$ ***A1***

[9 marks]

1 – (MAT/21_HL_Summer_2012_Q5) – *Statistics*

(a) 50

(b) Lower quartile is 4 so at least 26 obtained a 4
Lower bound is 26

Minimum is 2 but the rest could be 4

So upper bound is 49

2 – (MAT/22_HL_Summer_2012_Q10) – *Probability, Statistics*

(a) (i) $\mathrm{P}(X > 225) = 0.158\ldots$
expected number $= 450 \times 0.158\ldots = 71.4$

(ii) $\mathrm{P}(X < m) = 0.7$
$\Rightarrow m = 213$ (grams)

(b) $\dfrac{270-\mu}{\sigma} = 1.40\ldots$
$\dfrac{250-\mu}{\sigma} = -1.03\ldots$

Note: These could be seen in graphical form.

solving simultaneously
$\mu = 258,\ \sigma = 8.19$

(c)
$X \sim \mathrm{N}(80, 4^2)$
$\mathrm{P}(X > 82) = 0.3085\ldots$
recognition of the use of binomial distribution.
$X \sim \mathrm{B}(5, 0.3085\ldots)$
$\mathrm{P}(X = 3) = 0.140$

3 – (MAT/21_HL_Summer_2012_Q12) – *Probability, Statistics*

(a) (i) $X \sim \text{Po}(11)$

$\text{P}(X \le 11) = 0.579$

(ii) $\text{P}(X > 8 | X < 12) =$

$= \dfrac{\text{P}(8 < X < 12)}{\text{P}(X < 12)} \left(\text{or } \dfrac{\text{P}(X \le 11) - \text{P}(X \le 8)}{\text{P}(X \le 11)} \text{ or } \dfrac{0.3472...}{0.5792...} \right)$

$= 0.600$

(b) (i) $Y \sim \text{Po}(m)$

$\text{P}(Y > 3) = 0.24$

$\text{P}(Y \le 3) = 0.76$

$e^{-m}\left(1 + m + \dfrac{1}{2}m^2 + \dfrac{1}{6}m^3\right) = 0.76$

Note: At most two of the above lines can be implied.

Attempt to solve equation with GDC

$m = 2.49$

(ii) $A \sim \text{Po}(4.98)$

$\text{P}(A > 5) = 1 - \text{P}(A \le 5) = 0.380\ldots$

$W \sim \text{B}(4, 0.380\ldots)$

$\text{P}(W \ge 2) = 1 - \text{P}(W \le 1) = 0.490$

(c) $\text{P}(A < 25) = 0.8$, $\text{P}(A < 18) = 0.4$

$\dfrac{25 - \mu}{\sigma} = 0.8416\ldots$

$\dfrac{18 - \mu}{\sigma} = -0.2533\ldots$ (or -0.2534 from tables)

solving these equations

$\mu = 19.6$

4 – (MAT/20_HL_Winter_2012_Q4) – *Statistics*

$$\frac{\sum_{i=1}^{15} x_i}{15} = 11.5 \Rightarrow \sum_{i=1}^{15} x_i = 172.5$$

$$\text{new mean} = \frac{172.5 - 22.1}{14}$$

$$= 10.7428\ldots = 10.7 \text{ (3sf)}$$

$$\frac{\sum_{i=1}^{15} x_i^2}{15} - 11.5^2 = 9.3$$

$$\Rightarrow \sum_{i=1}^{15} x_i^2 = 2123.25$$

$$\text{new variance} = \frac{2123.25 - 22.1^2}{14} - (10.7428\ldots)^2$$

$$= 1.37 \text{ (3sf)}$$

5 – (MAT/20_HL_Winter_2012_Q11) – *Probability, Statistics*

(a) $2.2 \times 6 \times 60 = 792$

(b) $V \sim \text{Po}(2.2 \times 60)$

$\text{P}(V > 100) = 0.998$

(c) $(0.997801\ldots)^6 = 0.987$

(d) $A \sim \text{N}(\mu, \sigma^2)$

$\text{P}(A < 35) = 0.29$ and $\text{P}(A > 55) = 0.23 \Rightarrow \text{P}(A < 55) = 0.77$

$\text{P}\left(Z < \frac{35 - \mu}{\sigma}\right) = 0.29$ and $\text{P}\left(Z < \frac{55 - \mu}{\sigma}\right) = 0.77$

use of inverse normal

$\frac{35 - \mu}{\sigma} = -0.55338\ldots$ and $\frac{55 - \mu}{\sigma} = 0.738846\ldots$

solving simultaneously

$\mu = 43.564\ldots$ and $\sigma = 15.477\ldots$

$\mu = 43.6$ and $\sigma = 15.5$ (3sf)

(e) $0.29n = 100 \Rightarrow n = 344.82\ldots$

$\text{P}(A < 50) = 0.66121\ldots$

expected number of visitors under $50 = 228$

6 – (MAT/21_HL_Summer_2013_Q1) – *Statistics*

$$\frac{5\times 6+6k+7\times 3+8\times 1+9\times 2+10\times 1}{13+k}=6.5 \text{ (or equivalent)}$$

Note: Award ***(M1)(A1)*** for correct numerator, and ***(A1)*** for correct denominator.

$$0.5k=2.5\Rightarrow k=5$$

7 – (MAT/20_HL_Winter_2013_Q11) – *Probability, Statistics*

(a) (i) $X \sim \text{Po}(0.6)$

$\text{P}(X=0)=0.549 \ \left(=\text{e}^{-0.6}\right)$ ***A1***

(ii) $\text{P}(X \geq 3)=1-\text{P}(X \leq 2)$ ***(M1)(A1)***

$=1-\left(\text{e}^{-0.6}+\text{e}^{-0.6}\times 0.6+\text{e}^{-0.6}\times\frac{0.6^2}{2}\right)$

$=0.0231$ ***A1***

(iii) $Y \sim \text{Po}(2.4)$ ***(M1)***

$\text{P}(Y \leq 5)=0.964$ ***A1***

(iv) $Z \sim \text{B}(12,\ 0.451...)$ ***(M1)(A1)***

Note: Award ***M1*** for recognising binomial and ***A1*** for using correct parameters.

$\text{P}(Z=4)=0.169$ ***A1***

(b) (i) $k\int_1^3 \ln x\,\text{d}x=1$ ***(M1)***

$(k\times 1.2958...=1)$

$k=0.771702$ ***A1***

(ii) $\text{E}(X)=\int_1^3 kx\ln x\,\text{d}x$ ***(A1)***

attempting to evaluate their integral ***(M1)***

$=2.27$ ***A1***

(iii) $x=3$ ***A1***

(iv) $\int_1^m k\ln x\,\text{d}x=0.5$ ***(M1)***

$k[x\ln x-x]_1^m=0.5$

attempting to solve for m ***(M1)***

$m=2.34$ ***A1***

8 – (MAT/21_HL_Summer_2014_Q2) – *Statistics*

$X: \mathrm{N}\left(100, \sigma^2\right)$

$\mathrm{P}(X<124)=0.68$

$\frac{24}{\sigma}=0.4676....$

$\sigma=51.315\ldots$

$\text{variance}=2630$

9 – (MAT/22_HL_Summer_2014_Q2) – *Statistics, Probability*

(a) **METHOD 1**

$\mu=\frac{1}{2}\times(17.1+21.3)$

$\mu=19.2\,(\text{kg})$

finding z value for the upper quartile $=0.674489\text{K}$

$0.674489\text{K}=\frac{21.3-19.2}{\sigma}$ or $-0.674489\text{K}=\frac{17.1-19.2}{\sigma}$

$\sigma=3.11\,(\text{kg})$

METHOD 2

finding z value for the upper quartile $=0.674489\text{K}$
from symmetry the z value for a lower quartile is -0.674489K
forming two simultaneous equations:

$-0.674489\text{K}=\frac{17.1-\mu}{\sigma}$

$0.674489\text{K}=\frac{21.3-\mu}{\sigma}$

solving gives:
$\mu=19.2\,(\text{kg})$
$\sigma=3.11\,(\text{kg})$

(b) using $100\times\mathrm{P}(X>22)=100\times0.184241\text{K}$
$=18$

10 – (MAT/20_HL_Winter_2014_Q3) – *Statistics*

use of $\mu = \dfrac{\sum_{i=1}^{k} f_i x_i}{n}$ to obtain $\dfrac{2+x+y+10+17}{5} = 8$

$x + y = 11$

EITHER

use of $\sigma^2 = \dfrac{\sum_{i=1}^{k} f_i (x_i - \mu)^2}{n}$ to obtain $\dfrac{(-6)^2 + (x-8)^2 + (y-8)^2 + 2^2 + 9^2}{5} = 27.6$

$(x-8)^2 + (y-8)^2 = 17$

OR

use of $\sigma^2 = \dfrac{\sum_{i=1}^{k} f_i x_i^2}{n} - \mu^2$ to obtain $\dfrac{2^2 + x^2 + y^2 + 10^2 + 17^2}{5} - 8^2 = 27.6$

$x^2 + y^2 = 65$

THEN

attempting to solve the two equations
$x = 4$ and $y = 7$ (only as $x < y$)

11 – (MAT/20_HL_Winter_2015_Q3) – *Statistics*

(a) $\dfrac{0 \cdot 4 + 1 \cdot k + 2 \cdot 3 + 3 \cdot 2 + 4 \cdot 3 + 8 \cdot 1}{13 + k} = 1.95 \left(\dfrac{k + 32}{k + 13} = 1.95 \right)$

attempting to solve for k
$k = 7$

(b) (i) $\dfrac{7 + 32 + 22}{7 + 13 + 1} = 2.90 \left(= \dfrac{61}{21} \right)$

(ii) standard deviation $= 4.66$

12 – (MAT/21_HL_Summer_2016_Q13) – *Statistics, Probability*

(a) $\mathrm{E}(X) = 1 \times \frac{1}{6} + 2 \times \frac{2}{6} + 3 \times \frac{3}{6} = \frac{14}{6}\left(= \frac{7}{3} = 2.33\right)$

(b) (i) $3 \times \mathrm{P}(113) + 3 \times \mathrm{P}(122)$

$3 \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{2} + 3 \times \frac{1}{6} \times \frac{1}{3} \times \frac{1}{3} = \frac{7}{72} (= 0.0972)$

Note: Award ***M1*** for attempt to find at least four of the cases.

(ii) recognising 111 as a possibility (implied by $\frac{1}{216}$)

recognising 112 and 113 as possibilities (implied by $\frac{2}{216}$ and $\frac{3}{216}$)

seeing the three arrangements of 112 and 113

$\mathrm{P}(111) + 3 \times \mathrm{P}(112) + 3 \times \mathrm{P}(113)$

$= \frac{1}{216} + \frac{6}{216} + \frac{9}{216} = \frac{16}{216}\left(= \frac{2}{27} = 0.0741\right)$

(c) let the number of twos be X, $X \sim B\left(10, \frac{1}{3}\right)$

$\mathrm{P}(X < 4) = \mathrm{P}(X \le 3) = 0.559$

(d) let n be the number of balls drawn

$\mathrm{P}(X \ge 1) = 1 - \mathrm{P}(X = 0)$

$= 1 - \left(\frac{2}{3}\right)^n > 0.95$

$\left(\frac{2}{3}\right)^n < 0.05$

$n = 8$

(e) $8p_1 = 4.8 \Rightarrow p_1 = \frac{3}{5}$

$8p_2(1 - p_2) = 1.5$

$p_2^2 - p_2 - 0.1875 = 0$

$p_2 = \frac{1}{4}\left(\text{or } \frac{3}{4}\right)$

reject $\frac{3}{4}$ as it gives a total greater than one

$\mathrm{P}(1 \text{ or } 2) = \frac{17}{20}$ or $\mathrm{P}(3) = \frac{3}{20}$

recognising LCM as 20 so min total number is 20

the least possible number of 3's is 3

13 - (MAT/22_HL_Summer_2017_Q1) - *Probability, Statistics*

(a) $\mathrm{P}(5 \text{ or more}) = \frac{29}{75} (= 0.387)$ ***(M1)A1***

[2 marks]

(b) mean score $= \frac{2 \times 3 + 3 \times 15 + 4 \times 28 + 5 \times 17 + 6 \times 9 + 7 \times 3}{75}$ ***(M1)***

$= \frac{323}{75} (= 4.31)$ ***A1***

[2 marks]

14 – (MAT/22_HL_Summer_2017_Q3) – *Statistics, Probability*

(a) $\mathrm{P}(X < 250) = 0.0228$ ***(M1)A1***

[2 marks]

(b) $\dfrac{250-\mu}{1.5} = -2.878\ldots$ ***(M1)(A1)***

$\Rightarrow \mu = 254.32$ ***A1***

Notes: Only award ***A1*** here if the correct 2dp answer is seen.
Award ***M0*** for use of 1.5^2.

[3 marks]

(c) $\dfrac{250-253}{\sigma} = -2.878\ldots$ ***(A1)***

$\Rightarrow \sigma = 1.04$ ***A1***

[2 marks]

15 – (MAT/21_HL_Summer_2017_Q9) – *Probability, Statistics*

(a) $T \sim N\left(196, 24^2\right)$

$\mathrm{P}(T < 180) = 0.252$ ***(M1)A1***

[2 marks]

(b) $\mathrm{P}\left(T < T_1\right) = 0.05$ ***(M1)***

$T_1 = 157$ ***A1***

[2 marks]

(c) $F \sim N\left(210, \sigma^2\right)$

$\mathrm{P}(F < 235) = 0.79$ ***(M1)***

$\dfrac{235-210}{\sigma} = 0.806421$ or equivalent ***(M1)(A1)***

$\sigma = 31.0$ ***A1***

[4 marks]

16 – (MAT/20_HL_Winter_2017_Q4) – *Statistics*

let X be the random variable "amount of caffeine content in coffee"

$\mathrm{P}(X > 120) = 0.2$, $\mathrm{P}(X > 110) = 0.6$ ***(M1)***

$(\Rightarrow \mathrm{P}(X < 120) = 0.8$, $\mathrm{P}(X < 110) = 0.4)$

Note: Award ***M1*** for at least one correct probability statement.

$\dfrac{120-\mu}{\sigma} = 0.84162\ldots$, $\dfrac{110-\mu}{\sigma} = -0.253347\ldots$ ***(M1)(A1)(A1)***

Note: Award ***M1*** for attempt to find at least one appropriate z-value.

$120 - \mu = 0.84162\sigma$, $110 - \mu = -0.253347\sigma$

attempt to solve simultaneous equations ***(M1)***

$\mu = 112$, $\sigma = 9.13$ ***A1***

[6 marks]

17 – (MAT/22_HL_Summer_2017_Q10) – *Probability, Statistics*

(a) $\int_0^4 \left(\frac{x^2}{a}+b\right)\mathrm{d}x = 1 \Rightarrow \left[\frac{x^3}{3a}+bx\right]_0^4 = 1 \Rightarrow \frac{64}{3a}+4b=1$ ***M1A1***

$\int_2^4 \left(\frac{x^2}{a}+b\right)\mathrm{d}x = 0.75 \Rightarrow \frac{56}{3a}+2b=0.75$ ***M1A1***

Note: $\int_0^2 \left(\frac{x^2}{a}+b\right)\mathrm{d}x = 0.25 \Rightarrow \frac{8}{3a}+2b=0.25$ could be seen/used in place of either of the above equations.

evidence of an attempt to solve simultaneously
(or check given a, b values are consistent) ***M1***

$a=32,\ b=\frac{1}{12}$ ***AG***

[5 marks]

(b) $\mathrm{E}(X)=\int_0^4 x\left(\frac{x^2}{32}+\frac{1}{12}\right)\mathrm{d}x$ ***(M1)***

$\mathrm{E}(X)=\frac{8}{3}(=2.67)$ ***A1***

[2 marks]

(c) $\mathrm{E}\left(X^2\right)=\int_0^4 x^2\left(\frac{x^2}{32}+\frac{1}{12}\right)\mathrm{d}x$ ***(M1)***

$\mathrm{Var(X)}=\mathrm{E}\left(X^2\right)-\left[\mathrm{E}(X)\right]^2=\frac{16}{15}(=1.07)$ ***A1***

[2 marks]

(d) $\int_0^m \left(\frac{x^2}{32}+\frac{1}{12}\right)\mathrm{d}x = 0.5$ ***(M1)***

$\frac{m^3}{96}+\frac{m}{12}=0.5\left(\Rightarrow m^3+8m-48=0\right)$ ***(A1)***

$m=2.91$ ***A1***

[3 marks]

(e) $Y \sim B(8, 0.75)$ ***(M1)***

$E(Y)=8\times 0.75=6$ ***A1***

[2 marks]

(f) $\mathrm{P}(Y\geq 3)=0.996$ ***A1***

[1 mark]

18 – (MAT/21_HL_Summer_2018_Q8) – *Statistics*

METHOD 1

let p have no pets, q have one pet and r have two pets *(M1)*

$p+q+r+2=25$ *(A1)*

$0p+1q+2r+6=18$ *A1*

Note: Accept a statement that there are a total of 12 pets.

attempt to use variance equation, or evidence of trial and error *(M1)*

$\frac{0p+1q+4r+18}{25}-\left(\frac{18}{25}\right)^2=\left(\frac{24}{25}\right)^2$ *(A1)*

attempt to solve a system of linear equations *(M1)*

$p=14$ *A1*

METHOD 2

x	0	1	2	3
$\mathrm{P}(X=x)$	p	q	r	$\frac{2}{25}$

(M1)

$p+q+r+\frac{2}{25}=1$ *(A1)*

$q+2r+\frac{6}{25}=\frac{18}{25}\left(\Rightarrow q+2r=\frac{12}{25}\right)$ *A1*

$q+4r+\frac{18}{25}-\left(\frac{18}{25}\right)^2=\frac{576}{625}\left(\Rightarrow q+4r=\frac{18}{25}\right)$ *(M1)(A1)*

$q=\frac{6}{25}$, $r=\frac{3}{25}$ *(M1)*

$p=\frac{14}{25}$ *A1*

so 14 have no pets

[7 marks]

19 – (MathAA/21_HL_Summer_2021_Q1) – *Statistics*

(a) $a = 0.433156...,\ b = 4.50265...$

$a = 0.433,\ b = 4.50$ ***A1A1***

[2 marks]

(b) attempt to substitute $x = 18$ into their equation ***(M1)***

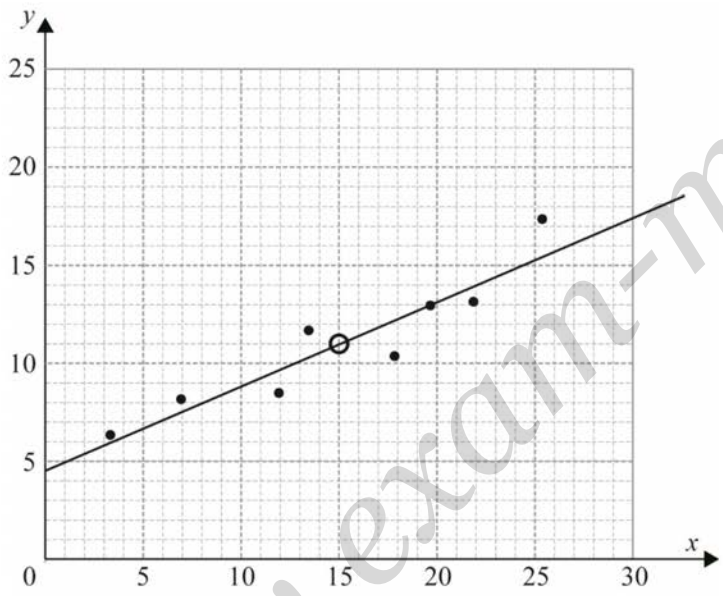
$y = 0.433 \times 18 + 4.50$
$= 12.2994...$
$= 12.3$ ***A1***

[2 marks]

(c) $\bar{x} = 15$, $\bar{y} = 11$ ***A1***

[1 mark]

(d)



A1A1

Note: Award marks as follows:

A1 for a straight line going through (15, 11)

A1 for intercepting the y-axis between their $b \pm 1.5$ (when their line is extended), which includes all the data for $3.3 \le x \le 25.3$.

If the candidate does not use a ruler, award ***A0A1*** where appropriate.

[2 marks]

20 – (MathAA/22_HL_Summer_2021_Q1) – *Statistics*

(a) (i) $a = 0.805084...$ and $b = 2.88135...$
$a = 0.805$ and $b = 2.88$ ***A1A1***

(ii) $r = 0.97777...$
$r = 0.978$ ***A1***

[3 marks]

(b) a represents the (average) increase in waiting time (0.805 mins) per additional customer (waiting to receive their coffee) ***R1***

[1 mark]

(c) attempt to substitute $x = 7$ into their equation ***(M1)***

$8.51693...$
8.52 (mins) ***A1***

[2 marks]

21 – (MathAA/20_HL_Winter_2021_Q1) – *Statistics*

(a) use of GDC to give ***(M1)***

$r = 0.883529...$

$r \approx 0.884$ ***A1***

Note: Award the ***(M1)*** for any correct value of r, a, b or $r^2 = 0.780624...$ seen in part (a) or part (b).

[2 marks]

(b) $a = 1.36609...$, $b = 64.5171...$

$a = 1.37$, $b = 64.5$ ***A1***

[1 mark]

(c) attempt to find their difference ***(M1)***

$5 \times 1.36609...$ OR $1.36609...(h+5) + 64.5171... - (1.36609...h + 64.5171...)$

$6.83045...$

$= 6.83 (6.85 \text{ from } 1.37)$

the student could have expected her score to increase by 7 marks. ***A1***

Note: Accept an increase of 6, 6.83 or 6.85.

[2 marks]

(d) Lucy is incorrect in suggesting there is a causal relationship.
This might be true, but the data can only indicate a correlation. ***R1***

Note: Accept 'Lucy is incorrect as correlation does not imply causation' or equivalent.

[1 mark]

(e) no effect ***A1***

[1 mark]

Total [7 marks]

22 – (MathAA/21_HL_Summer_2022_Q1) – *Statistics*

(a) **EITHER**

recognising that half the total frequency is 10 (may be seen in an ordered list or indicated on the frequency table) ***(A1)***

OR

$5+1+4=3+x$ ***(A1)***

OR

$\sum f = 20$ ***(A1)***

THEN

$x=7$ ***A1***

[2 marks]

(b) **METHOD 1**

1.58429…

1.58 ***A2***

METHOD 2

EITHER

$$\sigma^2 = \frac{5\times(2-4.3)^2+1\times(3-4.3)^2+4\times(4-4.3)^2+3\times(5-4.3)^2+7\times(6-4.3)^2}{20} \quad (=2.51)$$ ***(A1)***

OR

$$\sigma^2 = \frac{5\times2^2+1\times3^2+4\times4^2+3\times5^2+7\times6^2}{20} - 4.3^2 \quad (=2.51)$$ ***(A1)***

THEN

$\sigma=\sqrt{2.51}=1.58429...$

$=1.58$ ***A1***

[2 marks]

23 – (MathAA/22_HL_Summer_2022_Q4) – *Statistics*

(a) 0.28 (s) ***A1***

[1 mark]

(b) $\text{IQR} = 0.35 - 0.27\ (= 0.08)$ (s) ***(A1)***

substituting **their** IQR into correct expression for upper fence ***(A1)***

$0.35 + 1.5 \times 0.08\ (= 0.47)$ (s)

$0.46 < 0.47$ ***R1***

so 0.46 (s) is not an outlier ***AG***

[3 marks]

(c) **EITHER**

the median is closer to the lower quartile (positively skewed) ***R1***

OR

the distribution is positively skewed ***R1***

OR

the range of reaction times below the median is smaller than the range of reaction times above the median ***R1***

Note: These are sample answers from a range of acceptable correct answers. Award ***R1*** for any correct statement that explains this.

Do not award ***R1*** if there is also an incorrect statement, even if another statement in the answer is correct. Accept a correctly and clearly labelled diagram.

[1 mark]

(d) **EITHER**

the distribution for 'not sleeping well' is centred at a higher reaction time ***R1***

OR

the median reaction time after not sleeping well is equal to the upper quartile reaction time after sleeping well ***R1***

OR

75% of reaction times are <0.35 seconds after sleeping well, compared with 50% after not sleeping well ***R1***

OR

the sample size of 9 is too small to draw any conclusions ***R1***

Note: These are sample answers from a range of acceptable correct answers. Accept any relevant correct statement **that relates to the median and/or quartiles shown in the box plots**. Do not accept a comparison of means.

Do not award ***R1*** if there is also an incorrect statement, even if another statement in the answer is correct

Award ***R0*** to "correlation does not imply causation".

[1 mark]

24 – (MathAA/20_HL_Winter_2022_Q1) – *Statistics*

(a) $1.01206\ldots, 2.45230\ldots$

$a = 1.01$, $b = 2.45$ $(1.01x + 2.45)$ ***A1A1***

[2 marks]

(b) $0.981464\ldots$

$r = 0.981$ ***A1***

Note: A common error is to enter the data incorrectly into the GDC, and obtain the answers $a = 1.01700\ldots$, $b = 2.09814\ldots$ and $r = 0.980888\ldots$ Some candidates may write the 3 sf answers, ie. $a = 1.02$, $b = 2.10$ and $r = 0.981$ or 2 sf answers, ie. $a = 1.0$, $b = 2.1$ and $r = 0.98$. In these cases award ***A0A0*** for part (a) and ***A0*** for part (b). Even though some values round to an accepted answer, they come from incorrect working.

[1 mark]

(c) correct substitution of 78 into **their** regression equation ***(M1)***

$81.3930\ldots$, 81.23 from 3 sf answer

81 ***A1***

[2 marks]

25 – (MathAA/20_HL_Winter_2022_Q6) – *Statistics*

(a) $\int_0^b axe^x \mathrm{d}x = 1$ (seen anywhere) ***M1***

attempt to use integration by parts (either way around) ***(M1)***

$\left[axe^x\right]_0^b - \int_0^b ae^x \mathrm{d}x \left(=1\right)$ ***(A1)***

$\left[axe^x\right]_0^b - \left[ae^x\right]_0^b \left(=1\right)$ ***A1***

Note: Condone incorrect or absent limits up to this point.

$abe^b - ae^b + a = 1$

$a = \dfrac{1}{be^b - e^b + 1}$ ***A1***

[5 marks]

(b) $\int_0^m xe^x \mathrm{d}x = \dfrac{1}{2}$ ***(M1)***

$\left[xe^x\right]_0^m - \left[e^x\right]_0^m = \dfrac{1}{2}$

$me^m - e^m + 1 = \dfrac{1}{2}$ ***(A1)***

$m = 0.768039...$

$m = 0.768$ ***A1***

[3 marks]

1 – (MAT/21_HL_Summer_2012_Q2) – *Probability*

$$k\int_1^2 2^{\frac{1}{x}}\,\mathrm{d}x = 1 \Rightarrow k = \frac{1}{\int_1^2 2^{\frac{1}{x}}\,\mathrm{d}x} \quad (= 0.61556\ldots)$$

$$\mathrm{E}(X) = k\int_1^2 x2^{\frac{1}{x}}\,\mathrm{d}x = 2.39....k \text{ or } 1.47$$

2 – (MAT/22_HL_Summer_2012_Q2) – *Probability*

(a) $\mathrm{E}(X) = np$

$\Rightarrow 10 = 30p$

$\Rightarrow p = \frac{1}{3}$

(b) $\mathrm{P}(X = 10) = \binom{30}{10}\left(\frac{1}{3}\right)^{10}\left(\frac{2}{3}\right)^{20} = 0.153$

(c) $\mathrm{P}(X \geq 15) = 1 - \mathrm{P}(X \leq 14)$

$= 1 - 0.9565\ldots = 0.0435$

3 – (MAT/22_HL_Summer_2012_Q5) – *Probability*

(a) $\mathrm{P}(X = 5) = \mathrm{P}(X = 3) + \mathrm{P}(X = 4)$

$\frac{\mathrm{e}^{-m}m^5}{5!} = \frac{\mathrm{e}^{-m}m^3}{3!} + \frac{\mathrm{e}^{-m}m^4}{4!}$

$m^2 - 5m - 20 = 0$

$\Rightarrow m = \frac{5 + \sqrt{105}}{2} = (7.62)$

(b) $\mathrm{P}(X > 2) = 1 - \mathrm{P}(X \leq 2)$

$= 1 - 0.018\ldots$

$= 0.982$

4 - (MAT/21_HL_Summer_2012_Q7) - *Probability*

$X \sim \mathrm{Po}(m)$

$\mathrm{P}(X=2)=\mathrm{P}(X<2)$

$\frac{1}{2}m^2\mathrm{e}^{-m}=\mathrm{e}^{-m}(1+m)$

$m=2.73 \quad \left(1+\sqrt{3}\right)$

in four hours the expected value is 10.9 $\left(4+4\sqrt{3}\right)$

5 - (MAT/22_HL_Summer_2012_Q7) - *Probability*

(a) $\int_0^a \frac{1}{1+x^4}\,\mathrm{d}x=1$

$a=1.40$

(b) $\mathrm{E}(X)=\int_0^a \frac{x}{1+x^4}\,\mathrm{d}x$

$\left(=\frac{1}{2}\arctan(a^2)\right)$

$=0.548$

6 – (MAT/22_HL_Summer_2012_Q10) – *Probability, Statistics*

(a) (i) $\mathrm{P}(X > 225) = 0.158\ldots$

expected number $= 450 \times 0.158\ldots = 71.4$

(ii) $\mathrm{P}(X < m) = 0.7$

$\Rightarrow m = 213$ (grams)

(b) $\dfrac{270-\mu}{\sigma} = 1.40\ldots$

$\dfrac{250-\mu}{\sigma} = -1.03\ldots$

Note: These could be seen in graphical form.

solving simultaneously

$\mu = 258,\ \sigma = 8.19$

(c)

$X \sim \mathrm{N}(80, 4^2)$

$\mathrm{P}(X > 82) = 0.3085\ldots$

recognition of the use of binomial distribution.

$X \sim \mathrm{B}(5, 0.3085\ldots)$

$\mathrm{P}(X = 3) = 0.140$

7 – (MAT/21_HL_Summer_2012_Q12) – *Probability, Statistics*

(a) (i) $X \sim \text{Po}(11)$

$\text{P}(X \le 11) = 0.579$

(ii) $\text{P}(X > 8 | X < 12) =$

$= \dfrac{\text{P}(8 < X < 12)}{\text{P}(X < 12)} \left(\text{or } \dfrac{\text{P}(X \le 11) - \text{P}(X \le 8)}{\text{P}(X \le 11)} \text{ or } \dfrac{0.3472...}{0.5792...} \right)$

$= 0.600$

(b) (i) $Y \sim \text{Po}(m)$

$\text{P}(Y > 3) = 0.24$

$\text{P}(Y \le 3) = 0.76$

$e^{-m}\left(1 + m + \dfrac{1}{2}m^2 + \dfrac{1}{6}m^3\right) = 0.76$

Note: At most two of the above lines can be implied.

Attempt to solve equation with GDC

$m = 2.49$

(ii) $A \sim \text{Po}(4.98)$

$\text{P}(A > 5) = 1 - \text{P}(A \le 5) = 0.380\ldots$

$W \sim \text{B}(4, 0.380\ldots)$

$\text{P}(W \ge 2) = 1 - \text{P}(W \le 1) = 0.490$

(c) $\text{P}(A < 25) = 0.8$, $\text{P}(A < 18) = 0.4$

$\dfrac{25 - \mu}{\sigma} = 0.8416\ldots$

$\dfrac{18 - \mu}{\sigma} = -0.2533\ldots$ (or -0.2534 from tables)

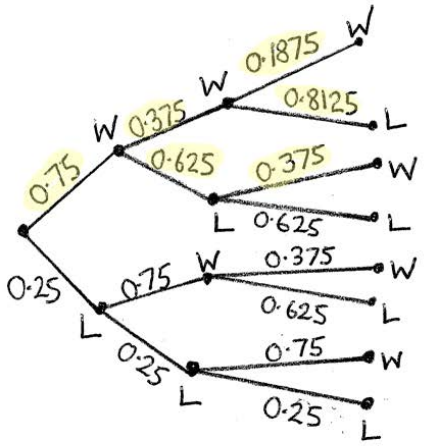
solving these equations

$\mu = 19.6$

8 – (MAT/20_HL_Winter_2012_Q7) – *Probability*

(a) $P(WWW) = 0.75 \times 0.375 \times 0.1875 = 0.0527$ (3sf) $\left(\frac{3}{4} \times \frac{3}{8} \times \frac{3}{16} = \frac{27}{512}\right)$

(b)



Note:	Award ***M1*** for any reasonable attempt to use a tree diagram showing that three games were played (do not award ***M1*** for tree diagrams that only show the first two games) and ***A1*** for the highlighted probabilities.

$$P(\text{wins 2 games} \mid \text{wins first game}) = \frac{P(WWL, WLW)}{P(\text{wins first game})}$$

$$= \frac{0.75 \times 0.375 \times 0.8125 + 0.75 \times 0.625 \times 0.375}{0.75}$$

$$= 0.539 \text{ (3sf)} \left(\text{or } \frac{69}{128}\right)$$

9 – (MAT/20_HL_Winter_2012_Q11) – *Probability, Statistics*

(a) $2.2 \times 6 \times 60 = 792$

(b) $V \sim \text{Po}(2.2 \times 60)$
$\text{P}(V > 100) = 0.998$

(c) $(0.997801\ldots)^6 = 0.987$

(d) $A \sim \text{N}(\mu, \sigma^2)$
$\text{P}(A < 35) = 0.29$ and $\text{P}(A > 55) = 0.23 \Rightarrow \text{P}(A < 55) = 0.77$
$\text{P}\left(Z < \frac{35-\mu}{\sigma}\right) = 0.29$ and $\text{P}\left(Z < \frac{55-\mu}{\sigma}\right) = 0.77$
use of inverse normal
$\frac{35-\mu}{\sigma} = -0.55338\ldots$ and $\frac{55-\mu}{\sigma} = 0.738846\ldots$
solving simultaneously
$\mu = 43.564...$ and $\sigma = 15.477...$
$\mu = 43.6$ and $\sigma = 15.5\,(3\text{sf})$

(e) $0.29n = 100 \Rightarrow n = 344.82\ldots$
$\text{P}(A < 50) = 0.66121\ldots$
expected number of visitors under $50 = 228$

10 – (MAT/21_HL_Summer_2013_Q3) – *Probability*

Let X represent the length of time a journey takes on a particular day.

(a) $P(X > 15) = 0.0912112819\ldots = 0.0912$

(b) Use of correct Binomial distribution
$N \sim B(5, 0.091...)$
$1 - 0.0912112819\ldots = 0.9087887181\ldots$
$1 - (0.9087887181\ldots)^5 = 0.380109935\ldots = 0.380$

11 - (MAT/22_HL_Summer_2013_Q3) - *Probability*

(a) $X \sim N(13.5, 9.5)$

$13.5 - \sqrt{9.5} < X < 13.5 + \sqrt{9.5}$

$10.4 < X < 16.6$

Note: Accept 6.16.

(b) $\mathrm{P}(X < 10) = 0.12807\ldots$

estimate is 1281 (correct to the nearest whole number).

12 – (MAT/22_HL_Summer_2013_Q7) – *Probability*

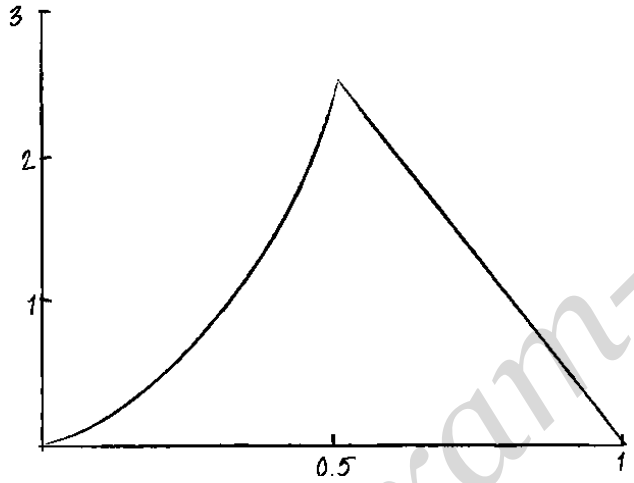
(a) $\int_0^{0.5} ax^2 \mathrm{d}x + \int_{0.5}^{1} 0.5a(1-x)\mathrm{d}x = 1$

$\frac{5a}{48}$ (or equivalent) or $a \times 0.104... = 1$

Note:	Award ***M1*** for considering two definite integrals. Award ***A1*** for equating to 1. Award ***A1*** for a correct equation. The ***A1A1*** can be awarded in any order.

$a = 9.6$

(b)



correct shape for $0 \le x \le 0.5$ and $f(0.5) \approx 2.4$

correct shape for $0.5 \le x \le 1$ and $f(1) = 0$

(c) attempting to find $\mathrm{P}(X < 0.6)$

direct GDC use or *eg* $\mathrm{P}(0 \le X \le 0.5) + \mathrm{P}(0.5 \le X \le 0.6)$ or $1 - \mathrm{P}(0.6 \le X \le 1)$

$\mathrm{P}(X < 0.6) = 0.616 \left(= \frac{77}{125}\right)$

13 – (MAT/22_HL_Summer_2013_Q9) – *Probability*

(a) $X \sim \text{Po}(1.2)$

$\text{P}(X=3)\times\text{P}(X=0)$

$=0.0867\ldots\times 0.3011\ldots$

$=0.0261$

(b) Three requests over two days can occur as $(3,0),(0,3),(2,1)$ or $(1,2)$.

using conditional probability, for example

$$\frac{\text{P}(3,0)}{\text{P}(3\text{ requests, }m=2.4)}=0.125 \text{ or } \frac{\text{P}(2,1)}{\text{P}(3\text{ requests, }m=2.4)}=0.375$$

expected income is

$2\times 0.125\times\text{US\$}120+2\times 0.375\times\text{US\$}180$

Note: Award ***M1*** for attempting to find the expected income including both $(3,0)$ and $(2,1)$ cases.

$=\text{US\$}30+\text{US\$}135$

$=\text{US\$}165$

14 – (MAT/21_HL_Summer_2013_Q10) – *Probability*

(a) $X \sim \text{Po}(0.25\text{T})$

Attempt to solve $\text{P}(X\le 3)=0.6$

$T=12.8453\ldots\ =\ 13\text{ (minutes)}$

Note: Award ***A1M1A0*** if T found correctly but not stated to the nearest minute.

(b) let X_1 be the number of cars that arrive during the first interval and X_2 be the number arriving during the second.

X_1 and X_2 are $\text{Po}(2.5)$

$\text{P (all get on)}=\text{P}(X_1\le 3)\times\text{P}(X_2\le 3)+\text{P}(X_1=4)\times\text{P}(X_2\le 2)$

$+\text{P}(X_1=5)\times\text{P}(X_2\le 1)+\text{P}(X_1=6)\times\text{P}(X_2=0)$

$=0.573922\ldots+0.072654\ldots+0.019192\ldots+0.002285\ldots$

$=0.668\ (053\ldots)$

15 – (MAT/22_HL_Summer_2013_Q11) – *Sequences - Series, Probability*

(a) (i) $\sum_{k=1}^{n}(2k-1)$ (or equivalent)

Note: Award ***A0*** for $\sum_{n=1}^{n}(2n-1)$ or equivalent.

(ii) **EITHER**

$2\times\frac{n(n+1)}{2}-n$

OR

$\frac{n}{2}\left(2+(n-1)2\right)$ (using $S_n=\frac{n}{2}\left(2u_1+(n-1)d\right)$)

OR

$\frac{n}{2}(1+2n-1)$ (using $S_n=\frac{n}{2}(u_1+u_n)$)

THEN

$=n^2$

(iii) $47^2-14^2=2013$

(b) (i) **EITHER**

a pentagon and five diagonals

OR

five diagonals (circle optional)

(ii) Each point joins to $n-3$ other points.
a correct argument for $n(n-3)$
a correct argument for $\frac{n(n-3)}{2}$

(iii) attempting to solve $\frac{1}{2}n(n-3)>1000000$ for n.

$n>1415.7$
$n=1416$

(c) (i) $np = 4$ and $npq = 3$

attempting to solve for n and p

$n = 16$ and $p = \frac{1}{4}$

(ii) $X \sqcup \mathrm{B}(16, 0.25)$

$\mathrm{P}(X = 1) = 0.0534538\ldots (= \binom{16}{1}(0.25)(0.75)^{15})$

$\mathrm{P}(X = 3) = 0.207876\ldots (= \binom{16}{3}(0.25)^3(0.75)^{13})$

$\mathrm{P}(X = 1) + \mathrm{P}(X = 3)$

$= 0.261$

16 – (MAT/20_HL_Winter_2013_Q4) – *Probability*

$\mathrm{P}\left(Z < \frac{780 - \mu}{\sigma}\right) = 0.92$ and $\mathrm{P}\left(Z < \frac{755 - \mu}{\sigma}\right) = 0.12$

use of inverse normal

$\Rightarrow \frac{780 - \mu}{\sigma} = 1.405\ldots$ and $\frac{755 - \mu}{\sigma} = -1.174\ldots$

solving simultaneously

Note: Award ***M1*** for attempting to solve an incorrect pair of equations *eg,* inverse normal not used.

$\mu = 766.385$

$\sigma = 9.6897$

$\mu = 12$ hrs 46 mins ($= 766$ mins)

$\sigma = 10$ mins

17 – (MAT/20_HL_Winter_2013_Q5) – *Probability*

(a) $\mathrm{P}(F)=\left(\frac{1}{7}\times\frac{7}{9}\right)+\left(\frac{6}{7}\times\frac{4}{9}\right)$

Note: Award ***M1*** for the sum of two products.

$=\frac{31}{63}\ (=0.4920\ldots)$

(b) Use of $\mathrm{P}(S\mid F)=\frac{\mathrm{P}(S\cap F)}{\mathrm{P}(F)}$ to obtain $\mathrm{P}(S\mid F)=\frac{\frac{1}{7}\times\frac{7}{9}}{\frac{31}{63}}$.

Note: Award ***M1*** only if the numerator results from the product of two probabilities.

$=\frac{7}{31}\ (=0.2258\ldots)$

18 – (MAT/20_HL_Winter_2013_Q11) – *Probability, Statistics*

(a) (i) $X \sim \text{Po}(0.6)$

$\text{P}(X=0)=0.549 \ \left(=\text{e}^{-0.6}\right)$ ***A1***

(ii) $\text{P}(X \geq 3)=1-\text{P}(X \leq 2)$ ***(M1)(A1)***

$=1-\left(\text{e}^{-0.6}+\text{e}^{-0.6}\times 0.6+\text{e}^{-0.6}\times\frac{0.6^2}{2}\right)$

$=0.0231$ ***A1***

(iii) $Y \sim \text{Po}(2.4)$ ***(M1)***

$\text{P}(Y \leq 5)=0.964$ ***A1***

(iv) $Z \sim \text{B}(12,\ 0.451...)$ ***(M1)(A1)***

Note: Award ***M1*** for recognising binomial and ***A1*** for using correct parameters.

$\text{P}(Z=4)=0.169$ ***A1***

(b) (i) $k\int_1^3 \ln x\,\text{d}x=1$ ***(M1)***

$(k\times 1.2958...=1)$

$k=0.771702$ ***A1***

(ii) $\text{E}(X)=\int_1^3 kx\ln x\,\text{d}x$ ***(A1)***

attempting to evaluate their integral ***(M1)***

$=2.27$ ***A1***

(iii) $x=3$ ***A1***

(iv) $\int_1^m k\ln x\,\text{d}x=0.5$ ***(M1)***

$k[x\ln x-x]_1^m=0.5$

attempting to solve for m ***(M1)***

$m=2.34$ ***A1***

19 – (MAT/22_HL_Summer_2014_Q2) – *Statistics, Probability*

(a) **METHOD 1**

$\mu = \frac{1}{2} \times (17.1 + 21.3)$

$\mu = 19.2\,(\text{kg})$

finding z value for the upper quartile $= 0.674489\text{K}$

$0.674489\text{K} = \frac{21.3 - 19.2}{\sigma}$ or $-0.674489\text{K} = \frac{17.1 - 19.2}{\sigma}$

$\sigma = 3.11\,(\text{kg})$

METHOD 2

finding z value for the upper quartile $= 0.674489\text{K}$
from symmetry the z value for a lower quartile is -0.674489K
forming two simultaneous equations:

$-0.674489\text{K} = \frac{17.1 - \mu}{\sigma}$

$0.674489\text{K} = \frac{21.3 - \mu}{\sigma}$

solving gives:

$\mu = 19.2\,(\text{kg})$

$\sigma = 3.11\,(\text{kg})$

(b) using $100 \times \text{P}(X > 22) = 100 \times 0.184241\text{K}$

$= 18$

20 – (MAT/22_HL_Summer_2014_Q6) – *Probability*

(a) (i) $0.6^3 \times 0.4^3$

Note: Award ***(M1)*** for use of the product of probabilities.

$= 0.0138$

(ii) binomial distribution X : $\text{B}(6,\ 0.6)$

Note: Award ***(M1)*** for recognizing the binomial distribution.

$\text{P}(X = 3) = {}^6C_3\,(0.6)^3\,(0.4)^3$

$= 0.276$

(b) $Y : \mathrm{B}(n, 0.4)$

$\mathrm{P}(Y \geq 1) > 0.995$

$1 - \mathrm{P}(Y = 0) > 0.995$

$\mathrm{P}(Y = 0) < 0.005$

Note: Award ***(M1)*** for any of the last three lines. Accept equalities.

$0.6^n < 0.005$

Note: Award ***(M1)*** for attempting to solve $0.6^n < 0.005$ using any method, *eg*, logs, graphically, use of solver. Accept an equality.

$n > 10.4$

$\therefore n = 11$

21 – (MAT/21_HL_Summer_2014_Q8) – *Binomial Theorem, Probability*

(a) $\left(A\binom{6}{5}2^5 B + 3\binom{6}{4}2^4 B^2 \right)x^5$

$= \left(192AB + 720B^2\right)x^5$

(b) **METHOD 1**

$x = \frac{1}{6},\ A = \frac{3}{6}\left(= \frac{1}{2}\right),\ B = \frac{4}{6}\left(= \frac{2}{3}\right)$

probability is $\frac{4}{81}$ $(= 0.0494)$

METHOD 2

$\mathrm{P}(5 \text{ eaten}) = \mathrm{P}(\text{M eats } 1)\,\mathrm{P}(\text{N eats } 4) + \mathrm{P}(\text{M eats } 0)\,\mathrm{P}(\text{N eats } 5)$

$= \frac{1}{2}\binom{6}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2 + \frac{1}{2}\binom{6}{5}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)$

$= \frac{4}{81}\ (= 0.0494)$

22 – (MAT/22_HL_Summer_2014_Q8) – *Probability*

(a) $\frac{\mu^2 e^{-\mu}}{2!}+\frac{\mu^3 e^{-\mu}}{3!}=\frac{\mu^5 e^{-\mu}}{5!}$

$\frac{\mu^2}{2}+\frac{\mu^3}{6}-\frac{\mu^5}{120}=0$

$\mu=5.55$

(b) $\sigma=\sqrt{5.55...}=2.35598...$

$\mathrm{P}(3.19\le X\le 7.9)$

$\mathrm{P}(4\le X\le 7)$

$=0.607$

23 – (MAT/21_HL_Summer_2014_Q9) – *Probability*

(a) mean for week is 40.88 ***(A1)***

$\mathrm{P}(S>40)=1-\mathrm{P}(S\le 40)=0.513$ ***A1***

(b) $\frac{\text{probability there were more than 10 on Monday AND more than 40 over the week}}{\text{probability there were more than 10 on Monday}}$

M1

possibilities for the numerator are:

there were more than 40 birds on the power line on Monday ***R1***
11 on Monday and more than 29 over the course of the next 6 days ***R1***
12 on Monday and more than 28 over the course of the next 6 days … until
40 on Monday and more than 0 over the course of the next 6 days ***R1***
hence if X is the number on the power line on Monday and Y, the number on the power line Tuesday – Sunday then the numerator is ***M1***

$\mathrm{P}(X>40)+\mathrm{P}(X=11)\times\mathrm{P}(Y>29)+\mathrm{P}(X=12)\times\mathrm{P}(Y>28)+\ldots$
$+\mathrm{P}(X=40)\times\mathrm{P}(Y>0)$

$=\mathrm{P}(X>40)+\sum_{r=11}^{40}\mathrm{P}(X=r)\mathrm{P}(Y>40-r)$

hence solution is $\frac{\mathrm{P}(X>40)+\sum_{r=11}^{40}\mathrm{P}(X=r)\mathrm{P}(Y>40-r)}{\mathrm{P}(X>10)}$ ***AG***

24 – (MAT/21_HL_Summer_2014_Q11) – *Probability*

(a) $\int_2^3 (ax+b)\,\mathrm{d}x (=1)$

$\left[\frac{1}{2}ax^2+bx\right]_2^3 (=1)$

$\frac{5}{2}a+b=1$

$5a+2b=2$

(b) (i) $\int_2^3 \left(ax^2+bx\right)\mathrm{d}x (=\mu)$

$\left[\frac{1}{3}ax^3+\frac{1}{2}bx^2\right]_2^3 (=\mu)$

$\frac{19}{3}a+\frac{5}{2}b=\mu$

eliminating b

eg

$\frac{19}{3}a+\frac{5}{2}\left(1-\frac{5}{2}a\right)=\mu$

$\frac{1}{12}a+\frac{5}{2}=\mu$

$a=12\mu-30$

Note: Elimination of b could be at different stages.

(ii) $b=1-\frac{5}{2}(12\mu-30)$

$=76-30\mu$

Note: This solution may be seen in part (i).

(c) (i) $\int_2^{2.3} (ax+b)\,\mathrm{d}x (=0.5)$

$\left[\frac{1}{2}ax^2+bx\right]_2^{2.3} (=0.5)$

$0.645a+0.3b(=0.5)$

$0.645(12\mu-30)+0.3(76-30\mu)=0.5$

$\mu=2.34\left(=\frac{295}{126}\right)$

(ii) $\mathrm{E}\left(X^2\right)=\int_2^3 x^2(ax+b)\,\mathrm{d}x$

$a=12\mu-30=-\frac{40}{21},\ b=76-30\mu=\frac{121}{21}$

$\mathrm{E}\left(X^2\right)=\int_2^3 x^2\left(-\frac{40}{21}x+\frac{121}{21}\right)\mathrm{d}x=5.539\ldots\left(=\frac{349}{63}\right)$

$\mathrm{Var}(X)=5.539\mathrm{K}-(2.341\mathrm{K})^2=0.05813\ldots$

$\sigma=0.241$

25 – (MAT/22_HL_Summer_2014_Q11) – *Probability*

(a) $a\int_0^{\frac{\pi}{2}} x\cos x\,\mathrm{d}x = 1$

integrating by parts:

$u = x \qquad v' = \cos x$

$u' = 1 \qquad v = \sin x$

$\int x\cos x\,\mathrm{d}x = x\sin x + \cos x$

$\left[x\sin x + \cos x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1$

$a = \dfrac{1}{\frac{\pi}{2} - 1}$

$= \dfrac{2}{\pi - 2}$

(b) $\mathrm{P}\left(X < \frac{\pi}{4}\right) = \frac{2}{\pi - 2}\int_0^{\frac{\pi}{4}} x\cos x\,\mathrm{d}x = 0.460$

Note: Accept $\frac{2}{\pi - 2}\left(= \frac{\pi\sqrt{2}}{8} + \frac{\sqrt{2}}{2} - 1\right)$ or equivalent

(c) (i) mode = 0.860

(x-value of a maximum on the graph over the given domain)

(ii) $\frac{2}{\pi - 2}\int_0^m x\cos x\,\mathrm{d}x = 0.5$

$\int_0^m x\cos x\,\mathrm{d}x = \frac{\pi - 2}{4}$

$m\sin m + \cos m - 1 = \frac{\pi - 2}{4}$

median = 0.826

(d) $\mathrm{P}\left(X<\frac{\pi}{8} \mid X<\frac{\pi}{4}\right)=\frac{\mathrm{P}\left(X<\frac{\pi}{8}\right)}{\mathrm{P}\left(X<\frac{\pi}{4}\right)}$

$=\frac{0.129912}{0.459826}$

$=0.283$

26 – (MAT/20_HL_Winter_2014_Q2) – *Probability*

(a) $\mathrm{P}(X>x)=0.99 \ (=\mathrm{P}(X<x)=0.01)$
$\Rightarrow x=54.6(\mathrm{cm})$

(b) $\mathrm{P}(60.15 \leq X \leq 60.25)$
$=0.0166$

27 – (MAT/20_HL_Winter_2014_Q11) – *Probability*

(a) (i) $\mathrm{P}(X=0)=0.549\left(=\mathrm{e}^{-0.6}\right)$

(ii) $\mathrm{P}(X \geq 3)=1-\mathrm{P}(X \leq 2)$
$\mathrm{P}(X \geq 3)=0.0231$

(b) **EITHER**

using $Y \sim \mathrm{Po}(3)$

OR

using $(0.549)^5$

THEN

$\mathrm{P}(Y=0)=0.0498\left(=\mathrm{e}^{-3}\right)$

(c) $\mathrm{P}(X=0)$ (most likely number of complaints received is zero)

EITHER

calculating $\mathrm{P}(X=0)=0.549$ and $\mathrm{P}(X=1)=0.329$

OR

sketching an appropriate (discrete) graph of $\mathrm{P}(X=x)$ against x

OR

finding $\mathrm{P}(X=0)=e^{-0.6}$ and stating that $\mathrm{P}(X=0)>0.5$

OR

using $\mathrm{P}(X=x)=\mathrm{P}(X=x-1)\times\frac{\mu}{x}$ where $\mu<1$

(d) $\mathrm{P}(X=0)=0.8(\Rightarrow e^{-\lambda}=0.8)$

$\lambda=0.223\left(=\ln\frac{5}{4},=-\ln\frac{4}{5}\right)$

28 – (MAT/20_HL_Winter_2014_Q12) – *Probability*

(a) $\mathrm{P}(\text{Ava wins on her first turn}) = \frac{1}{3}$

(b) $\mathrm{P}(\text{Barry wins on his first turn}) = \left(\frac{2}{3}\right)^2$

$= \frac{4}{9} (= 0.444)$

(c) $\mathrm{P}(\text{Ava wins in one of her first three turns})$

$= \frac{1}{3} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\frac{1}{3} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\frac{1}{3}$

Note: Award ***M1*** for adding probabilities, award ***A1*** for a correct second term and award ***A1*** for a correct third term.
Accept a correctly labelled tree diagram, awarding marks as above.

$= \frac{103}{243} (= 0.424)$

(d) $\mathrm{P}(\text{Ava eventually wins}) = \frac{1}{3} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\frac{1}{3} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\frac{1}{3} + \ldots$

using $S_\infty = \frac{a}{1-r}$ with $a = \frac{1}{3}$ and $r = \frac{2}{9}$

Note: Award ***(M1)*** for using $S_\infty = \frac{a}{1-r}$ and award ***(A1)*** for $a = \frac{1}{3}$ and $r = \frac{2}{9}$.

$= \frac{3}{7} (= 0.429)$

29 – (MAT/21_HL_Summer_2015_Q2) – *Probability*

(a) $X \sim N\left(210,\ 22^2\right)$

$\mathrm{P}\left(X < 180\right) = 0.0863$

(b) $\mathrm{P}\left(X < T\right) = 0.9 \Rightarrow T = 238$ (mins)

30 – (MAT/21_HL_Summer_2015_Q3) – *Probability*

(a) $W \sim B(1000,\ 0.1)$ (accept $C_k^{1000}\left(0.1\right)^k\left(0.9\right)^{1000-k}$)

Note: First ***A1*** is for recognizing the binomial, second ***A1*** for both parameters if stated explicitly in this part of the question.

(b) $\mu(=1000\times 0.1) = 100$

(c) $\mathrm{P}(W > 89) = \mathrm{P}(W \geq 90)\ \left(= 1 - \mathrm{P}(W \leq 89)\right)$

$= 0.867$

31 – (MAT/22_HL_Summer_2015_Q4) – *Probability*

(a) (i) $X \sim Po(5)$

$\mathrm{P}(X \geq 8) = 0.133$

(ii) $7 \times 0.133\ldots$

≈ 0.934 days

Note: Accept "1 day".

(b) $7 \times 5 = 35\ \left(Y \sim Po(35)\right)$

$\mathrm{P}(Y \leq 29) = 0.177$

32 - (MAT/21_HL_Summer_2015_Q7) - *Probability*

(a) $2\frac{e^{-m}m^4}{4!} = \frac{e^{-m}m^5}{5!}$

$\frac{2}{4!} = \frac{m}{5!}$ or other simplification

Note: accept a labelled graph showing clearly the solution to the equation. Do not accept simple verification that $m = 10$ is a solution.

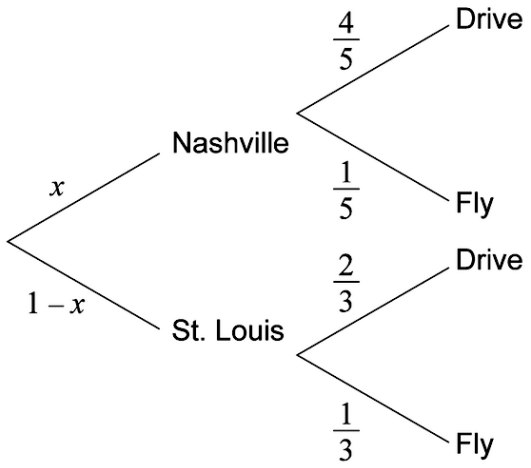
$\Rightarrow m = 10$

(b) $P(X = 6 \mid X \le 11) = \frac{P(X = 6)}{P(X \le 11)}$

$= \frac{0.063055...}{0.696776...}$

$= 0.0905$

33 – (MAT/22_HL_Summer_2015_Q9) – *Probability*



(a) attempt to set up the problem using a tree diagram and/or an equation, with the unknown x

$$\frac{4}{5}x+\frac{2}{3}(1-x)=\frac{13}{18}$$
$$\frac{4x}{5}-\frac{2x}{3}=\frac{13}{18}-\frac{2}{3}$$

$$\frac{2x}{15}=\frac{1}{18}$$
$$x=\frac{5}{12}$$

(b) attempt to set up the problem using conditional probability

EITHER

$$\frac{\frac{5}{12}\times\frac{1}{5}}{1-\frac{13}{18}}$$

OR

$$\frac{\frac{5}{12}\times\frac{1}{5}}{\frac{1}{12}+\frac{7}{36}}$$

THEN

$$=\frac{3}{10}$$

34 – (MAT/22_HL_Summer_2015_Q10) – *Probability*

(a) (i) $P(110 < X < 130) = = 0.49969... = 0.500 = 50.0\%$

Note: Accept 50

Note: Award ***M1A0*** for 0.50 (0.500)

(ii) $P(X > 130) = (1 - 0.707\ldots) = 0.293\ldots$
expected number of turnips $= 29.3$

Note: Accept 29.

(iii) no of turnips weighing more than 130 is $Y \sim B(100,\ 0.293)$
$P(Y \geq 30) = 0.478$

(b) (i) $X \sim N\left(144,\ \sigma^2\right)$

$P(X \leq 130) = \frac{1}{15} = 0.0667$

$P\left(Z \leq \frac{130 - 144}{\sigma}\right) = 0.0667$

$\frac{14}{\sigma} = 1.501$

$\sigma = 9.33\,g$

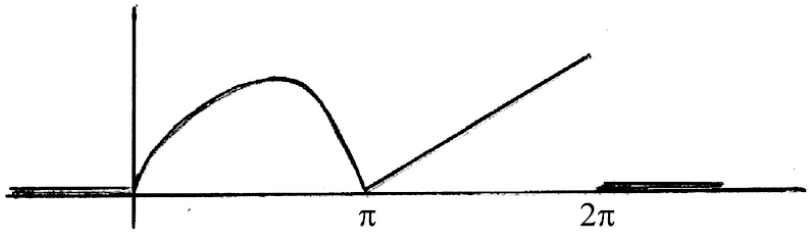
(ii) $P(X > 150 \mid X > 130) = \frac{P(X > 150)}{P(X > 130)}$

$= \frac{0.26008\ldots}{1 - 0.06667} = 0.279$

expected number of turnips $= 55.7$

35 – (MAT/21_HL_Summer_2015_Q11) – *Probability*

(a)



Award ***A1*** for sine curve from 0 to π, award ***A1*** for straight line from π to 2π

(b) $\int_0^{\pi} \frac{\sin x}{4} \mathrm{d}x = \frac{1}{2}$

(c) **METHOD 1**

require $\frac{1}{2} + \int_{\pi}^{2\pi} a(x-\pi)\,\mathrm{d}x = 1$

$\Rightarrow \frac{1}{2} + a\left[\frac{(x-\pi)^2}{2}\right]_{\pi}^{2\pi} = 1$ (or $\frac{1}{2} + a\left[\frac{x^2}{2} - \pi x\right]_{\pi}^{2\pi} = 1$)

$\Rightarrow a\frac{\pi^2}{2} = \frac{1}{2}$

$\Rightarrow a = \frac{1}{\pi^2}$

Note: Must obtain the exact value. Do not accept answers obtained with calculator.

METHOD 2

$0.5 +$ area of triangle $= 1$

area of triangle $= \frac{1}{2}\pi \times a\pi = 0.5$

Note: Award ***M1*** for correct use of area formula $= 0.5$, ***A1*** for $a\pi$.

$a = \frac{1}{\pi^2}$

(d) median is π

(e) $\mu = \int_0^{\pi} x \cdot \frac{\sin x}{4} \mathrm{d}x + \int_{\pi}^{2\pi} x \cdot \frac{x-\pi}{\pi^2} \mathrm{d}x$

$= 3.40339... = 3.40$ (or $\frac{\pi}{4} + \frac{5\pi}{6} = \frac{13}{12}\pi$)

(f) For $\mu = 3.40339...$

EITHER

$\sigma^2 = \int_0^{\pi} x^2 \cdot \frac{\sin x}{4} \mathrm{d}x + \int_{\pi}^{2\pi} x^2 \cdot \frac{x-\pi}{\pi^2} \mathrm{d}x - \mu^2$

OR

$\sigma^2 = \int_0^{\pi} (x-\mu)^2 \cdot \frac{\sin x}{4} \mathrm{d}x + \int_{\pi}^{2\pi} (x-\mu)^2 \cdot \frac{x-\pi}{\pi^2} \mathrm{d}x$

THEN

$= 3.866277... = 3.87$

(g) $\int_{\frac{\pi}{2}}^{\pi} \frac{\sin x}{4} \mathrm{d}x + \int_{\pi}^{\frac{3\pi}{2}} \frac{x-\pi}{\pi^2} \mathrm{d}x = 0.375$ (or $\frac{1}{4} + \frac{1}{8} = \frac{3}{8}$)

(h) $\mathrm{P}\left(\pi \le X \le 2\pi \middle| \frac{\pi}{2} \le X \le \frac{3\pi}{2}\right) = \frac{\mathrm{P}\left(\pi \le X \le \frac{3\pi}{2}\right)}{\mathrm{P}\left(\frac{\pi}{2} \le X \le \frac{3\pi}{2}\right)}$

$= \frac{\int_{\pi}^{\frac{3\pi}{2}} \frac{(x-\pi)}{\pi^2} \mathrm{d}x}{0.375} = \frac{0.125}{0.375}$ (or $= \frac{\frac{1}{8}}{\frac{3}{8}}$ from diagram areas)

$= \frac{1}{3}$ (0.333)

36 – (MAT/20_HL_Winter_2015_Q1) – *Probability*

(a) $0.818 = 0.65 + 0.48 - \mathrm{P}(A \cap B)$

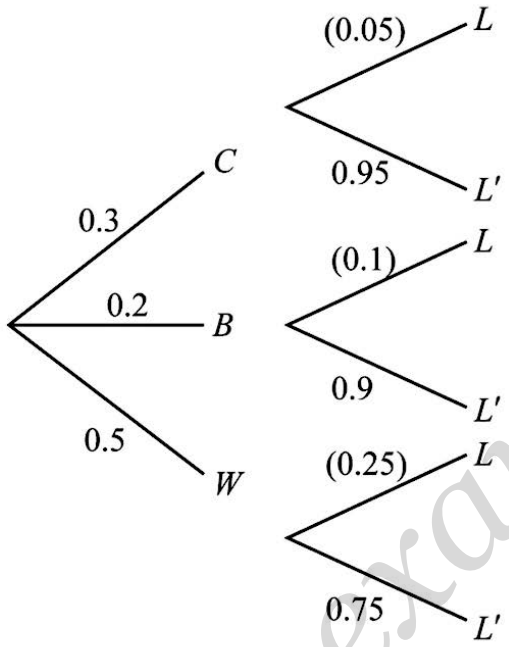
$\mathrm{P}(A \cap B) = 0.312$

(b) $\mathrm{P}(A)\,\mathrm{P}(B) = 0.312 \left(= 0.48 \times 0.65\right)$

since $\mathrm{P}(A)\,\mathrm{P}(B) = \mathrm{P}(A \cap B)$ then A and B are independent

37 – (MAT/20_HL_Winter_2015_Q6) – *Probability*

EITHER



Note: Award ***M1*** for a two-level tree diagram, ***A1*** for correct first level probabilities, and ***A1*** for correct second level probabilities.

OR

$$\mathrm{P}(B \mid L') = \frac{\mathrm{P}(L' \mid B)\,\mathrm{P}(B)}{\mathrm{P}(L' \mid B)\,\mathrm{P}(B) + \mathrm{P}(L' \mid C)\,\mathrm{P}(C) + \mathrm{P}(L' \mid W)\,\mathrm{P}(W)} \left(= \frac{\mathrm{P}(B \cap L')}{\mathrm{P}(L')}\right)$$

THEN

$$\mathrm{P}(B \mid L') = \frac{0.9 \times 0.2}{0.9 \times 0.2 + 0.95 \times 0.3 + 0.75 \times 0.5} \left(= \frac{0.18}{0.84}\right)$$

$$= 0.214 \left(= \frac{3}{14}\right)$$

38 – (MAT/20_HL_Winter_2015_Q8) – *Probability*

(a) $A\int_1^5 \sin(\ln x)\,\mathrm{d}x = 1$

$A = 0.323$ (3 dp only)

(b) either a graphical approach or $f'(x) = \dfrac{\cos(\ln x)}{x} = 0$

$x = 4.81\left(= \mathrm{e}^{\frac{\pi}{2}}\right)$

Note: Do not award ***A1FT*** for a candidate working in degrees.

(c) $\mathrm{P}(X \le 3 \mid X \ge 2) = \dfrac{\mathrm{P}(2 \le X \le 3)}{\mathrm{P}(X \ge 2)}\left(= \dfrac{\int_2^3 \sin(\ln(x))\mathrm{d}x}{\int_2^5 \sin(\ln(x))\mathrm{d}x}\right)$

$= 0.288$

39 – (MAT/20_HL_Winter_2015_Q11) – *Probability*

(a) (i) let W be the weight of a worker and $W \sim \mathrm{N}\left(\mu, \sigma^2\right)$

$\mathrm{P}\left(Z < \dfrac{62-\mu}{\sigma}\right) = 0.3$ and $\mathrm{P}\left(Z < \dfrac{98-\mu}{\sigma}\right) = 0.75$

$\dfrac{62-\mu}{\sigma} = \Phi^{-1}(0.3)\ (= -0.524\ldots)$ and

$\dfrac{98-\mu}{\sigma} = \Phi^{-1}(0.75)\ (= 0.674\ldots)$

or linear equivalents

(ii) attempting to solve simultaneously

$\mu = 77.7,\ \sigma = 30.0$

(b) $\mathrm{P}(W > 100) = 0.229$

(c) let X represent the number of workers over $100\,\mathrm{kg}$ in a lift of ten passengers

$X \sim \mathrm{B}(10,\ 0.229\ldots)$

$\mathrm{P}(X \ge 4) = 0.178$

(d) $\mathrm{P}(X < 4 | X \geq 1) = \dfrac{\mathrm{P}(1 \leq X \leq 3)}{\mathrm{P}(X \geq 1)}$

Note: Award the ***M1*** for a clear indication of conditional probability.

$= 0.808$

(e) $L \sim \mathrm{Po}(50)$
$\mathrm{P}(L > 60) = 1 - \mathrm{P}(L \leq 60)$
$= 0.0722$

(f) 400 workers require at least 40 elevators
$\mathrm{P}(L \geq 40) = 1 - \mathrm{P}(L \leq 39)$
$= 0.935$

40 – (MAT/22_HL_Summer_2016_Q2) – *Probability*

(a) $\mathrm{P}(0 \leq X \leq 2) = 0.242$

(b) **METHOD 1**

$\mathrm{P}(|X| > 1) = \mathrm{P}(X < -1) + \mathrm{P}(X > 1)$
$= 0.02275\ldots + 0.84134\ldots$
$= 0.864$

METHOD 2

$\mathrm{P}(|X| > 1) = 1 - \mathrm{P}(-1 < X < 1)$
$= 1 - 0.13590\ldots$
$= 0.864$

(c) $c = 3.30$

41 – (MAT/21_HL_Summer_2016_Q6) – *Probability*

let the heights of the students be X

$\mathrm{P}(X < 1.62) = 0.4$, $\mathrm{P}(X > 1.79) = 0.25$

Note: Award ***M1*** for either of the probabilities above.

$\mathrm{P}\left(Z < \frac{1.62 - \mu}{\sigma}\right) = 0.4$, $\mathrm{P}\left(Z < \frac{1.79 - \mu}{\sigma}\right) = 0.75$

Note: Award ***M1*** for either of the expressions above.

$\frac{1.62 - \mu}{\sigma} = -0.2533\ldots$, $\frac{1.79 - \mu}{\sigma} = 0.6744\ldots$

Note: ***A1*** for both values correct.

$\mu = 1.67(\mathrm{m})$, $\sigma = 0.183(\mathrm{m})$

Note: Accept answers that round to $1.7(\mathrm{m})$ and $0.18(\mathrm{m})$.

Note: Accept answers in centimetres.

42 – (MAT/22_HL_Summer_2016_Q6) – *Probability*

(a) $X \sim \mathrm{Po}(0.5)$

$\mathrm{P}(X \geq 1) = 0.393 \ \left(= 1 - \mathrm{e}^{-0.5}\right)$

(b) $\mathrm{P}(X = 0) = 0.607\ldots$

$\mathrm{E}(P) = (0.607\ldots \times 5) - (0.393\ldots \times 3)$

the expected profit is $\$1.85$ per glass sheet

(c) $Y \sim \mathrm{Po}(2)$

$\mathrm{P}(Y = 0) = 0.135 \ \left(= \mathrm{e}^{-2}\right)$

43 – (MAT/21_HL_Summer_2016_Q10) – *Probability*

$\mathrm{P}(3 \text{ in the first hour}) = \dfrac{\lambda^3 e^{-\lambda}}{3!}$

number to arrive in the four hours follows $Po(4\lambda)$

$\mathrm{P}(5 \text{ arrive in total}) = \dfrac{(4\lambda)^5 e^{-4\lambda}}{5!}$

attempt to find $\mathrm{P}(2 \text{ arrive in the next three hours})$

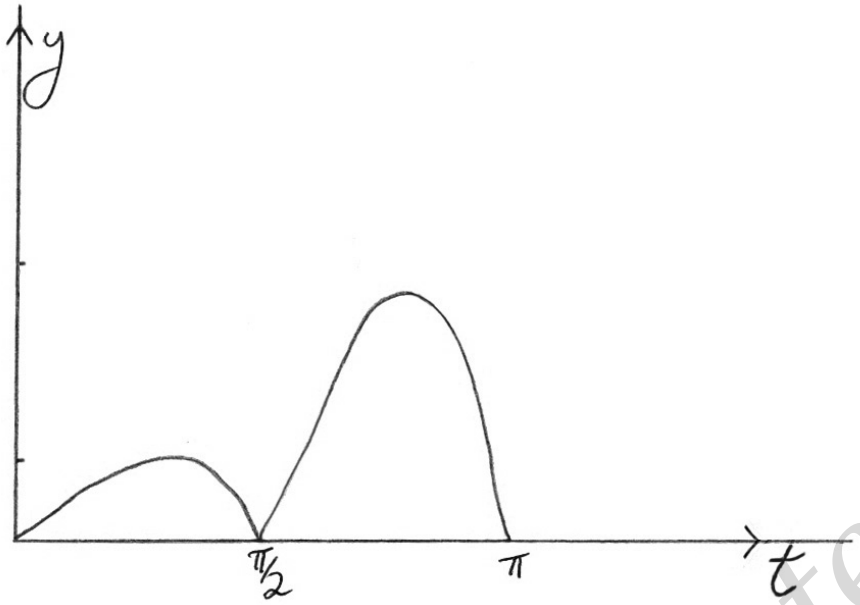
$= \dfrac{(3\lambda)^2 e^{-3\lambda}}{2!}$

use of conditional probability formula

$$\mathrm{P}(3 \text{ in the first hour given } 5 \text{ in total}) = \frac{\dfrac{\lambda^3 e^{-\lambda}}{3!} \times \dfrac{(3\lambda)^2 e^{-3\lambda}}{2!}}{\dfrac{(4\lambda)^5 e^{-4\lambda}}{5!}}$$

$$\frac{\left(\dfrac{9}{2!3!}\right)}{\left(\dfrac{4^5}{5!}\right)} = \frac{45}{512} = 0.0879$$

44 – (MAT/22_HL_Summer_2016_Q10) – *Probability*



(a) two enclosed regions ($0 \le t \le \frac{\pi}{2}$ and $\frac{\pi}{2} \le t \le \pi$) bounded by the curve and the t-axis

correct non-symmetrical shape for $0 \le t \le \frac{\pi}{2}$ and

$\frac{\pi}{2} <$ mode of $T < \pi$ clearly apparent

(b) $\text{mode} = 2.46$

(c) $\mathrm{E}(T) = \frac{1}{\pi}\int_0^{\pi} t^2 \left|\sin 2t\right| \mathrm{d}t$

$= 2.04$

(d) **EITHER**

$$\mathrm{Var}(T) = \int_0^{\pi} (t - 2.03788\ldots)^2 \left(\frac{t|\sin 2t|}{\pi} \right) \mathrm{d}t$$

OR

$$\mathrm{Var}(T) = \int_0^{\pi} t^2 \left(\frac{t|\sin 2t|}{\pi} \right) \mathrm{d}t - (2.03788\ldots)^2$$

THEN

$$\mathrm{Var}(T) = 0.516$$

(e) $\frac{1}{\pi} \int_{2.03788\ldots}^{2.456590\ldots} t|\sin 2t| \mathrm{d}t = 0.285$

(f) (i) attempting integration by parts

($u = t$, $\mathrm{d}u = \mathrm{d}t$, $\mathrm{d}v = \sin 2t \ \mathrm{d}t$ and $v = -\frac{1}{2}\cos 2t$)

$$\frac{1}{\pi}\left[t\left(-\frac{1}{2}\cos 2t \right) \right]_0^T - \frac{1}{\pi}\int_0^T \left(-\frac{1}{2}\cos 2t \right) \mathrm{d}t$$

Note: Award ***A1*** if the limits are not included.

$$= \frac{\sin 2T}{4\pi} - \frac{T\cos 2T}{2\pi}$$

(ii) $\frac{\sin \pi}{4\pi} - \frac{\frac{\pi}{2}\cos \pi}{2\pi} = \frac{1}{4}$

as $\mathrm{P}\left(0 \le T \le \frac{\pi}{2} \right) = \frac{1}{4}$ (or equivalent), then the lower quartile of T is $\frac{\pi}{2}$.

45 – (MAT/21_HL_Summer_2016_Q13) – *Statistics, Probability*

(a) $\mathrm{E}(X) = 1 \times \frac{1}{6} + 2 \times \frac{2}{6} + 3 \times \frac{3}{6} = \frac{14}{6}\left(= \frac{7}{3} = 2.33\right)$

(b) (i) $3 \times \mathrm{P}(113) + 3 \times \mathrm{P}(122)$

$3 \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{2} + 3 \times \frac{1}{6} \times \frac{1}{3} \times \frac{1}{3} = \frac{7}{72} (= 0.0972)$

Note: Award ***M1*** for attempt to find at least four of the cases.

(ii) recognising 111 as a possibility (implied by $\frac{1}{216}$)

recognising 112 and 113 as possibilities (implied by $\frac{2}{216}$ and $\frac{3}{216}$)

seeing the three arrangements of 112 and 113

$\mathrm{P}(111) + 3 \times \mathrm{P}(112) + 3 \times \mathrm{P}(113)$

$= \frac{1}{216} + \frac{6}{216} + \frac{9}{216} = \frac{16}{216}\left(= \frac{2}{27} = 0.0741\right)$

(c) let the number of twos be X, $X \sim B\left(10, \frac{1}{3}\right)$

$\mathrm{P}(X < 4) = \mathrm{P}(X \le 3) = 0.559$

(d) let n be the number of balls drawn

$\mathrm{P}(X \ge 1) = 1 - \mathrm{P}(X = 0)$

$= 1 - \left(\frac{2}{3}\right)^n > 0.95$

$\left(\frac{2}{3}\right)^n < 0.05$

$n = 8$

(e) $8p_1 = 4.8 \Rightarrow p_1 = \frac{3}{5}$

$8p_2(1 - p_2) = 1.5$

$p_2^2 - p_2 - 0.1875 = 0$

$p_2 = \frac{1}{4}\left(\text{or } \frac{3}{4}\right)$

reject $\frac{3}{4}$ as it gives a total greater than one

$\mathrm{P}(1 \text{ or } 2) = \frac{17}{20}$ or $\mathrm{P}(3) = \frac{3}{20}$

recognising LCM as 20 so min total number is 20

the least possible number of 3's is 3

46 – (MAT/20_HL_Winter_2016_Q1) – *Probability*

(a) $\mathrm{E}\left(X^2\right) = \sum x^2 \cdot \mathrm{P}(X = x) = 10.37 \ (= 10.4$ 3 sf) ***(M1)A1***

[2 marks]

(b) **METHOD 1**

$\mathrm{sd}(X) = 1.44069\ldots$ ***(M1)(A1)***

$\mathrm{Var}(X) = 2.08 \ \ (= 2.0756)$ ***A1***

METHOD 2

$\mathrm{E}(X) = 2.88 \ (= 0.06 + 0.27 + 0.5 + 0.98 + 0.63 + 0.44)$ ***(A1)***

use of $\mathrm{Var}(X) = \mathrm{E}\left(X^2\right) - \left(\mathrm{E}(X)\right)^2$ ***(M1)***

Note: Award ***(M1)*** only if $\left(\mathrm{E}(X)\right)^2$ is used correctly.

$\left(\mathrm{Var}(X) = 10.37 - 8.29\right)$

$\mathrm{Var}(X) = 2.08 \ (= 2.0756)$ ***A1***

Note: Accept 2.11.

METHOD 3

$\mathrm{E}(X) = 2.88 \ (= 0.06 + 0.27 + 0.5 + 0.98 + 0.63 + 0.44)$ ***(A1)***

use of $\mathrm{Var}(X) = \mathrm{E}\left(\left(X - \mathrm{E}(X)\right)^2\right)$ ***(M1)***

$(0.679728 + \ldots + 0.549152)$

$\mathrm{Var}(X) = 2.08 \ (= 2.0756)$ ***A1***

[3 marks]

47 – (MAT/20_HL_Winter_2016_Q3) – *Probability*

(a) **METHOD 1**

$\mathrm{P}(X = x+1) = \dfrac{\mu^{x+1}}{(x+1)!}\mathrm{e}^{-\mu}$ ***A1***

$= \dfrac{\mu}{x+1} \times \dfrac{\mu^{x}}{x!}\mathrm{e}^{-\mu}$ ***M1A1***

$= \dfrac{\mu}{x+1} \times \mathrm{P}(X = x)$ ***AG***

METHOD 2

$\dfrac{\mu}{x+1} \times \mathrm{P}(X = x) = \dfrac{\mu}{x+1} \times \dfrac{\mu^{x}}{x!}\mathrm{e}^{-\mu}$ ***A1***

$= \dfrac{\mu^{x+1}}{(x+1)!}\mathrm{e}^{-\mu}$ ***M1A1***

$= \mathrm{P}(X = x+1)$ ***AG***

METHOD 3

$\dfrac{\mathrm{P}(X = x+1)}{\mathrm{P}(X = x)} = \dfrac{\dfrac{\mu^{x+1}}{(x+1)!}\mathrm{e}^{-\mu}}{\dfrac{\mu^{x}}{x!}\mathrm{e}^{-\mu}}$ ***(M1)***

$= \dfrac{\mu^{x+1}}{\mu^{x}} \times \dfrac{x!}{(x+1)!}$ ***A1***

$= \dfrac{\mu}{x+1}$ ***A1***

and so $\mathrm{P}(X = x+1) = \dfrac{\mu}{x+1} \times \mathrm{P}(X = x)$ ***AG***

[3 marks]

(b) $\mathrm{P}(X = 3) = \dfrac{\mu}{3} \cdot \mathrm{P}(X = 2) \left(0.112777 = \dfrac{\mu}{3} \cdot 0.241667 \right)$ ***A1***

attempting to solve for μ ***(M1)***

$\mu = 1.40$ ***A1***

[3 marks]

48 – (MAT/20_HL_Winter_2016_Q8) – *Probability*

(a) $\mathrm{P}(X < 42.52) = 0.6940$ ***(M1)***

either $\mathrm{P}\left(Z < \frac{30.31-\mu}{\sigma}\right) = 0.1180$ or $\mathrm{P}\left(Z < \frac{42.52-\mu}{\sigma}\right) = 0.6940$ ***(M1)***

$\frac{30.31-\mu}{\sigma} = \underbrace{\Phi^{-1}(0.1180)}_{-1.1850\ldots}$ ***(A1)***

$\frac{42.52-\mu}{\sigma} = \underbrace{\Phi^{-1}(0.6940)}_{0.5072\ldots}$ ***(A1)***

attempting to solve simultaneously ***(M1)***

$\mu = 38.9$ and $\sigma = 7.22$ ***A1***

[6 marks]

(b) $\mathrm{P}(\mu - 1.2\sigma < X < \mu + 1.2\sigma)$ (or equivalent *eg.* $2\mathrm{P}(\mu < X < \mu + 1.2\sigma)$) ***(M1)***

$= 0.770$ ***A1***

Note: Award ***(M1)A1*** for $\mathrm{P}(-1.2 < Z < 1.2) = 0.770$.

[2 marks]

49 – (MAT/20_HL_Winter_2016_Q11) – *Probability*

(a) $\mathrm{P}(X=3)=(0.1)^3$ ***A1***

$=0.001$ ***AG***

$\mathrm{P}(X=4)=\mathrm{P}\left(VV\overline{V}V\right)+\mathrm{P}\left(V\overline{V}VV\right)+\mathrm{P}\left(\overline{V}VVV\right)$ ***(M1)***

$=3\times(0.1)^3\times 0.9$ (or equivalent) ***A1***

$=0.0027$ ***AG***

[3 marks]

(b) **METHOD 1**

attempting to form equations in a and b ***M1***

$\dfrac{9+3a+b}{2000}=\dfrac{1}{1000}\quad(3a+b=-7)$ ***A1***

$\dfrac{16+4a+b}{2000}\times\dfrac{9}{10}=\dfrac{27}{10\,000}\quad(4a+b=-10)$ ***A1***

attempting to solve simultaneously ***(M1)***

$a=-3,\ b=2$ ***A1***

METHOD 2

$\mathrm{P}(X=n)=\dbinom{n-1}{2}\times 0.1^3\times 0.9^{n-3}$ ***M1***

$=\dfrac{(n-1)(n-2)}{2000}\times 0.9^{n-3}$ ***(M1)A1***

$=\dfrac{n^2-3n+2}{2000}\times 0.9^{n-3}$ ***A1***

$a=-3,\ b=2$ ***A1***

Note: Condone the absence of 0.9^{n-3} in the determination of the values of a and b.

[5 marks]

(c) **METHOD 1**

EITHER

$$\mathrm{P}(X=n)=\frac{n^2-3n+2}{2000}\times 0.9^{n-3}$$ ***(M1)***

OR

$$\mathrm{P}(X=n)=\binom{n-1}{2}\times 0.1^3\times 0.9^{n-3}$$ ***(M1)***

THEN

$$=\frac{(n-1)(n-2)}{2000}\times 0.9^{n-3}$$ ***A1***

$$\mathrm{P}(X=n-1)=\frac{(n-2)(n-3)}{2000}\times 0.9^{n-4}$$ ***A1***

$$\frac{\mathrm{P}(X=n)}{\mathrm{P}(X=n-1)}=\frac{(n-1)(n-2)}{(n-2)(n-3)}\times 0.9$$ ***A1***

$$=\frac{0.9(n-1)}{n-3}$$ ***AG***

METHOD 2

$$\frac{\mathrm{P}(X=n)}{\mathrm{P}(X=n-1)}=\frac{\dfrac{n^2-3n+2}{2000}\times 0.9^{n-3}}{\dfrac{(n-1)^2-3(n-1)+2}{2000}\times 0.9^{n-4}}$$ ***(M1)***

$$=\frac{0.9\left(n^2-3n+2\right)}{\left(n^2-5n+6\right)}$$ ***A1A1***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator.

$$=\frac{0.9(n-1)(n-2)}{(n-2)(n-3)}$$ ***A1***

$$=\frac{0.9(n-1)}{n-3}$$ ***AG***

[4 marks]

(d) (i) attempting to solve $\frac{0.9(n-1)}{n-3}=1$ for n ***M1***

$n=21$ ***A1***

$\frac{0.9(n-1)}{n-3}<1 \Rightarrow n>21$ ***R1***

$\frac{0.9(n-1)}{n-3}>1 \Rightarrow n<21$ ***R1***

X has two modes ***AG***

Note: Award ***R1R1*** for a clearly labelled graphical representation of the two inequalities (using $\frac{\mathrm{P}(X=n)}{\mathrm{P}(X=n-1)}$).

(ii) the modes are 20 and 21 ***A1***

[5 marks]

(e) **METHOD 1**

$Y \sim \mathrm{B}(x, 0.1)$ ***(A1)***

attempting to solve $\mathrm{P}(Y \geq 3) > 0.5$ (or equivalent *eg* $1-\mathrm{P}(Y \leq 2) > 0.5$) for x ***(M1)***

Note: Award ***(M1)*** for attempting to solve an equality (obtaining $x=26.4$).

$x=27$ ***A1***

METHOD 2

$\sum_{n=0}^{x} \mathrm{P}(X=n) > 0.5$ ***(A1)***

attempting to solve for x ***(M1)***

$x=27$ ***A1***

[3 marks]

50 – (MAT/21_HL_Summer_2017_Q1) – *Probability*

(a) use of $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ ***M1***

$0.5 = k + 3k - k^2$ ***A1***

$k^2 - 4k + 0.5 = 0$

$k = 0.129$ ***A1***

Note: Do not award the final ***A1*** if two solutions are given.

[3 marks]

(b) use of $\mathrm{P}(A' \cap B) = \mathrm{P}(B) - \mathrm{P}(A \cap B)$ or alternative ***(M1)***

$\mathrm{P}(A' \cap B) = 3k - k^2$ ***(A1)***

$= 0.371$ ***A1***

[3 marks]

51 – (MAT/22_HL_Summer_2017_Q1) – *Probability, Statistics*

(a) $\mathrm{P}(5 \text{ or more}) = \frac{29}{75} (= 0.387)$ ***(M1)A1***

[2 marks]

(b) mean score $= \frac{2 \times 3 + 3 \times 15 + 4 \times 28 + 5 \times 17 + 6 \times 9 + 7 \times 3}{75}$ ***(M1)***

$= \frac{323}{75} (= 4.31)$ ***A1***

[2 marks]

52 – (MAT/22_HL_Summer_2017_Q3) – *Statistics, Probability*

(a) $\mathrm{P}(X < 250) = 0.0228$ ***(M1)A1***

[2 marks]

(b) $\frac{250 - \mu}{1.5} = -2.878\ldots$ ***(M1)(A1)***

$\Rightarrow \mu = 254.32$ ***A1***

Notes: Only award ***A1*** here if the correct 2dp answer is seen.
Award ***M0*** for use of 1.5^2.

[3 marks]

(c) $\frac{250 - 253}{\sigma} = -2.878\ldots$ ***(A1)***

$\Rightarrow \sigma = 1.04$ ***A1***

[2 marks]

53 – (MAT/21_HL_Summer_2017_Q5) – *Probability*

(a) $\lambda = 4 \times 0.5$ ***(M1)***

$\lambda = 2$ ***(A1)***

$\mathrm{P}(X \le 2) = 0.677$ ***A1***

[3 marks]

(b) $Y \sim B(10, 0.677)$ ***(M1)(A1)***

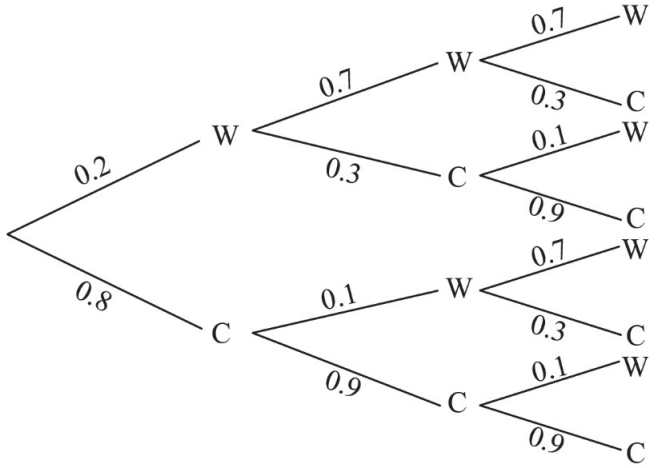
$\mathrm{P}(Y = 7) = 0.263$ ***A1***

Note: Award ***M1*** for clear recognition of binomial distribution.

[3 marks]

54 – (MAT/22_HL_Summer_2017_Q5) – *Probability*

(a)



M1A2

Note: Award ***M1*** for 3 stage tree-diagram, ***A2*** for 0.8,0.9,0.3 probabilities correctly placed.

[3 marks]

(b) $0.2 \times 0.7 \times 0.3 + 0.2 \times 0.3 \times 0.9 + 0.8 \times 0.1 \times 0.3 + 0.8 \times 0.9 \times 0.9 = 0.768$

(M1)A1

[2 marks]

(c) $\text{P(1st July is calm} \mid \text{3rd July is windy)} = \dfrac{\text{P(1st July is calm and 3rd July is windy)}}{\text{P(3rd July is windy)}}$

(M1)

$= \dfrac{0.8 \times 0.1 \times 0.7 + 0.8 \times 0.9 \times 0.1}{1 - 0.768}$

OR $\dfrac{0.8 \times 0.1 \times 0.7 + 0.8 \times 0.9 \times 0.1}{0.2 \times 0.7 \times 0.7 + 0.2 \times 0.3 \times 0.1 + 0.8 \times 0.1 \times 0.7 + 0.8 \times 0.9 \times 0.1}$

OR $\dfrac{0.128}{0.232}$ ***(A1)(A1)***

Note: Award ***A1*** for correct numerator, ***A1*** for correct denominator.

$= 0.552$ ***A1***

[4 marks]

55 – (MAT/21_HL_Summer_2017_Q9) – *Probability, Statistics*

(a) $T \sim N\left(196,\ 24^2\right)$

$\mathrm{P}(T < 180) = 0.252$ ***(M1)A1***

[2 marks]

(b) $\mathrm{P}\left(T < T_1\right) = 0.05$ ***(M1)***

$T_1 = 157$ ***A1***

[2 marks]

(c) $F \sim N\left(210,\ \sigma^2\right)$

$\mathrm{P}(F < 235) = 0.79$ ***(M1)***

$\dfrac{235 - 210}{\sigma} = 0.806421$ or equivalent ***(M1)(A1)***

$\sigma = 31.0$ ***A1***

[4 marks]

56 – (MAT/22_HL_Summer_2017_Q10) – *Probability, Statistics*

(a) $\int_0^4 \left(\dfrac{x^2}{a} + b\right)\mathrm{d}x = 1 \Rightarrow \left[\dfrac{x^3}{3a} + bx\right]_0^4 = 1 \Rightarrow \dfrac{64}{3a} + 4b = 1$ ***M1A1***

$\int_2^4 \left(\dfrac{x^2}{a} + b\right)\mathrm{d}x = 0.75 \Rightarrow \dfrac{56}{3a} + 2b = 0.75$ ***M1A1***

Note: $\int_0^2 \left(\dfrac{x^2}{a} + b\right)\mathrm{d}x = 0.25 \Rightarrow \dfrac{8}{3a} + 2b = 0.25$ could be seen/used in place of either of the above equations.

evidence of an attempt to solve simultaneously
(or check given a, b values are consistent) ***M1***

$a = 32,\ b = \dfrac{1}{12}$ ***AG***

[5 marks]

(b) $\mathrm{E}(X) = \int_0^4 x\left(\dfrac{x^2}{32} + \dfrac{1}{12}\right)\mathrm{d}x$ ***(M1)***

$\mathrm{E}(X) = \dfrac{8}{3}(= 2.67)$ ***A1***

[2 marks]

(c) $\mathrm{E}\left(X^2\right) = \int_0^4 x^2\left(\dfrac{x^2}{32} + \dfrac{1}{12}\right)\mathrm{d}x$ ***(M1)***

$\mathrm{Var(X)} = \mathrm{E}\left(X^2\right) - \left[\mathrm{E}(X)\right]^2 = \dfrac{16}{15}(= 1.07)$ ***A1***

[2 marks]

(d) $\int_0^m \left(\frac{x^2}{32} + \frac{1}{12}\right) dx = 0.5$ ***(M1)***

$\frac{m^3}{96} + \frac{m}{12} = 0.5 \left(\Rightarrow m^3 + 8m - 48 = 0\right)$ ***(A1)***

$m = 2.91$ ***A1***

[3 marks]

(e) $Y \sim B(8, 0.75)$ ***(M1)***

$E(Y) = 8 \times 0.75 = 6$ ***A1***

[2 marks]

(f) $\mathrm{P}(Y \geq 3) = 0.996$ ***A1***

[1 mark]

57 – (MAT/20_HL_Winter_2017_Q2) – *Probability*

(a) $\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

$\Rightarrow 0.75 = \frac{0.6}{\mathrm{P}(B)}$ ***(M1)***

$\Rightarrow \mathrm{P}(B) \left(= \frac{0.6}{0.75}\right) = 0.8$ ***A1***

[2 marks]

(b) $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$\Rightarrow 0.95 = \mathrm{P}(A) + 0.8 - 0.6$ ***(M1)***

$\Rightarrow \mathrm{P}(A) = 0.75$ ***A1***

[2 marks]

(c) **METHOD 1**

$\mathrm{P}(A' \mid B) = \frac{\mathrm{P}(A' \cap B)}{\mathrm{P}(B)} = \frac{0.2}{0.8} = 0.25$ ***A1***

$\mathrm{P}(A' \mid B) = \mathrm{P}(A')$ ***R1***

hence A' and B are independent ***AG***

Note: If there is evidence that the student has calculated $\mathrm{P}(A' \cap B) = 0.2$ by assuming independence in the first place, award ***A0R0***.

METHOD 2

EITHER

$\mathrm{P}(A) = \mathrm{P}(A|B)$ ***A1***

OR

$\mathrm{P}(A) \times \mathrm{P}(B) = 0.75 \times 0.80 = 0.6 = \mathrm{P}(A \cap B)$ ***A1***

THEN

A and B are independent ***R1***

hence A' and B are independent ***AG***

METHOD 3

$\mathrm{P}(A') \times \mathrm{P}(B) = 0.25 \times 0.80 = 0.2$ ***A1***

$\mathrm{P}(A') \times \mathrm{P}(B) = \mathrm{P}(A' \cap B)$ ***R1***

hence A' and B are independent ***AG***

[2 marks]

58 – (MAT/20_HL_Winter_2017_Q6) – *Probability*

(a) let X be the number of bananas eaten in one day

$X \sim \mathrm{Po}(0.2)$

$\mathrm{P}(X \geq 1) = 1 - \mathrm{P}(X = 0)$ ***(M1)***

$= 0.181 \left(= 1 - \mathrm{e}^{-0.2}\right)$ ***A1***

[2 marks]

(b) **EITHER**

let Y be the number of bananas eaten in one week

$Y \sim \mathrm{Po}(1.4)$ ***(A1)***

$\mathrm{P}(Y = 0) = 0.246596\ldots \left(= \mathrm{e}^{-1.4}\right)$ ***(A1)***

OR

let Z be the number of days in one week at least one banana is eaten

$Z \sim \mathrm{B}(7, 0.181\ldots)$ ***(A1)***

$\mathrm{P}(Z = 0) = 0.246596\ldots$ ***(A1)***

THEN

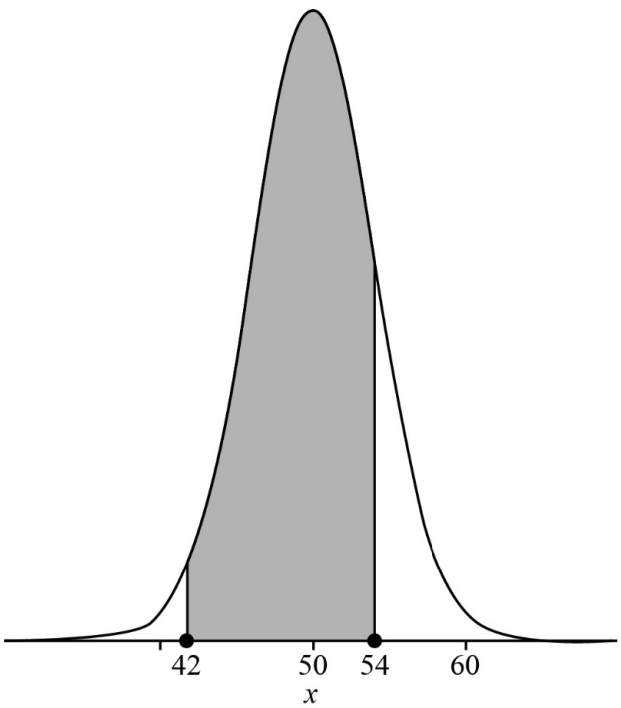
$52 \times 0.246596\ldots$ ***(M1)***

$= 12.8 \left(= 52\mathrm{e}^{-1.4}\right)$ ***A1***

[4 marks]

59 – (MAT/22_HL_Summer_2018_Q3) – *Probability*

(a)



normal curve centred on 50 ***A1***

vertical lines at $x = 42$ and $x = 54$, with shading in between ***A1***

[2 marks]

(b) $\mathrm{P}(42 < X < 54)\ \left(= \mathrm{P}(-2 < Z < 1)\right)$ ***(M1)***

$= 0.819$ ***A1***

[2 marks]

(c) $\mathrm{P}(\mu - k\sigma < X < \mu + k\sigma) = 0.5 \Rightarrow \mathrm{P}(X < \mu + k\sigma) = 0.75$ ***(M1)***

$k = 0.674$ ***A1***

Note: Award ***M1A0*** for $k = -0.674$.

[2 marks]

60 – (MAT/21_HL_Summer_2018_Q4) – *Probability*

(a) $\mathrm{P}(L \geq 5) = 0.910$ ***(M1)A1***

[2 marks]

(b) X is the number of wolves found to be at least 5 years old

recognising binomial distribution ***M1***

$X \sim \mathrm{B}(8,\ 0.910\ldots)$

$\mathrm{P}(X > 6) = 1 - \mathrm{P}(X \leq 6)$ ***(M1)***

$= 0.843$ ***A1***

Note: Award ***M1A0*** for finding $\mathrm{P}(X \geq 6)$.

[3 marks]

61 – (MAT/21_HL_Summer_2018_Q6) – *Probability*

X is number of squirrels in reserve

$X \sim \text{Po}(179.2)$ ***A1***

Note: Award ***A1*** if 179.2 or 56×3.2 seen or implicit in future calculations.

recognising conditional probability ***M1***

$\text{P}(X > 190 \mid X \geq 168)$

$= \dfrac{\text{P}(X > 190)}{\text{P}(X \geq 168)} \left(= \dfrac{0.19827\ldots}{0.80817\cdots} \right)$ ***(A1)(A1)***

$= 0.245$ ***A1***

62 – (MAT/22_HL_Summer_2018_Q8) – *Probability*

(a) $np = 3.5$ ***(A1)***

$p \leq 1 \Rightarrow$ least $n = 4$ ***A1***

[2 marks]

(b) $(1-p)^n + np(1-p)^{n-1} = 0.09478$ ***M1A1***

attempt to solve above equation with $np = 3.5$ ***(M1)***

$n = 12,\ p = \dfrac{7}{24}\ (= 0.292)$ ***A1A1***

Note: Do not accept n as a decimal.

[5 marks]

63 – (MAT/22_HL_Summer_2018_Q9) – *Probability*

(a) (i) $X \sim \text{Po}(5.3)$

$\text{P}(X = 4) = \text{e}^{-5.3}\dfrac{5.3^4}{4!}$ ***(M1)***

$= 0.164$ ***A1***

(ii) **METHOD 1**

listing probabilities (table or graph) ***M1***
mode $X = 5$ (with probability 0.174) ***A1***

Note: Award ***M0A0*** for 5 (taxis) or mode = 5 with no justification.

METHOD 2

mode is the integer part of mean ***R1***
$E(X) = 5.3 \Rightarrow \text{mode} = 5$ ***A1***

Note: Do not allow ***R0A1***.

(iii) attempt at conditional probability ***(M1)***
$\dfrac{\text{P}(X = 7)}{\text{P}(X \geq 6)}$ or equivalent $\left(= \dfrac{0.1163...}{0.4365...}\right)$ ***A1***
$= 0.267$ ***A1***

[7 marks]

(b) **METHOD 1**

the possible arrivals are $(2,0)$, $(1,1)$, $(0,2)$ ***(A1)***
$Y \sim \text{Po}(0.65)$ ***A1***
attempt to compute, using sum and product rule, ***(M1)***
$0.070106\ldots \times 0.52204\ldots + 0.026455\ldots \times 0.33932\ldots + 0.0049916\ldots \times 0.11028\ldots$
(A1)(A1)

Note: Award ***A1*** for one correct product and ***A1*** for two other correct products.

$= 0.0461$ ***A1***

[6 marks]

METHOD 2

recognising a sum of 2 independent Poisson variables *eg* $Z = X + Y$ ***R1***

$\lambda = 5.3 + \dfrac{1.3}{2}$ ***A1***

$\text{P}(Z = 2) = 0.0461$ ***(M1)A3***

[6 marks]

64 – (MAT/21_HL_Summer_2018_Q10) – *Probability*

(a) $a\left[\int_0^{0.5} 3x\,\mathrm{d}x + \int_{0.5}^{2}(2-x)\mathrm{d}x\right] = 1$ ***M1***

Note: Award the ***M1*** for the total integral equalling 1, or equivalent.

$a\left(\frac{3}{2}\right) = 1$ ***(M1)A1***

$a = \frac{2}{3}$ ***AG***

[3 marks]

(b) **EITHER**

$\int_0^{0.5} 2x\,\mathrm{d}x + \frac{2}{3}\int_{0.5}^{1}(2-x)\,\mathrm{d}x$ ***(M1)(A1)***

$= \frac{2}{3}$ ***A1***

OR

$\frac{2}{3}\int_1^2 (2-x)\,\mathrm{d}x = \frac{1}{3}$ ***(M1)***

so $\mathrm{P}(X<1) = \frac{2}{3}$ ***(M1)A1***

[3 marks]

(c) $\mathrm{P}(s<X<0.8) = \int_s^{0.5} 2x\,\mathrm{d}x + \frac{2}{3}\int_{0.5}^{0.8}(2-x)\,\mathrm{d}x$ ***M1A1***

$= \left[x^2\right]_s^{0.5} + 0.27$

$0.25 - s^2 + 0.27$ ***(A1)***

$\mathrm{P}(2s<X<0.8) = \frac{2}{3}\int_{2s}^{0.8}(2-x)\,\mathrm{d}x$ ***A1***

$= \frac{2}{3}\left[2x - \frac{x^2}{2}\right]_{2s}^{0.8}$

$\frac{2}{3}\left(1.28 - (4s - 2s^2)\right)$

equating

$0.25 - s^2 + 0.27 = \frac{4}{3}\left(1.28 - (4s - 2s^2)\right)$ ***(A1)***

attempt to solve for s ***(M1)***

$s = 0.274$ ***A1***

[7 marks]

65 – (MAT/20_HL_Winter_2018_Q3) – *Probability*

(a) use of inverse normal (implied by $\pm 0.1509\ldots$ or $\pm 1.554\ldots$) ***(M1)***

$\mathrm{P}(X < 16) = 0.56$

$\Rightarrow \dfrac{16-\mu}{\sigma} = 0.1509\ldots$ ***(A1)***

$\mathrm{P}(X < 17) = 0.94$

$\Rightarrow \dfrac{17-\mu}{\sigma} = 1.554\ldots$ ***(A1)***

attempt to solve a pair of simultaneous equations ***(M1)***

$\mu = 15.9,\ \sigma = 0.712$ ***A1A1***

[6 marks]

(b) correctly shaded diagram or intent to find $\mathrm{P}(X \geq 15)$ ***(M1)***

$= 0.895$ ***A1***

Note: Accept answers rounding to 0.89 or 0.90. Award ***M1A0*** for the answer 0.9.

[2 marks]

66 – (MAT/20_HL_Winter_2018_Q10) – *Probability*

(a) (i) $\mathrm{P}(X < 60)$

$= \mathrm{P}(X \le 59)$ ***(M1)***

$= 0.102$ ***A1***

(ii) standard deviation $= \sqrt{70}\ (= 8.37)$ ***(M1)A1***

[4 marks]

(b) (i) use of midpoints (accept consistent use of $45, 55$ etc.) ***(M1)***

$$\frac{44.5 \times 2 + 54.5 \times 15 + 64.5 \times 40 + 74.5 \times 53 + 94.5 + 104.5 \times 3 + 114.5 \times 6}{2 + 15 + 40 + 53 + 0 + 1 + 3 + 6}$$

(M1)

$= \frac{8530}{120} (= 71.1)$ ***A1***

Note: If 45, 55, etc. are used consistently instead of midpoints (implied by the answer 71.58…) award ***M1M1A0.***

(ii) 13.9 ***(M1)A1***

[5 marks]

(c) valid reason given to include the examples below ***R1***

variance is 192 which is not close to the mean (accept not equal to)

standard deviation too high (using parts (a)(ii) and (b)(ii))

relative frequency of $X \le 59$ is 0.142 which is too high (using part (a)(i))

Poisson would give a frequency of roughly 14 for $80 \le X \le 89$

Note: Reasons which do not use values found in previous parts must be backed up with numerical evidence.

[1 mark]

(d) $\mathrm{P}(Y > 10) = 0.99$

$1 - \mathrm{P}(Y \le 10) = 0.99 \Rightarrow \mathrm{P}(Y \le 10) = 0.01$ ***(M1)***

attempt to solve a correct equation ***(M1)***

$\lambda = 20.1$ ***A1***

[3 marks]

(e) in 1 day, no of emails is $X \sim \mathrm{Po}(\lambda)$

in 2 days, no of emails is $Y \sim \mathrm{Po}(2\lambda)$ ***(A1)***

$\mathrm{P}(10 \text{ on first day} \mid 20 \text{ in 2 days})$ ***(M1)***

$$= \frac{\mathrm{P}(X=10) \times \mathrm{P}(X=10)}{\mathrm{P}(Y=20)}$$ ***(M1)***

$$= \frac{\left(\frac{\lambda^{10}e^{-\lambda}}{10!}\right)^2}{\frac{(2\lambda)^{20}e^{-2\lambda}}{20!}}$$ ***A1***

$$= \frac{\lambda^{20}e^{-2\lambda}}{2^{20}\lambda^{20}e^{-2\lambda}} \times \frac{20!}{(10!)^2}$$ ***A1***

$$= \frac{20!}{2^{20}(10!)^2}$$

which is independent of λ ***AG***

[5 marks]

67 – (MAT/22_HL_Summer_2019_Q2) – *Probability*

(a) $X \sim \mathrm{N}\left(820, 230^2\right)$ ***(M1)***

Note: Award ***M1*** for an attempt to use normal distribution. Accept labelled normal graph.

$\Rightarrow \mathrm{P}\left(X > 1000\right) = 0.217$ ***A1***

[2 marks]

(b) $Y \sim \mathrm{B}\left(24, 0.217...\right)$ ***(M1)***

Note: Award ***M1*** for recognition of binomial distribution with parameters.

$\mathrm{P}\left(Y \le 10\right) - \mathrm{P}\left(Y \le 4\right)$ ***(M1)***

Note: Award ***M1*** for an attempt to find $\mathrm{P}(5 \le Y \le 10)$ or $\mathrm{P}\left(Y \le 10\right) - \mathrm{P}\left(Y \le 4\right)$.

$= 0.613$ ***A1***

[3 marks]

68 – (MAT/21_HL_Summer_2019_Q3) – *Probability*

(a) (i) 6.75 ***A1***

(ii) 2.22 ***A1***

[2 marks]

(b) (i) 8.75 ***A1***

(ii) 2.22 ***A1***

[2 marks]

(c) the order is 3, 4, 6, 7, 7, 8, 9, 10
median is currently 7 ***A1***

Note: This can be indicated by a diagram/list, rather than actually stated.

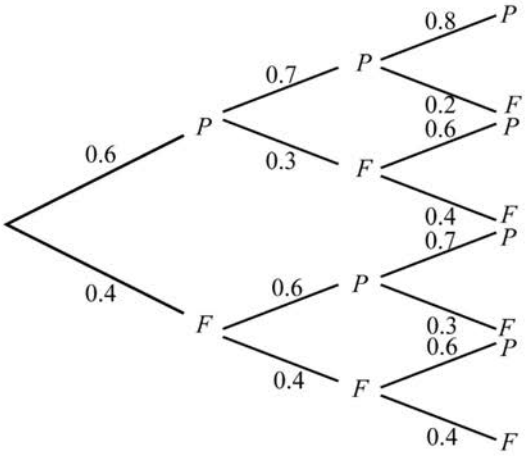
with 9 numbers the middle value (median) will be the 5th value ***R1***
which will correspond to 7 regardless of whether the position of the median
moves up or down ***R1***

Note: Accept answers using data 5, 6, 8, 9, 9, 10, 11, 12 (ie from part (b)).

[3 marks]

69 – (MAT/22_HL_Summer_2019_Q3) – *Probability*

(a)



A1A1A1

Note: Award ***A1*** for each correct column of probabilities.

[3 marks]

(b) probability (at least twice) =

EITHER

$(0.6\times0.7\times0.8)+(0.6\times0.7\times0.2)+(0.6\times0.3\times0.6)+(0.4\times0.6\times0.7)$ ***(M1)***

OR

$(0.6\times0.7)\ +\ (0.6\times0.3\times0.6)\ +\ (0.4\times0.6\times0.7)$ ***(M1)***

Note: Award ***M1*** for summing all required probabilities.

THEN

$=0.696$ ***A1***

[2 marks]

(c) P(passes third paper given only one paper passed before)

$=\dfrac{\text{P (passes third AND only one paper passed before)}}{\text{P (passes once in first two papers)}}$ ***(M1)***

$=\dfrac{(0.6\times0.3\times0.6)+(0.4\times0.6\times0.7)}{(0.6\times0.3)+(0.4\times0.6)}$ ***A1***

$=0.657$ ***A1***

[3 marks]

70 – (MAT/21_HL_Summer_2019_Q9) – *Probability*

(a) (i) use of formula or Venn diagram ***(M1)***

$0.72 + 0.45 - 1$ ***(A1)***

$= 0.17$ ***A1***

(ii) $0.72 - 0.17 = 0.55$ ***A1***

[4 marks]

(b) (i) $200 \times 0.45 = 90$ ***A1***

(ii) let X be the number of customers who order cake

$X \sim \mathrm{B}(200, 0.45)$ ***(M1)***

$\mathrm{P}(X > 100) = \mathrm{P}(X \geq 101)\left(= 1 - \mathrm{P}(X \leq 100)\right)$ ***(M1)***

$= 0.0681$ ***A1***

[4 marks]

(c) (i) $0.46 \times 0.8 = 0.368$ ***A1***

(ii) **METHOD 1**

$0.368 + 0.54 \times \mathrm{P}(S \mid F) = 0.72$ ***M1A1A1***

Note: Award ***M1*** for an appropriate tree diagram. Award ***A1*** for LHS, ***A1*** for RHS.

$\mathrm{P}(S \mid F) = 0.652$ ***A1***

METHOD 2

$\mathrm{P}(S \mid F) = \dfrac{\mathrm{P}(S \cap F)}{\mathrm{P}(F)}$ ***(M1)***

$= \dfrac{0.72 - 0.368}{0.54}$ ***A1A1***

Note: Award ***A1*** for numerator, ***A1*** for denominator.

$\mathrm{P}(S \mid F) = 0.652$ ***A1***

[5 marks]

71 – (MAT/22_HL_Summer_2019_Q10) – *Probability*

(a) $X \sim \mathrm{Po}(2.1)$

$\mathrm{P}(X = 0) = 0.122\left(= e^{-2.1}\right)$ ***(M1)A1***

[2 marks]

(b)

y	0	1	2	3	4
$P(Y=y)$	0.122... $\left(=e^{-2.1}\right)$	0.257... $\left(=e^{-2.1}2.1\right)$	0.270... $\left(=\frac{e^{-2.1}2.1^2}{2!}\right)$	0.189... $\left(=\frac{e^{-2.1}2.1^3}{3!}\right)$	0.161...

A1A1A1A1

Note: Award ***A1*** for each correct probability for $Y=1, 2, 3, 4$. Accept 0.162 for $P(Y=4)$.

[4 marks]

(c) $E(Y)=\sum yP(Y=y)$ ***(M1)***

$=1\times 0.257...+2\times 0.270...+3\times 0.189...+4\times 0.161...$ ***(A1)***

$=2.01$ ***A1***

[3 marks]

(d) let T be the no of days per year that Steffi does not visit

$T\sim B(365, 0.122...)$ ***(M1)***

require $0.45\le P(T\le n)<0.55$ ***(M1)***

$P(T\le 44)=0.51$

$n=44$ ***A1***

[3 marks]

(e) **METHOD 1**

let V be the discrete random variable "number of times Steffi is not fed per day"

$E(V)=1\times P(X=5)+2\times P(X=6)+3\times P(X=7)+\cdots$ ***M1***

$=1\times 0.0416...+2\times 0.0145...+3\times 0.00437...+\cdots$ ***A1***

$=0.083979...$ ***A1***

expected no of occasions per year $>0.083979...\times 365=30.7$ ***A1***

hence Steffi can expect not to be fed on at least 30 occasions ***AG***

Note: Candidates may consider summing more than three terms in their calculation for $E(V)$.

[4 marks]

METHOD 2

$E(X)-E(Y)=0.0903...$ ***M1A1***

$0.0903...\times 365$ ***M1***

$=33.0>30$ ***A1AG***

[4 marks]

72 – (MAT/20_HL_Winter_2019_Q2) – *Probability*

(a) $X \sim \mathrm{Po}(1.3)$

$\mathrm{P}(X \geq 2) = 0.373$ ***(M1)A1***

[2 marks]

(b) $V \sim \mathrm{B}(5, 0.373)$ ***(M1)A1***

Note: Award ***(M1)*** for recognition of binomial or equivalent, ***A1*** for correct parameters.

$\mathrm{P}(V = 4) = 0.0608$ ***(M1)A1***

[4 marks]

73 – (MAT/20_HL_Winter_2019_Q7) – *Probability*

$T \sim \mathrm{N}(11.6, 0.8^2)$

$\mathrm{P}(T < 10.7 \mid T < 11)$ ***(M1)***

$= \dfrac{\mathrm{P}(T < 10.7 \cap T < 11)}{\mathrm{P}(T < 11)}$ ***(M1)***

$= \dfrac{\mathrm{P}(T < 10.7)}{\mathrm{P}(T < 11)}$ ***(M1)***

$\mathrm{P}(T < 10.7) = 0.1302...$ ***(A1)***

$\mathrm{P}(T < 11) = 0.2266...$ ***(A1)***

$\mathrm{P}(T < 10.7 \mid T < 11) = 0.575$ ***A1***

Note: Accept only 0.575.

[6 marks]

74 – (MAT/20_HL_Winter_2019_Q10) – *Probability*

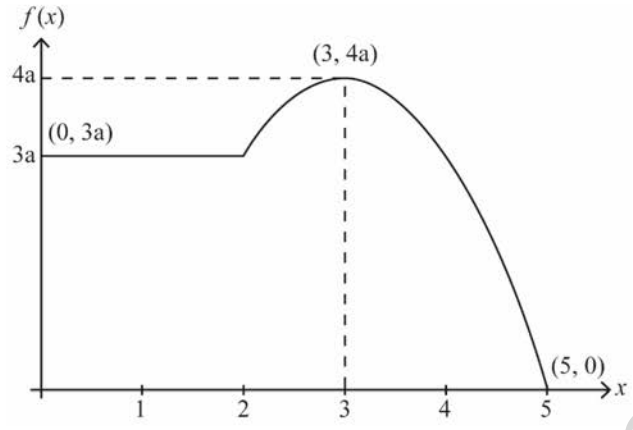
(a) $\left(\mathrm{P}(1 < X < 3) =\right) \int_1^2 3a\,\mathrm{d}x + a\int_2^3 -x^2 + 6x - 5\,\mathrm{d}x$ ***(M1)(A1)(A1)***

$= 3a + \frac{11}{3}a$

$= \frac{20}{3}a\,(= 6.67a)$ ***A1***

[4 marks]

(b)



A4

award ***A1*** for $(0, 3a)$, ***A1*** for continuity at $(2, 3a)$, ***A1*** for maximum at $(3, 4a)$, ***A1*** for $(5, 0)$

Note: Award ***A3*** if correct four points are not joined by a straight line and a quadratic curve.

[4 marks]

(c) (i) $\mathrm{P}(0 \le X \le 5) = 6a + a\int_2^5 -x^2 + 6x - 5\,\mathrm{d}x$ ***(M1)***

$= 15a$ ***(A1)***

$15a = 1$ ***(M1)***

$\Rightarrow a = \frac{1}{15}\,(= 0.0667)$ ***A1***

(ii) $\mathrm{E}(X) = \frac{1}{5}\int_0^2 x\,\mathrm{d}x + \frac{1}{15}\int_2^5 -x^3 + 6x^2 - 5x\,\mathrm{d}x$ ***(M1)(A1)***

$= 2.35$ ***A1***

(iii) attempt to use $\int_0^m f(x)\mathrm{d}x = 0.5$ ***(M1)***

$0.4 + a\int_2^m -x^2 + 6x - 5\,\mathrm{d}x = 0.5$ ***(A1)***

$a\int_2^m -x^2 + 6x - 5\,\mathrm{d}x = 0.1$

attempt to solve integral using GDC and/or analytically ***(M1)***

$\frac{1}{15}\left[-\frac{1}{3}x^3 + 3x^2 - 5x\right]_2^m = 0.1$

$m = 2.44$ ***A1***

[11 marks]

75 – (MAT/20_HL_Winter_2020_Q2) – *Probability*

let X be the random variable "number of books Jenna reads per week."

then $X \sim \mathrm{Po}(2.6)$

$\mathrm{P}(X \geq 4) = 0.264\ (0.263998\ldots)$ ***(M1)(A1)***

$0.263998\ldots \times 52$ ***(M1)***

$= 13.7$ ***A1***

Note: Accept 14 weeks.

[4 marks]

76 – (MAT/20_HL_Winter_2020_Q5) – *Probability*

(a) (i) people's holidays are independent of each other ***R1***

the proportion is constant (at 0.15) ***R1***

(ii) $X \sim \mathrm{B}(16, 0.15)$

$\mathrm{P}(X \geq 3) = 0.439$ ***(M1)A1***

[4 marks]

(b) probability of at least one $= 1 -$ probability of none

$\Rightarrow 1 - 0.85^n > 0.999$ **OR** $0.85^n < 0.001$ ***(A1)***

attempt to solve inequality ***(M1)***

$n \geq 42.503\ldots$

so least possible $n = 43$ ***A1***

[3 marks]

77 – (MAT/20_HL_Winter_2020_Q9) – *Probability*

(a) $X \sim \mathrm{N}\left(102, 8^2\right)$

$\mathrm{P}\left(X < 100\right) = 0.401$ ***(M1)A1***

[2 marks]

(b) $\mathrm{P}\left(X > w\right) = 0.444$ ***(M1)***

$\Rightarrow w = 103$ (g) ***A1***

[2 marks]

(c) $\mathrm{P}\left(X > 110 \mid X > 105\right) = \dfrac{\mathrm{P}\left(X > 110 \cap X > 105\right)}{\mathrm{P}\left(X > 105\right)}$ ***(M1)***

$= \dfrac{\mathrm{P}\left(X > 110\right)}{\mathrm{P}\left(X > 105\right)}$ ***(A1)***

$= \dfrac{0.15865...}{0.35383...}$

$= 0.448$ ***A1***

[3 marks]

(d) **EITHER**

$\mathrm{P}\left(90 < X < 114\right) = 0.866...$ ***(A1)***

OR

$\mathrm{P}\left(-1.5 < Z < 1.5\right) = 0.866...$ ***(A1)***

THEN

$0.866... \times 500$ ***(M1)***
$= 433$ ***A1***

[3 marks]

(e) $p = \mathrm{P}\left(X < 95\right) = 0.19078...$ ***(A1)***

recognising $Y \sim \mathrm{B}\left(80, p\right)$ ***(M1)***

now using $Y \sim \mathrm{B}(80, 0.19078...)$ ***(M1)***

$\mathrm{P}(Y \geq 20) = 0.116$ ***A1***

[4 marks]

78 – (MathAA/21_HL_Summer_2021_Q2) – *Probability*

Note: In this question, do not penalise incorrect use of strict inequality signs.

Let X = mass of a bag of sugar

(a) evidence of identifying the correct area ***(M1)***

$\mathrm{P}(X < 995) = 0.0765637...$

$= 0.0766$ ***A1***

[2 marks]

(b) 0.0766×100

≈ 8 ***A1***

[1 mark]

Note: Accept 7.66 .

(c) recognition that $\mathrm{P}(X > 1005 | X \geq 995)$ is required ***(M1)***

$$\frac{\mathrm{P}(X \geq 995 \cap X > 1005)}{\mathrm{P}(X \geq 995)}$$

$$\frac{\mathrm{P}(X > 1005)}{\mathrm{P}(X \geq 995)}$$ ***(A1)***

$$\frac{0.07656...}{1 - 0.07656...} \left(= \frac{0.07656...}{0.9234...} \right)$$

$= 0.0829$ ***A1***

[3 marks]

79 – (MathAA/22_HL_Summer_2021_Q3) – *Probability*

(a) **EITHER**

$\mathrm{P}(S)+\mathrm{P}(T)+\mathrm{P}(S'\cap T')-\mathrm{P}(S\cap T)=1$ OR $\mathrm{P}(S\cup T)=\mathrm{P}((S'\cap T')')$ ***(M1)***

$0.7+0.2+0.18-\mathrm{P}(S\cap T)=1$ ***OR*** $\mathrm{P}(S\cup T)=1-0.18$

OR

a clearly labelled Venn diagram ***(M1)***

THEN

$\mathrm{P}(S\cap T)=0.08$ (accept 8%) ***A1***

Note: To obtain the ***M1*** for the Venn diagram all labels must be correct and in the correct sections. For example, do not accept 0.7 in the area corresponding to $S\cap T'$.

[2 marks]

(b) **EITHER**

$\mathrm{P}(T\cap S')=\mathrm{P}(T)-\mathrm{P}(T\cap S)(=0.2-0.08)$ OR

$\mathrm{P}(T\cap S')=\mathrm{P}(T\cup S)-\mathrm{P}(S)(=0.82-0.7)$ ***(M1)***

OR

a clearly labelled Venn diagram including $\mathrm{P}(S)$, $\mathrm{P}(T)$ and $\mathrm{P}(S\cap T)$ ***(M1)***

THEN

$=0.12$ (accept 12%) ***A1***

[2 marks]

(c) $\mathrm{P}(G\cap T)=\mathrm{P}(T|G)\mathrm{P}(G)$ (0.25×0.48) ***(M1)***

$=0.12$ ***A1***

[2 marks]

(d) **METHOD 1**

$\mathrm{P}(G)\times\mathrm{P}(T)(=0.48\times 0.2)=0.096$ ***A1***

$\mathrm{P}(G)\times\mathrm{P}(T)\neq\mathrm{P}(G\cap T)\Rightarrow$ G and T are not independent ***R1***

METHOD 2

$\mathrm{P}(T\,|\,G)=0.25$ ***A1***

$\mathrm{P}(T\,|\,G)\neq\mathrm{P}(T)\Rightarrow$ G and T are not independent ***R1***

Note: Do not award ***A0R1.***

[2 marks]

80 – (MathAA/22_HL_Summer_2021_Q6) – *Probability*

(a) $\mathrm{E}(X) = (n+1)\int_0^1 x^{n+1}\,\mathrm{d}x$ ***M1***

$= (n+1)\left[\dfrac{x^{n+2}}{n+2}\right]_0^1$ ***A1***

leading to $\mathrm{E}(X) = \dfrac{n+1}{n+2}$ ***AG***

[2 marks]

(b) **METHOD 1**

use of $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$ ***M1***

$\mathrm{Var}(X) = (n+1)\int_0^1 x^{n+2}\,\mathrm{d}x - \left(\dfrac{n+1}{n+2}\right)^2$

$= (n+1)\left[\dfrac{1}{n+3}x^{n+3}\right]_0^1 - \left(\dfrac{n+1}{n+2}\right)^2$

$= \dfrac{n+1}{n+3} - \left(\dfrac{n+1}{n+2}\right)^2$ ***A1***

$= \dfrac{(n+1)(n+2)^2 - (n+1)^2(n+3)}{(n+2)^2(n+3)}$ ***M1***

EITHER

$= \dfrac{(n+1)\left(n^2+4n+4-\left(n^2+4n+3\right)\right)}{(n+2)^2(n+3)}$ ***A1***

OR

$= \dfrac{\left(n^3+5n^2+8n+4\right)-\left(n^3+5n^2+7n+3\right)}{(n+2)^2(n+3)}$ ***A1***

THEN

so $\mathrm{Var}(X) = \dfrac{n+1}{(n+2)^2(n+3)}$ ***AG***

METHOD 2

use of $\mathrm{Var}(X)=\mathrm{E}(X-\mathrm{E}(X))^2$ ***M1***

$$\mathrm{Var}(X)=(n+1)\int_0^1\left(x-\frac{n+1}{n+2}\right)^2 x^n\,\mathrm{d}x$$

$$=(n+1)\left[\frac{1}{n+3}x^{n+3}-\frac{2(n+1)}{(n+2)^2}x^{n+2}+\frac{n+1}{(n+2)^2}x^{n+1}\right]_0^1$$

$$=\frac{n+1}{n+3}-\left(\frac{n+1}{n+2}\right)^2$$ ***A1***

$$=\frac{(n+1)\left((n+2)^2-(n+1)(n+3)\right)}{(n+2)^2(n+3)}$$ ***M1***

EITHER

$$=\frac{(n+1)\left(n^2+4n+4-\left(n^2+4n+3\right)\right)}{(n+2)^2(n+3)}$$ ***A1***

OR

$$=\frac{\left(n^3+5n^2+8n+4\right)-\left(n^3+5n^2+7n+3\right)}{(n+2)^2(n+3)}$$ ***A1***

THEN

so $\mathrm{Var}(X)=\dfrac{n+1}{(n+2)^2(n+3)}$ ***AG***

[4 marks]

81 – (MathAA/21_HL_Summer_2021_Q7) – *Probability*

(a) recognition of the need to integrate $\frac{x}{\sqrt{(x^2+k)^3}}$ ***(M1)***

$\int \frac{x}{\sqrt{(x^2+k)^3}}\,\mathrm{d}x\,(=1)$

EITHER

$u = x^2 + k \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$ (or equivalent) ***(A1)***

$\int \frac{x}{\sqrt{(x^2+k)^3}}\,\mathrm{d}x = \frac{1}{2}\int u^{-\frac{3}{2}}\mathrm{d}u$

$= -u^{-\frac{1}{2}}(+c)\left(= -(x^2+k)^{-\frac{1}{2}}(+c)\right)$ ***A1***

OR

$\int \frac{x}{\sqrt{(x^2+k)^3}}\,\mathrm{d}x = \frac{1}{2}\int \frac{2x}{\sqrt{(x^2+k)^3}}\,\mathrm{d}x$ ***(A1)***

$= -(x^2+k)^{-\frac{1}{2}}(+c)$ ***A1***

THEN

attempt to use correct limits for their integrand and set equal to 1 ***M1***

$\left[-u^{-\frac{1}{2}}\right]_k^{16+k} = 1$ OR $\left[-(x^2+k)^{-\frac{1}{2}}\right]_0^4 = 1$

$-(16+k)^{-\frac{1}{2}} + k^{-\frac{1}{2}} = 1\left(\Rightarrow \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{16+k}} = 1\right)$ ***A1***

$\sqrt{16+k} - \sqrt{k} = \sqrt{k}\sqrt{16+k}$ ***AG***

[5 marks]

(b) attempt to solve $\sqrt{16+k} - \sqrt{k} = \sqrt{k}\sqrt{16+k}$ ***(M1)***

$k = 0.645038...$

$= 0.645$ ***A1***

[2 marks]

82 – (MathAA/22_HL_Summer_2021_Q10) – *Probability*

(a) use of inverse normal to find z-score ***(M1)***

$z = 2.0537...$

$2.0537... = \frac{82-75}{\sigma}$ ***(A1)***

$\sigma = 3.408401...$

$\sigma = 3.41$ ***A1***

[3 marks]

(b) evidence of identifying the correct area under the normal curve ***(M1)***

$\mathrm{P}(T > 80) = 0.071193...$

$\mathrm{P}(T > 80) = 0.0712$ ***A1***

[2 marks]

(c) recognition that $\mathrm{P}(80 < T < 82)$ is required ***(M1)***

$P(T < 82 | T > 80) = \frac{\mathrm{P}(80 < T < 82)}{\mathrm{P}(T > 80)} = \left(\frac{0.051193...}{0.071193...}\right)$ ***(M1)(A1)***

$= 0.719075...$

$= 0.719$ ***A1***

[4 marks]

(d) recognition of binomial probability ***(M1)***

$X \sim \mathrm{B}(64, 0.071193...)$ or $\mathrm{E}(X) = 64 \times 0.071193...$ ***(A1)***

$\mathrm{E}(X) = 4.556353...$

$\mathrm{E}(X) = 4.56$ (flights) ***A1***

[3 marks]

(e) $\mathrm{P}(X > 6) = P(X \geq 7) = 1 - \mathrm{P}(X \leq 6)$ ***(M1)***

$= 1 - 0.83088...$ ***(A1)***

$= 0.1691196...$

$= 0.169$ ***A1***

[3 marks]

83 – (MathAA/20_HL_Winter_2021_Q3) – *Probability*

(a) recognize that the variable has a Binomial distribution ***(M1)***

$X \sim \mathrm{B}(30, 0.05)$

attempt to find $\mathrm{P}(X \geq 1)$ ***(M1)***

$1 - \mathrm{P}(X = 0)$ OR $1 - 0.95^{30}$ OR $1 - 0.214638...$ OR $0.785361...$

Note: The two ***M*** marks are independent of each other.

$\mathrm{P}(X \geq 1) = 0.785$ ***A1***

[3 marks]

(b) recognition of conditional probability ***(M1)***

$\mathrm{P}(X \leq 2 \mid X \geq 1)$ OR $\mathrm{P}(\text{at most 2 defective} \mid \text{at least 1 defective})$

Note: Recognition must be shown in context either in words or symbols but not just $\mathrm{P}(A \mid B)$.

$\dfrac{\mathrm{P}(1 \leq X \leq 2)}{\mathrm{P}(X \geq 1)}$ OR $\dfrac{\mathrm{P}(X = 1) + \mathrm{P}(X = 2)}{\mathrm{P}(X \geq 1)}$ ***(A1)***

$\dfrac{0.597540...}{0.785361...}$ OR $\dfrac{0.812178... - 0.214638...}{0.785361...}$ OR $\dfrac{0.338903... + 0.258636...}{0.785361...}$ ***(A1)***

$= 0.760847...$

$\mathrm{P}(X \leq 2 \mid X \geq 1) = 0.761$ ***A1***

[4 marks]

Total [7 marks]

84 – (MathAA/20_HL_Winter_2021_Q7) – *Probability*

(a) recognises that $\int_0^m \arccos x \, dx = 0.5$ ***(M1)***

$m \arccos m - \sqrt{1-m^2} - \left(0 - \sqrt{1}\right) = 0.5$

$m = 0.360034...$

$m = 0.360$ ***A1***

[2 marks]

(b) **METHOD 1**

attempts to find at least one endpoint (limit) both in terms of m (or their m) and a ***(M1)***

$\mathrm{P}\left(m - a \le X \le m + a\right) = 0.3$

$\int_{0.360034... - a}^{0.360034... + a} \arccos x \, dx = 0.3$ ***(A1)***

Note: Award ***(A1)*** for $\int_{m-a}^{m+a} \arccos x \, dx = 0.3$.

$\left[x \arccos x - \sqrt{1-x^2}\right]_{0.360034... - a}^{0.360034... + a}$

attempts to solve their equation for a ***(M1)***

Note: The above ***(M1)*** is dependent on the first ***(M1)***.

$a = 0.124861...$

$a = 0.125$ ***A1***

METHOD 2

$\int_a^a \arccos\left|x - 0.360034...\right| dx \ \left(= 0.3\right)$ ***(M1)(A1)***

Note: Only award ***(M1)*** if at least one limit has been translated correctly.

Note: Award ***(M1)(A1)*** for $\int_a^a \arccos\left|x - m\right| dx \ \left(= 0.3\right)$.

attempts to solve their equation for a ***(M1)***

$a = 0.124861...$

$a = 0.125$ ***A1***

METHOD 3

EITHER

$\int_{a}^{a} \arccos(x+0.360034...)\,\mathrm{d}x \ (=0.3)$ ***(M1)(A1)***

Note: Only award ***(M1)*** if at least one limit has been translated correctly.

Note: Award ***(M1)(A1)*** for $\int_{a}^{a} \arccos(x+m)\,\mathrm{d}x \ (=0.3)$.

OR

$\int_{2(0.360034...)-a}^{2(0.360034...)+a} \arccos(x-0.360034...)\,\mathrm{d}x \ (=0.3)$ ***(M1)(A1)***

Note: Only award ***(M1)*** if at least one limit has been translated correctly.

Note: Award ***(M1)(A1)*** for $\int_{2m-a}^{2m+a} \arccos(x-m)\,\mathrm{d}x \ (=0.3)$.

THEN

attempts to solve their equation for a ***(M1)***

Note: The above ***(M1)*** is dependent on the first ***(M1)***.

$a = 0.124861...$

$a = 0.125$ ***A1***

[4 marks]

Total [6 marks]

85 – (MathAA/21_HL_Summer_2022_Q3) – *Probability*

(a) $0.41+k-0.28+0.46+0.29-2k^2=1$ OR $k-2k^2+0.01=0.13$ (or equivalent) ***A1***

$2k^2-k+0.12=0$ ***AG***

[1 mark]

(b) one of 0.2 OR 0.3 ***(M1)***

$k=0.3$ ***A1***

reasoning to reject $k=0.2$ eg $\mathrm{P}(1)=k-0.28\geq 0$ therefore $k\neq 0.2$ ***R1***

[3 marks]

(c) attempting to use the expected value formula ***(M1)***

$\mathrm{E}(X)=0\times 0.41+1\times(0.3-0.28)+2\times 0.46+3\times(0.29-2\times 0.3^2)$

$=1.27$ ***A1***

Note: Award ***M1A0*** if additional values are given.

[2 marks]

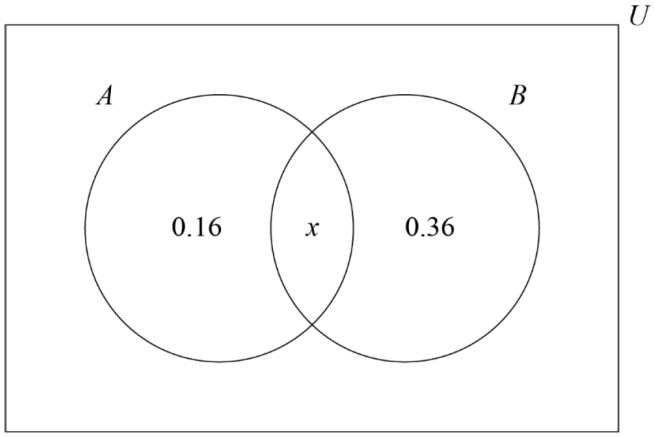
86 – (MathAA/21_HL_Summer_2022_Q5) – *Probability*

(a) **METHOD 1**

EITHER

one of $\mathrm{P}(A)=x+0.16$ OR $\mathrm{P}(B)=x+0.36$ ***A1***

OR



A1

THEN

attempt to equate their $\mathrm{P}(A\cap B)$ with their expression for $\mathrm{P}(A)\times\mathrm{P}(B)$ ***M1***

$\mathrm{P}(A\cap B)=\mathrm{P}(A)\times\mathrm{P}(B)\Rightarrow x=(x+0.16)\times(x+0.36)$ ***A1***

$x=0.24$ ***A1***

METHOD 2

attempt to form at least one equation in $\mathrm{P}(A)$ and $\mathrm{P}(B)$ using independence ***M1***

$\left(\mathrm{P}(A\cap B')=\mathrm{P}(A)\times\mathrm{P}(B')\Rightarrow\right)\ \mathrm{P}(A)\times\left(1-\mathrm{P}(B)\right)=0.16$ OR

$\left(\mathrm{P}(A'\cap B)=\mathrm{P}(A')\times\mathrm{P}(B)\Rightarrow\right)\ \left(1-\mathrm{P}(A)\right)\times\mathrm{P}(B)=0.36$

$\mathrm{P}(A)=0.4$ AND $\mathrm{P}(B)=0.6$ ***A1***

$\mathrm{P}(A\cap B)=\mathrm{P}(A)\times\mathrm{P}(B)=0.4\times0.6$ ***(A1)***

$x=0.24$ ***A1***

[4 marks]

(b) **METHOD 1**

recognising $\mathrm{P}(A' \mid B') = \mathrm{P}(A')$ ***(M1)***

$= 1 - 0.16 - 0.24$

$= 0.6$ ***A1***

METHOD 2

$\mathrm{P}(B) = 0.36 + 0.24 (= 0.6)$

$\mathrm{P}(A' \mid B') = \dfrac{\mathrm{P}(A' \cap B')}{\mathrm{P}(B')} \quad \left(= \dfrac{0.24}{0.4}\right)$ ***(A1)***

$= 0.6$ ***A1***

[2 marks]

87 – (MathAA/21_HL_Summer_2022_Q11) – *Probability*

(a) $\mathrm{P}(C < 61)$ ***(M1)***

$= 0.365112...$

$= 0.365$ ***A1***

[2 marks]

(b) recognition of binomial eg $X \sim \mathrm{B}(12, 0.365...)$ ***(M1)***

$\mathrm{P}(X = 5) = 0.213666...$

$= 0.214$ ***A1***

[2 marks]

(c) (i) Let CM represent 'chocolate muffin' and BM represent 'banana muffin'

$\mathrm{P}(B<61)=0.0197555...$ ***(A1)***

EITHER

$\mathrm{P}(CM)\times\mathrm{P}(C<61\,|\,CM)+\mathrm{P}(BM)\times\mathrm{P}(B<61\,|\,BM)$ (or equivalent in words) ***(M1)***

OR

tree diagram showing two ways to have a muffin weigh < 61 ***(M1)***

THEN

$(0.6\times0.365...)+(0.4\times0.0197...)$ ***(A1)***

$=0.226969...$

$=\ 0.227$ ***A1***

(ii) recognizing conditional probability ***(M1)***

Note: Recognition must be shown in context either in words or symbols, not just $\mathrm{P}(A\,|\,B)$.

$\dfrac{0.6\times0.365112...}{0.226969...}$ ***(A1)***

$=0.965183...$

$=0.965$ ***A1***

[7 marks]

(d) **METHOD 1**

$\mathrm{P}(CM)\times\mathrm{P}(C<61\,|\,CM)+\mathrm{P}(BM)\times\mathrm{P}(B<61\,|\,BM)=0.157$ ***(M1)***

$\left(0.6\times\mathrm{P}(C<61)\right)+(0.4\times0.0197555...)=0.157$

$\mathrm{P}(C<61)=0.248496...$ ***(A1)***

attempt to solve for σ using GDC ***(M1)***

Note: Award ***(M1)*** for a graph or table of values to show their $\mathrm{P}(C<61)$ with a variable standard deviation.

$\sigma=1.47225...$

$\sigma=1.47$ (g) ***A2***

METHOD 2

$\mathrm{P}(CM)\times\mathrm{P}(C<61\mid CM)+\mathrm{P}(BM)\times\mathrm{P}(B<61\mid BM)=0.157$ ***(M1)***

$\left(0.6\times\mathrm{P}(C<61)\right)+(0.4\times0.0197555...)=0.157$

$\mathrm{P}(C<61)=0.248496...$ ***(A1)***

use of inverse normal to find z score of their $\mathrm{P}(C<61)$ ***(M1)***

$z=-0.679229...$

correct substitution ***(A1)***

$\frac{61-62}{\sigma}=-0.679229...$

$\sigma=1.47225...$

$\sigma=1.47$ (g) ***A1***

[5 marks]

88 – (MathAA/22_HL_Summer_2022_Q3) – *Probability*

$\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)=0.68$

substitution of $\mathrm{P}(A)\cdot\mathrm{P}(B)$ for $\mathrm{P}(A\cap B)$ in $\mathrm{P}(A\cup B)$ ***(M1)***

$\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A)\mathrm{P}(B)\ (=0.68)$

substitution of $3\mathrm{P}(B)$ for $\mathrm{P}(A)$ ***(M1)***

$3\mathrm{P}(B)+\mathrm{P}(B)-3\mathrm{P}(B)\mathrm{P}(B)=0.68$ (or equivalent) ***(A1)***

Note: The first two marks are independent of each other.

attempts to solve their quadratic equation ***(M1)***

$\mathrm{P}(B)=0.2,\ 1.133...\ \left(\frac{1}{5},\frac{17}{15}\right)$

$\mathrm{P}(B)=0.2\ \left(=\frac{1}{5}\right)$ ***A2***

Note: Award ***A1*** if both answers are given as final answers for $\mathrm{P}(B)$.

[6 marks]

89 – (MathAA/22_HL_Summer_2022_Q8) – *Probability*

Rachel: $R \sim \mathrm{N}\left(56.5, 3^2\right)$

$\mathrm{P}\left(R \geq 60\right) = 0.1216...$ ***(A1)***

Sophia: $S \sim \mathrm{N}\left(57.5, 1.8^2\right)$

$\mathrm{P}\left(S \geq 60\right) = 0.0824...$ ***(A1)***

recognises binomial distribution with $n = 5$ ***(M1)***

let N_R represent the number of Rachel's throws that are longer than 60 metres

$N_R \sim \mathrm{B}\left(5, 0.1216...\right)$

either $\mathrm{P}\left(N_R \geq 1\right) = 0.4772...$ or $\mathrm{P}\left(N_R = 0\right) = 0.5227...$ ***(A1)***

let N_S represent the number of Sophia's throws that are longer than 60 metres

$N_S \sim \mathrm{B}\left(5, 0.0824...\right)$

either $\mathrm{P}\left(N_S \geq 1\right) = 0.3495...$ or $\mathrm{P}\left(N_S = 0\right) = 0.6504...$ ***(A1)***

EITHER

uses $\mathrm{P}\left(N_R \geq 1\right)\mathrm{P}\left(N_S = 0\right) + \mathrm{P}\left(N_S \geq 1\right)\mathrm{P}\left(N_R = 0\right)$ ***(M1)***

$\mathrm{P}\left(\text{one of Rachel or Sophia qualify}\right) = \left(0.4772... \times 0.6504...\right) + \left(0.3495... \times 0.5227...\right)$

OR

uses $\mathrm{P}\left(N_R \geq 1\right) + \mathrm{P}\left(N_S \geq 1\right) - 2 \times \mathrm{P}\left(N_R \geq 1\right) \times \mathrm{P}\left(N_S \geq 1\right)$ ***(M1)***

$\mathrm{P}\left(\text{one of Rachel or Sophia qualify}\right) = 0.4772... + 0.3495... - 2 \times 0.4772... \times 0.3495...$

THEN

$= 0.4931...$

$= 0.493$ ***A1***

[7 marks]

Note: ***M*** marks are not dependent on the previous ***A*** marks.

90 – (MathAA/20_HL_Winter_2022_Q10) – *Probability*

Note: Do not penalize for inclusion or non-inclusion of endpoints for probabilities using a normal distribution. For example, for $\mathrm{P}(T < 55 \mid T > 40)$ accept $\mathrm{P}(T \le 55 \mid T > 40)$, $\mathrm{P}(T \le 55 \mid T \ge 40)$, etc.

(a) recognising to find $\mathrm{P}(T > 40)$ ***(M1)***

$\mathrm{P}(T > 40) = 0.574136...$

$\mathrm{P}(T > 40) = 0.574$ ***A1***

[2 marks]

(b) attempt to multiply four independent probabilities using their $\mathrm{P}(T > 40)$ and $\mathrm{P}(T < 40)$ ***(M1)***

$(1-p)^3 \cdot p$ OR $(1-0.574136...)^3 \cdot 0.574136...$ OR $(0.425863...)^3 \cdot 0.574136...$ ***(A1)***

$0.0443430...$

0.0443 , 0.0444 from 3 sf values ***A1***

[3 marks]

(c) (i) recognizing conditional probability ***(M1)***

$\mathrm{P}(T < 55 \mid T > 40)$

Note: Award ***(M1)*** for an expression or description in context. Accept $\mathrm{P}(T > 40 \mid T < 55)$ but do not accept just $\mathrm{P}(A \mid B)$.

$\dfrac{\mathrm{P}(40 < T < 55)}{\mathrm{P}(T > 40)}$ ***(A1)***

$\dfrac{0.461944...}{0.574136...}$ ***(A1)***

$\mathrm{P}(T < 55 \mid T > 40) = 0.804590...$

$= 0.805$ ***A1***

(ii) recognizing binomial probability ***(M1)***

$X \sim \mathrm{B}(n, p)$

$n = 10$ and $p = 0.804589...$ ***(A1)***

$0.0242111..., 0.0240188...$ using $p = 0.805$

$\mathrm{P}(X = 5) = 0.0242$ ***A1***

[7 marks]

(d) Let $\mathrm{P}(T < a) = x$

recognition that probabilities sum to 1 (seen anywhere) ***(M1)***

EITHER

expressing the three regions in one variable ***(M1)***

$x + 0.904 + 2x$ OR $\mathrm{P}(T < a) + 0.904 + 2\mathrm{P}(T < a)$ OR $\frac{1}{2}\mathrm{P}(T > b) + 0.904 + \mathrm{P}(T > b)$

OR x and $2x$ correctly indicated on labelled bell diagram

$\mathrm{P}(T < a) + 0.904 + 2\mathrm{P}(T < a) = 1$ OR $\frac{1}{2}\mathrm{P}(T > b) + 0.904 + \mathrm{P}(T > b) = 1$ (or equivalent) ***(A1)***

OR

expressing either $\mathrm{P}(T < a)$ or $\mathrm{P}(T > b)$ only in terms of $\mathrm{P}(a \le T \le b)$ ***(M1)***

$(\mathrm{P}(T < a) =)\frac{1}{3}(1 - \mathrm{P}(a \le T \le b))$ OR $(\mathrm{P}(T > b) =)\frac{2}{3}\cdot(1 - \mathrm{P}(a \le T \le b))$

$x = \frac{1}{3}(1 - 0.904)(= 0.032)$ OR $\mathrm{P}(T > b) = \frac{2}{3}(1 - 0.904)(= 0.064)$ ***(A1)***

THEN

$\mathrm{P}(T < a) = 0.032$

$a = 22.18167...$

$a = 22.2$ (accept 22.1) ***A1***

[4 marks]

1 - (MAT/21_HL_Summer_2012_Q1) - *Differentiation*

$$\frac{dy}{dx} = 3x^2 - 12x + k$$

For use of discriminant $b^2 - 4ac = 0$ or completing the square $3(x-2)^2 + k - 12$

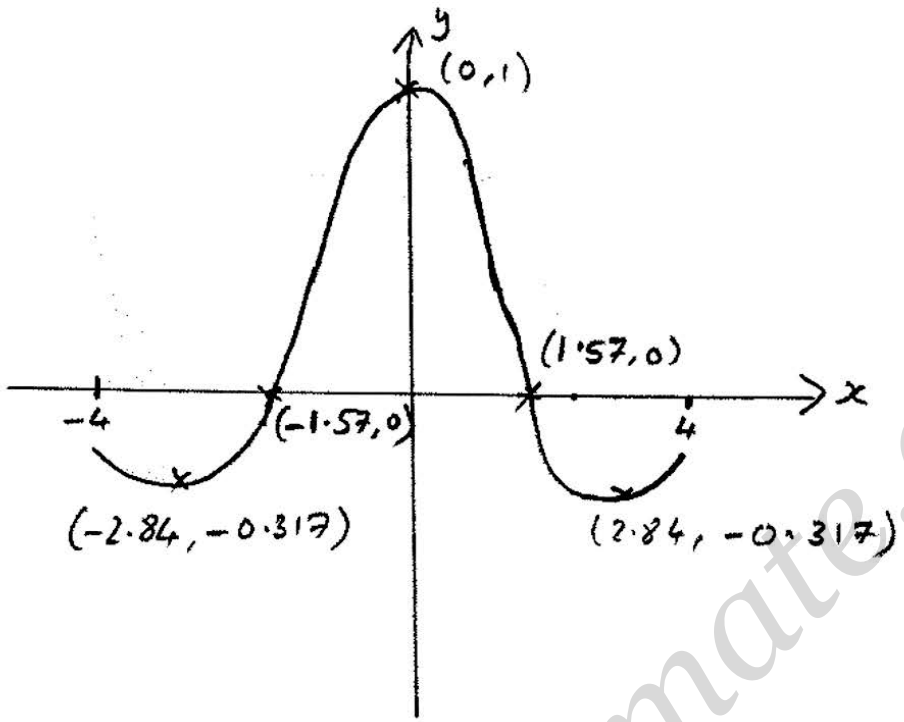
$144 - 12k = 0$

Note:	Accept trial and error, sketches of parabolas with vertex (2,0) or use of second derivative.

$k = 12$

2 – (MAT/22_HL_Summer_2012_Q6) – *Differentiation*

(a)



Note:	Award *A1* for correct shape. Do not penalise if too large a domain is used, *A1* for correct x-intercepts, *A1* for correct coordinates of two minimum points, *A1* for correct coordinates of maximum point. Accept answers which correctly indicate the position of the intercepts, maximum point and minimum points.

(b) gradient at $x = 1$ is -0.786

(c) gradient of normal is $\frac{-1}{-0.786} (= 1.272\ldots)$

when $x = 1$, $y = 0.3820...$

Equation of normal is $y - 0.382 = 1.27(x - 1)$

$(\Rightarrow y = 1.27x - 0.890)$

3 - (MAT/21_HL_Summer_2012_Q10) - *Differentiation*

attempt to find intersections

intersections are $\left(\frac{10}{m+2}, \frac{10m}{m+2}\right)$ and $\left(\frac{10m}{2m-1}, -\frac{10}{2m-1}\right)$

area of triangle $= \frac{1}{2} \times \frac{\sqrt{100+100m^2}}{(m+2)} \times \frac{\sqrt{100+100m^2}}{(2m-1)}$

$= \frac{50(1+m^2)}{(m+2)(2m-1)}$

minimum when $m = 3$

4 – (MAT/21_HL_Summer_2012_Q11) – *Differentiation*

(a) $(3.79, -5)$

(b) $p = 1.57$ or $\frac{\pi}{2}$, $q = 6.00$

(c) $f'(x) = 3\cos x - 4\sin x$
$3\cos x - 4\sin x = 3 \Rightarrow x = 4.43\ldots$
$(y = -4)$

Coordinates are $(4.43, -4)$

(d) $m_{\text{normal}} = -\frac{1}{m_{\text{tangent}}}$

gradient at P is -4 so gradient of normal at P is $\frac{1}{4}$

gradient at Q is 4 so gradient of normal at Q is $-\frac{1}{4}$

equation of normal at P is $y - 3 = \frac{1}{4}(x - 1.570\ldots)$ $(\text{or } y = 0.25x + 2.60\ldots)$

equation of normal at Q is $y - 3 = -\frac{1}{4}\left(x - 5.999\ldots\right)\left(\text{or } y = -0.25x + \underbrace{4.499\ldots}\right)$

Note: Award the previous two ***M1*** even if the gradients are incorrect in $y - b = m(x - a)$ where (a, b) are coordinates of P and Q (or in $y = mx + c$ with c determined using coordinates of Pand Q.

intersect at $(3.79, 3.55)$

Note: Award ***N2*** for 3.79 without other working.

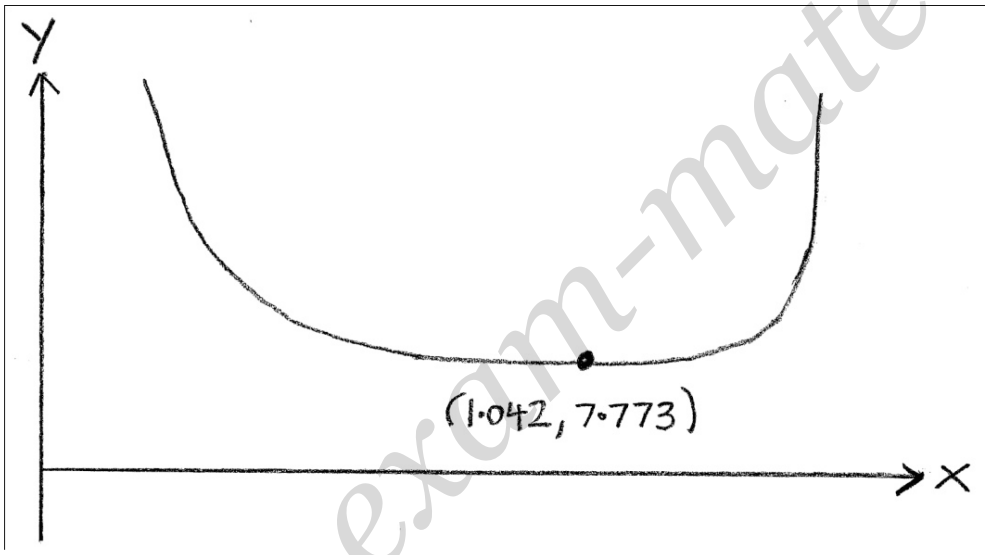
5 – (MAT/20_HL_Winter_2012_Q12) – *Differentiation*

(a) $L = \text{CA} + \text{AD}$

$$\sin\alpha = \frac{a}{\text{CA}} \Rightarrow \text{CA} = \frac{a}{\sin\alpha}$$
$$\cos\alpha = \frac{b}{\text{AD}} \Rightarrow \text{AD} = \frac{b}{\cos\alpha}$$
$$L = \frac{a}{\sin\alpha} + \frac{b}{\cos\alpha}$$

(b) $a = 5$ and $b = 1 \Rightarrow L = \dfrac{5}{\sin\alpha} + \dfrac{1}{\cos\alpha}$

METHOD 1



minimum from graph $\Rightarrow$ $L = 7.77$
minimum of L gives the max length of the painting

METHOD 2

$$\frac{\mathrm{d}L}{\mathrm{d}\alpha} = \frac{-5\cos\alpha}{\sin^2\alpha} + \frac{\sin\alpha}{\cos^2\alpha}$$
$$\frac{\mathrm{d}L}{\mathrm{d}\alpha} = 0 \Rightarrow \frac{\sin^3\alpha}{\cos^3\alpha} = 5 \Rightarrow \tan\alpha = \sqrt[3]{5} \quad (\alpha = 1.0416\ldots)$$
minimum of L gives the max length of the painting
maximum length $= 7.77$

(c) $\dfrac{\mathrm{d}L}{\mathrm{d}\alpha} = \dfrac{-3k\cos\alpha}{\sin^2\alpha} + \dfrac{k\sin\alpha}{\cos^2\alpha}$ (or equivalent)

(d) $\dfrac{\mathrm{d}L}{\mathrm{d}\alpha} = \dfrac{-3k\cos^3\alpha + k\sin^3\alpha}{\sin^2\alpha\cos^2\alpha}$

$\dfrac{\mathrm{d}L}{\mathrm{d}\alpha} = 0 \Rightarrow \dfrac{\sin^3\alpha}{\cos^3\alpha} = \dfrac{3k}{k} \Rightarrow \tan\alpha = \sqrt[3]{3} \quad (\alpha = 0.96454\ldots)$

$\tan\alpha = \sqrt[3]{3} \Rightarrow \dfrac{1}{\cos\alpha} = \sqrt{1+\sqrt[3]{9}} \qquad (1.755\ldots)$

and $\dfrac{1}{\sin\alpha} = \dfrac{\sqrt{1+\sqrt[3]{9}}}{\sqrt[3]{3}} \qquad (1.216\ldots)$

$L = 3k\left(\dfrac{\sqrt{1+\sqrt[3]{9}}}{\sqrt[3]{3}}\right) + k\sqrt{1+\sqrt[3]{9}} \qquad (L = 5.405598\ldots k)$

(e) $L \le 8 \Rightarrow k \ge 1.48$

the minimum value is 1.48

6 – (MAT/22_HL_Summer_2013_Q12) – *Differentiation, Integration*

(a) (i) **METHOD 1**

$$\frac{dy}{dx} = -\sin x + \cos x$$

$$y\frac{dy}{dx} = (\cos x + \sin x)(-\sin x + \cos x)$$

$$= \cos^2 x - \sin^2 x$$

$$= \cos 2x$$

METHOD 2

$$y^2 = (\sin x + \cos x)^2$$

$$2y\frac{dy}{dx} = 2(\cos x + \sin x)(\cos x - \sin x)$$

$$y\frac{dy}{dx} = \cos^2 x - \sin^2 x$$

$$= \cos 2x$$

(ii) attempting to separate variables $\int y\,dy = \int \cos 2x\,dx$

$$\frac{1}{2}y^2 = \frac{1}{2}\sin 2x + C$$

Note:	Award ***A1*** for a correct LHS and ***A1*** for a correct RHS.

$$y = \pm(\sin 2x + A)^{\frac{1}{2}}$$

(iii) $\sin 2x + A \equiv (\cos x + \sin x)^2$

$(\cos x + \sin x)^2 = \cos^2 x + 2\sin x\cos x + \sin^2 x$

use of $\sin 2x \equiv 2\sin x\cos x$.

$A = 1$

(b) (i) substituting $x=\frac{\pi}{4}$ and $y=2$ into $y=(\sin 2x+A)^{\frac{1}{2}}$

so $g(x)=(\sin 2x+3)^{\frac{1}{2}}$.

range g is $\left[\sqrt{2},2\right]$

Note: Accept $[1.41, 2]$. Award ***A1*** for each correct endpoint and ***A1*** for the correct closed interval.

(ii) $\int_0^{\frac{\pi}{2}}(\sin 2x+3)^{\frac{1}{2}}\,\mathrm{d}x$

$=2.99$

(iii) $\pi\int_0^{\frac{\pi}{2}}(\sin 2x+3)\,\mathrm{d}x-\pi(1)\left(\frac{\pi}{2}\right)$ (or equivalent)

Note: Award ***(M1)(A1)(A1)*** for $\pi\int_0^{\frac{\pi}{2}}(\sin 2x+2)\mathrm{d}x$

$=17.946-4.935\ \left(=\frac{\pi}{2}(3\pi+2)-\pi\left(\frac{\pi}{2}\right)\right)$

$=13.0$

Note: Award ***A1*** for $\pi(\pi+1)$.

7 – (MAT/21_HL_Summer_2013_Q13) – *Differentiation*

(a) $f(2) = 9$

$f^{-1}(x) = (x-1)^{\frac{1}{3}}$

$(f^{-1})'(x) = \frac{1}{3}(x-1)^{-\frac{2}{3}}$

$(f^{-1})'(9) = \frac{1}{12}$

$f'(x) = 3x^2$

$\frac{1}{f'(2)} = \frac{1}{3 \times 4} = \frac{1}{12}$

Note: The last ***M1*** and ***A1*** are independent of previous marks.

(b) $g'(x) = e^{x^2} + 2x^2 e^{x^2}$

$g'(x) > 0$ as each part is positive

(c) to find the x-coordinate on $y = g(x)$ solve

$2 = xe^{x^2}$

$x = 0.89605022078\ldots$

gradient $= (g^{-1})'(2) = \dfrac{1}{g'(0.896\ldots)}$

$= \dfrac{1}{e^{(0.896\ldots)^2}\left(1 + 2\times(0.896\ldots)^2\right)} = 0.172$ to 3sf

(using the $\dfrac{dy}{dx}$ function on gdc $g'(0.896\ldots) = 5.7716028\ldots$

$\dfrac{1}{g'(0.896\ldots)} = 0.173$)

(d) (i) $(x^3+1)e^{(x^3+1)^2} = 2$

$x = -0.470191\ldots$

(ii) **METHOD 1**

$(g\circ f)'(x) = 3x^2e^{(x^3+1)^2}\left(2(x^3+1)^2+1\right)$

$(g\circ f)'(-0.470191\ldots) = 3.85755\ldots$

$h'(2) = \dfrac{1}{3.85755\ldots} = 0.259\ (232\ldots)$

Note: The solution can be found without the student obtaining the explicit form of the composite function.

METHOD 2

$h(x) = (f^{-1}\circ g^{-1})(x)$

$h'(x) = (f^{-1})'\left(g^{-1}(x)\right)\times(g^{-1})'(x)$

$= \dfrac{1}{3}\left(g^{-1}(x)-1\right)^{-\frac{2}{3}}\times(g^{-1})'(x)$

$h'(2) = \dfrac{1}{3}\left(g^{-1}(2)-1\right)^{-\frac{2}{3}}\times(g^{-1})'(2)$

$= \dfrac{1}{3}(0.89605\ldots-1)^{-\frac{2}{3}}\times 0.171933\ldots$

$= 0.259\ (232\ldots)$

8 – (MAT/22_HL_Summer_2013_Q13) – *Differentiation*

(a) **EITHER**

$$\theta = \pi - \arctan\left(\frac{8}{x}\right) - \arctan\left(\frac{13}{20-x}\right) \text{ (or equivalent)}$$

Note: Accept $\theta = 180° - \arctan\left(\frac{8}{x}\right) - \arctan\left(\frac{13}{20-x}\right)$ (or equivalent).

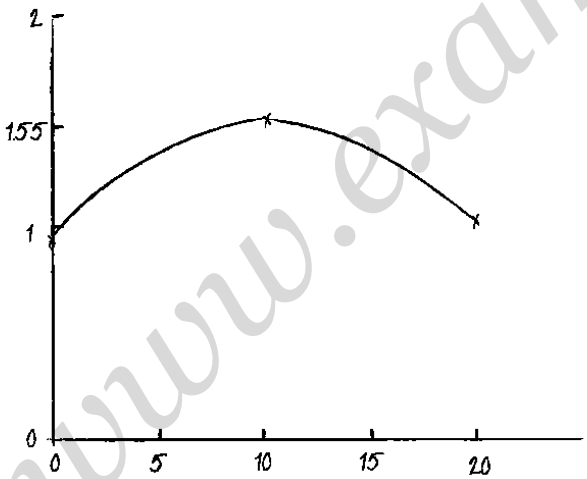
OR

$$\theta = \arctan\left(\frac{x}{8}\right) + \arctan\left(\frac{20-x}{13}\right) \text{ (or equivalent)}$$

(b) (i) $\theta = 0.994 \ \left(= \arctan\frac{20}{13}\right)$

(ii) $\theta = 1.19 \ \left(= \arctan\frac{5}{2}\right)$

(c) correct shape.
correct domain indicated.



(d) attempting to differentiate one $\arctan(f(x))$ term

EITHER

$$\theta = \pi - \arctan\left(\frac{8}{x}\right) - \arctan\left(\frac{13}{20-x}\right)$$

$$\frac{d\theta}{dx} = \frac{8}{x^2} \times \frac{1}{1+\left(\frac{8}{x}\right)^2} - \frac{13}{(20-x)^2} \times \frac{1}{1+\left(\frac{13}{20-x}\right)^2}$$

OR

$$\theta = \arctan\left(\frac{x}{8}\right) + \arctan\left(\frac{20-x}{13}\right)$$

$$\frac{d\theta}{dx} = \frac{\frac{1}{8}}{1+\left(\frac{x}{8}\right)^2} + \frac{-\frac{1}{13}}{1+\left(\frac{20-x}{13}\right)^2}$$

THEN

$$= \frac{8}{x^2+64} - \frac{13}{569-40x+x^2}$$

$$= \frac{8(569-40x+x^2)-13(x^2+64)}{(x^2+64)(x^2-40x+569)}$$

$$= \frac{5(744-64x-x^2)}{(x^2+64)(x^2-40x+569)}$$

(e) Maximum light intensity at P occurs when $\frac{d\theta}{dx} = 0$.

either attempting to solve $\frac{d\theta}{dx} = 0$ for x or using the graph of either θ or $\frac{d\theta}{dx}$

$x = 10.05$ (m)

(f) $\frac{dx}{dt} = 0.5$

At $x = 10$, $\frac{d\theta}{dx} = 0.000453$ $(= \frac{5}{11\,029})$.

use of $\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt}$

$\frac{d\theta}{dt} = 0.000227$ $(= \frac{5}{22058})$ (rad s^{-1})

9 – (MAT/22_HL_Summer_2014_Q9) – *Differentiation*

METHOD 1

volume of a cone is $V = \frac{1}{3}\pi r^2 h$

given $h = r$, $V = \frac{1}{3}\pi h^3$

$\frac{dV}{dh} = \pi h^2$

when $h = 4$, $\frac{dV}{dt} = \pi \times 4^2 \times 0.5$ (using $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$)

$\frac{dV}{dt} = 8\pi \ (= 25.1) \ (\text{cm}^3\,\text{min}^{-1})$

METHOD 2

volume of a cone is $V = \frac{1}{3}\pi r^2 h$

given $h = r$, $V = \frac{1}{3}\pi h^3$

$\frac{dV}{dt} = \frac{1}{3}\pi \times 3h^2 \times \frac{dh}{dt}$

when $h = 4$, $\frac{dV}{dt} = \pi \times 4^2 \times 0.5$

$\frac{dV}{dt} = 8\pi \ (= 25.1) \ (\text{cm}^3\,\text{min}^{-1})$

METHOD 3

$V = \frac{1}{3}\pi r^2 h$

$\frac{dV}{dt} = \frac{1}{3}\pi\left(2r\,h\frac{dr}{dt} + r^2\frac{dh}{dt}\right)$

Note: Award ***M1*** for attempted implicit differentiation and ***A1*** for each correct term on the RHS.

when $h = 4$, $r = 4$, $\frac{dV}{dt} = \frac{1}{3}\pi\left(2 \times 4 \times 4 \times 0.5 + 4^2 \times 0.5\right)$

$\frac{dV}{dt} = 8\pi \ (= 25.1) \ (\text{cm}^3\,\text{min}^{-1})$

10 – (MAT/21_HL_Summer_2014_Q10) – *Functions - Roots, Differentiation*

(a) $x \to -\infty \Rightarrow y \to -\frac{1}{2}$ so $y = -\frac{1}{2}$ is an asymptote

$e^x - 2 = 0 \Rightarrow x = \ln 2$ so $x = \ln 2 (= 0.693)$ is an asymptote

(b) (i) $f'(x) = \dfrac{2\left(e^x - 2\right)e^{2x} - \left(e^{2x} + 1\right)e^x}{\left(e^x - 2\right)^2}$

$= \dfrac{e^{3x} - 4e^{2x} - e^x}{\left(e^x - 2\right)^2}$

(ii) $f'(x) = 0$ when $e^{3x} - 4e^{2x} - e^x = 0$

$e^x\left(e^{2x} - 4e^x - 1\right) = 0$

$e^x = 0,\ e^x = -0.236,\ e^x = 4.24\ \left(\text{or } e^x = 2 \pm \sqrt{5}\right)$

Note: Award *A1* for zero, *A1* for other two solutions.
Accept any answers which show a zero, a negative and a positive.

as $e^x > 0$ exactly one solution

Note: Do not award marks for purely graphical solution.

(iii) $(1.44, 8.47)$

(c) $f'(0) = -4$

so gradient of normal is $\frac{1}{4}$

$f(0) = -2$

so equation of L_1 is $y = \frac{1}{4}x - 2$

(d) $f'(x) = \frac{1}{4}$

so $x = 1.46$

$f(1.46) = 8.47$

equation of L_2 is $y - 8.47 = \frac{1}{4}(x - 1.46)$

(or $y = \frac{1}{4}x + 8.11$)

11 – (MAT/22_HL_Summer_2014_Q10) – *Differentiation*

(a) **METHOD 1**

expanding the brackets first:

$x^4 + 2x^2y^2 + y^4 = 4xy^2$

$4x^3 + 4xy^2 + 4x^2y\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 4y^2 + 8xy\frac{dy}{dx}$

Note:	Award ***M1*** for an attempt at implicit differentiation. Award ***A1*** for each side correct.

$\frac{dy}{dx} = \frac{-x^3 - xy^2 + y^2}{yx^2 - 2xy + y^3}$ or equivalent

METHOD 2

$2\left(x^2 + y^2\right)\left(2x + 2y\frac{dy}{dx}\right) = 4y^2 + 8xy\frac{dy}{dx}$

Note:	Award ***M1*** for an attempt at implicit differentiation. Award ***A1*** for each side correct.

$\left(x^2 + y^2\right)\left(x + y\frac{dy}{dx}\right) = y^2 + 2xy\frac{dy}{dx}$

$x^3 + x^2y\frac{dy}{dx} + y^2x + y^3\frac{dy}{dx} = y^2 + 2xy\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{-x^3 - xy^2 + y^2}{yx^2 - 2xy + y^3}$ or equivalent

(b) **METHOD 1**

at $(1, 1)$, $\frac{dy}{dx}$ is undefined

$y = 1$

METHOD 2

gradient of normal $= -\frac{1}{\frac{dy}{dx}} = -\frac{\left(yx^2 - 2xy + y^3\right)}{\left(-x^3 - xy^2 + y^2\right)}$

at $(1, 1)$ gradient $= 0$

$y = 1$

12 – (MAT/22_HL_Summer_2014_Q12) – *Differentiation*

(a) $C = \text{AX} \times 5k + \text{XB} \times k$ ***(M1)***

Note: Award ***(M1)*** for attempting to express the cost in terms of AX, XB and k.

$= 5k\sqrt{450^2 + x^2} + (1000 - x)k$ ***A1***

$= 5k\sqrt{202500 + x^2} + (1000 - x)k$ ***AG***

(b) (i) $\dfrac{\mathrm{d}C}{\mathrm{d}x} = k\left[\dfrac{5 \times 2x}{2\sqrt{202500 + x^2}} - 1\right] = k\left(\dfrac{5x}{\sqrt{202500 + x^2}} - 1\right)$ ***M1A1***

(ii) attempting to solve $\frac{dC}{dx}=0$ *M1*

$\frac{5x}{\sqrt{202500+x^2}}=1$ *(A1)*

$x=91.9\text{ (m)}\left(=\frac{75\sqrt{6}}{2}\text{(m)}\right)$ *A1*

METHOD 1
for example,

at $x=91$ $\frac{dC}{dx}=-0.00895k<0$ *M1*

at $x=92$ $\frac{dC}{dx}=0.001506k>0$ *A1*

Note: Award *M1* for attempting to find the gradient either side of $x=91.9$ and *A1* for two correct values.

thus $x=91.9$ gives a minimum *AG*

METHOD 2

$$\frac{d^2C}{dx^2}=\frac{1012500k}{\left(x^2+202500\right)^{\frac{3}{2}}}$$

at $x=91.9$ $\frac{d^2C}{dx^2}=0.010451k>0$ *(M1)A1*

Note: Award *M1* for attempting to find the second derivative and *A1* for the correct value.

Note: If $\frac{d^2C}{dx^2}$ is obtained and its value at $x=91.9$ is not calculated, award *(M1)A1* for correct reasoning *eg*, both numerator and denominator are positive at $x=91.9$.

thus $x=91.9$ gives a minimum *AG*

METHOD 3

Sketching the graph of either C versus x or $\frac{dC}{dx}$ versus x. *M1*

Clearly indicating that $x=91.9$ gives the minimum on their graph. *A1*

(c) $C_{\min} = 3205k$

Note:	Accept $3200k$. Accept $3204\ k$.

(d) $\arctan\left(\dfrac{450}{91.855865\text{K}}\right) = 78.463\text{K}^{\circ}$

$180 - 78.463\text{K} = 101.537\text{K}$

$= 102^{\circ}$

(e) (i) when $\theta = 120^{\circ},\ x = 260\,(\text{m})\ \left(\dfrac{450}{\sqrt{3}}(\text{m})\right)$

(ii) $\dfrac{133.728\text{K}}{3204.5407685\text{K}} \times 100\%$

$= 4.17\,(\%)$

13 – (MAT/20_HL_Winter_2014_Q10) – *Differentiation*

(a) use of $A = \frac{1}{2}qr\sin\theta$ to obtain $A = \frac{1}{2}(x+2)(5-x)^2\sin 30^\circ$

$= \frac{1}{4}(x+2)(25-10x+x^2)$

$A = \frac{1}{4}(x^3 - 8x^2 + 5x + 50)$

(b) (i) $\frac{dA}{dx} = \frac{1}{4}\left(3x^2 - 16x + 5\right) = \frac{1}{4}(3x-1)(x-5)$

(ii) **METHOD 1**

EITHER

$\frac{dA}{dx} = \frac{1}{4}\left(3\left(\frac{1}{3}\right)^2 - 16\left(\frac{1}{3}\right) + 5\right) = 0$

OR

$\frac{dA}{dx} = \frac{1}{4}\left(3\left(\frac{1}{3}\right) - 1\right)\left(\left(\frac{1}{3}\right) - 5\right) = 0$

THEN

so $\frac{dA}{dx} = 0$ when $x = \frac{1}{3}$

METHOD 2

solving $\frac{dA}{dx} = 0$ for x

$-2 < x < 5 \Rightarrow x = \frac{1}{3}$

so $\frac{dA}{dx} = 0$ when $x = \frac{1}{3}$

METHOD 3

a correct graph of $\frac{dA}{dx}$ versus x

the graph clearly showing that $\frac{dA}{dx} = 0$ when $x = \frac{1}{3}$

so $\frac{dA}{dx} = 0$ when $x = \frac{1}{3}$

(c) (i) $\frac{d^2A}{dx^2} = \frac{1}{2}(3x-8)$

for $x = \frac{1}{3}$, $\frac{d^2A}{dx^2} = -3.5 (< 0)$

so $x = \frac{1}{3}$ gives the maximum area of triangle PQR

(ii) $A_{max} = \frac{343}{27} (= 12.7) (\text{cm}^2)$

(iii) $\text{PQ} = \frac{7}{3} (\text{cm})$ and $\text{PR} = \left(\frac{14}{3}\right)^2 (\text{cm})$

$\text{QR}^2 = \left(\frac{7}{3}\right)^2 + \left(\frac{14}{3}\right)^4 - 2\left(\frac{7}{3}\right)\left(\frac{14}{3}\right)^2 \cos 30^\circ$

$= 391.702\ldots$

$\text{QR} = 19.8 (\text{cm})$

14 – (MAT/20_HL_Winter_2014_Q13) – *Differentiation, Integration*

(a) attempting to use $V = \pi\int_a^b x^2 \mathrm{d}y$

attempting to express x^2 in terms of y *ie* $x^2 = 4(y+16)$

for $y = h$, $V = 4\pi\int_0^h y + 16\,\mathrm{d}y$

$V = 4\pi\left(\frac{h^2}{2} + 16h\right)$

(b) (i) **METHOD 1**

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t}$

$\frac{\mathrm{d}V}{\mathrm{d}h} = 4\pi(h+16)$

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{4\pi(h+16)} \times \frac{-250\sqrt{h}}{\pi(h+16)}$

Note: Award ***M1*** for substitution into $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t}$.

$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{250\sqrt{h}}{4\pi^2(h+16)^2}$

METHOD 2

$\frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi(h+16)\frac{\mathrm{d}h}{\mathrm{d}t}$ (implicit differentiation)

$\frac{-250\sqrt{h}}{\pi(h+16)} = 4\pi(h+16)\frac{\mathrm{d}h}{\mathrm{d}t}$ (or equivalent)

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{4\pi(h+16)} \times \frac{-250\sqrt{h}}{\pi(h+16)}$

$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{250\sqrt{h}}{4\pi^2(h+16)^2}$

(ii) $\frac{\mathrm{d}t}{\mathrm{d}h} = -\frac{4\pi^2(h+16)^2}{250\sqrt{h}}$

$t = \int -\frac{4\pi^2(h+16)^2}{250\sqrt{h}}\,\mathrm{d}h$

$t = \int -\frac{4\pi^2(h^2+32h+256)}{250\sqrt{h}}\,\mathrm{d}h$

$t = \frac{-4\pi^2}{250}\int\left(h^{\frac{3}{2}} + 32h^{\frac{1}{2}} + 256h^{-\frac{1}{2}}\right)\mathrm{d}h$

(iii) **METHOD 1**

$$t = \frac{-4\pi^2}{250}\int_{48}^{0}\left(h^{\frac{3}{2}} + 32h^{\frac{1}{2}} + 256h^{-\frac{1}{2}}\right)\mathrm{d}h$$ ***(M1)***

$t = 2688.756\ldots$ (s) ***(A1)***

45 minutes (correct to the nearest minute) ***A1***

METHOD 2

$$t = \frac{-4\pi^2}{250}\left(\frac{2}{5}h^{\frac{5}{2}} + \frac{64}{3}h^{\frac{3}{2}} + 512h^{\frac{1}{2}}\right) + c$$

when $t = 0, h = 48 \Rightarrow c = 2688.756\ldots\left(c = \frac{4\pi^2}{250}\left(\frac{2}{5}\times 48^{\frac{5}{2}} + \frac{64}{3}\times 48^{\frac{3}{2}} + 512\times 48^{\frac{1}{2}}\right)\right)$

when $h = 0$, $t = 2688.756\ldots\left(t = \frac{4\pi^2}{250}\left(\frac{2}{5}\times 48^{\frac{5}{2}} + \frac{64}{3}\times 48^{\frac{3}{2}} + 512\times 48^{\frac{1}{2}}\right)\right)$(s)

45 minutes (correct to the nearest minute)

(c) **EITHER**

the depth stabilises when $\frac{\mathrm{d}V}{\mathrm{d}t} = 0$ *ie* $8.5 - \frac{250\sqrt{h}}{\pi(h+16)} = 0$ ***R1***

attempting to solve $8.5 - \frac{250\sqrt{h}}{\pi(h+16)} = 0$ for h ***(M1)***

OR

the depth stabilises when $\frac{\mathrm{d}h}{\mathrm{d}t} = 0$ *ie* $\frac{1}{4\pi(h+16)}\left(8.5 - \frac{250\sqrt{h}}{\pi(h+16)}\right) = 0$ ***R1***

attempting to solve $\frac{1}{4\pi(h+16)}\left(8.5 - \frac{250\sqrt{h}}{\pi(h+16)}\right) = 0$ for h ***(M1)***

THEN

$h = 5.06$ (cm) ***A1***

15 – (MAT/21_HL_Summer_2015_Q5) – *Differentiation*

$V = 200\pi r^2$

Note: Allow $V = \pi h r^2$ if value of h is substituted later in the question.

EITHER

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 200\pi 2r\frac{\mathrm{d}r}{\mathrm{d}t}$$

Note: Award ***M1*** for an attempt at implicit differentiation.

at $r = 2$ we have $30 = 200\pi 4\frac{\mathrm{d}r}{\mathrm{d}t}$

OR

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\frac{\mathrm{d}V}{\mathrm{d}t}}{\frac{\mathrm{d}V}{\mathrm{d}r}}$$

$$\frac{\mathrm{d}V}{\mathrm{d}r} = 400\pi r$$

$r = 2$ we have $\frac{\mathrm{d}V}{\mathrm{d}r} = 800\pi$

THEN

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{30}{800\pi}\left(= \frac{3}{80\pi} \quad = 0.0119\right) \ (\mathrm{cm\,s^{-1}})$$

16 – (MAT/21_HL_Summer_2015_Q6) – *Differentiation*

$$f'(x) = 3x^2 + e^x$$

Note: Accept labelled diagram showing the graph $y = f'(x)$ above the x-axis; do not accept unlabelled graphs nor graph of $y = f(x)$.

EITHER

this is always > 0
so the function is (strictly) increasing
and thus $1-1$

OR

this is always > 0 (accept $\neq 0$)
so there are no turning points
and thus $1-1$

17 – (MAT/21_HL_Summer_2015_Q9) – *Differentiation*

$x = 0 \Rightarrow y = 1$

$y'(0) = 1.367879...$

Note: The exact answer is $y'(0) = \frac{e+1}{e} = 1 + \frac{1}{e}$.

so gradient of normal is $\frac{-1}{1.367879...} \left(= -0.731058...\right)$

equation of normal is $y = -0.731058...x + c$

gives $y = -0.731x + 1$

18 – (MAT/22_HL_Summer_2015_Q11) – *Differentiation*

(a) attempt at implicit differentiation

$$2x-5x\frac{\mathrm{d}y}{\mathrm{d}x}-5y+2y\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

Note: ***A1*** for differentiation of x^2-5xy, ***A1*** for differentiation of y^2 and 7.

$$2x-5y+\frac{\mathrm{d}y}{\mathrm{d}x}(2y-5x)=0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{5y-2x}{2y-5x}$$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{5\times1-2\times6}{2\times1-5\times6}=\frac{1}{4}$

gradient of normal $=-4$

equation of normal $y=-4x+c$

substitution of $(6, 1)$

$y=-4x+25$

Note: Accept $y-1=-4(x-6)$

(c) setting $\frac{5y-2x}{2y-5x}=1$

$y=-x$

substituting into original equation

$x^2+5x^2+x^2=7$

$7x^2=7$

$x=\pm1$

points $(1, -1)$ and $(-1, 1)$

distance $=\sqrt{8}\left(=2\sqrt{2}\right)$

19 – (MAT/20_HL_Winter_2015_Q13) – *Differentiation*

(a) $f'(x) = 30e^{-\frac{x^2}{400}} \cdot -\frac{2x}{400}\left(= -\frac{3x}{20}e^{-\frac{x^2}{400}}\right)$

Note: Award ***M1*** for attempting to use the chain rule.

$$f''(x) = -\frac{3}{20}e^{-\frac{x^2}{400}} + \frac{3x^2}{4000}e^{-\frac{x^2}{400}}\left(= \frac{3}{20}e^{-\frac{x^2}{400}}\left(\frac{x^2}{200} - 1\right)\right)$$

Note: Award ***M1*** for attempting to use the product rule.

(b) the roof function has maximum gradient when $f''(x) = 0$

Note: Award ***(M1)*** for attempting to find $f''\left(-\sqrt{200}\right)$.

EITHER
$= 0$

OR
$f''(x) = 0 \Rightarrow x = \pm\sqrt{200}$

THEN
valid argument for maximum such as reference to an appropriate graph or change in the sign of $f''(x)$ *eg* $f''(-15) = 0.010...(> 0)$ and $f''(-14) = -0.001...(< 0)$

$\Rightarrow x = -\sqrt{200}$

(c) $A = 2a \cdot 30\mathrm{e}^{-\frac{a^2}{400}} \left(= 60a\mathrm{e}^{-\frac{a^2}{400}} = -400 f'(a) \right)$ ***(M1)(A1)***

EITHER

$\frac{\mathrm{d}A}{\mathrm{d}a} = 60a\mathrm{e}^{-\frac{a^2}{400}} \cdot -\frac{a}{200} + 60\mathrm{e}^{-\frac{a^2}{400}} = 0 \Rightarrow a = \sqrt{200} \quad \left(-400 f''(a) = 0 \Rightarrow a = \sqrt{200}\right)$

M1A1

OR

by symmetry *eg* $a = -\sqrt{200}$ found in (b) or $A_{\max}$ coincides with $f''(a) = 0$ ***R1***

$\Rightarrow a = \sqrt{200}$ ***A1***

THEN

$A_{\max} = 60 \cdot \sqrt{200}\mathrm{e}^{-\frac{200}{400}}$ ***M1***

$= 600\sqrt{2}\mathrm{e}^{-\frac{1}{2}}$ ***AG***

(d) (i) perimeter $= 4a + 60\mathrm{e}^{-\frac{a^2}{400}}$ ***A1A1***

Note: Condone use of x.

(ii) $I(a) = \dfrac{4a + 60\mathrm{e}^{-\frac{a^2}{400}}}{60a\mathrm{e}^{-\frac{a^2}{400}}}$ ***(A1)***

graphing $I(a)$ or other valid method to find the minimum ***(M1)***

$a = 12.6$ ***A1***

(iii) area under roof $= \int_{-20}^{20} 30\mathrm{e}^{-\frac{x^2}{400}}\,\mathrm{d}x$ ***M1***

$= 896.18\ldots$ ***(A1)***

area of living space $= 60 \cdot (12.6\ldots) \cdot \mathrm{e}^{-\frac{(12.6\ldots)^2}{400}} = 508.56\ldots$ ***(A1)***

percentage of empty space $= 43.3\%$ ***A1***

20 – (MAT/22_HL_Summer_2016_Q7) – *Differentiation*

(a) $3x^2 + 3y^2\frac{dy}{dx} = 4\left(y + x\frac{dy}{dx}\right)$

$\left(3y^2 - 4x\right)\frac{dy}{dx} = 4y - 3x^2$

$\frac{dy}{dx} = \frac{4y - 3x^2}{3y^2 - 4x}$

(b) $\frac{dy}{dx} = 0 \Rightarrow 4y - 3x^2 = 0$

substituting $x = k$ and $y = \frac{3}{4}k^2$ into $x^3 + y^3 = 4xy$

$k^3 + \frac{27}{64}k^6 = 3k^3$

attempting to solve $k^3 + \frac{27}{64}k^6 = 3k^3$ for k

$k = 1.68\left(= \frac{4}{3}\sqrt[3]{2}\right)$

21 – (MAT/22_HL_Summer_2016_Q11) – *Differentiation*

(a) **EITHER**

$\alpha = \arctan\frac{7}{10} - \arctan\frac{5}{10}$ $(= 34.992\ldots^\circ - 26.5651\ldots^\circ)$ ***(M1)(A1)(A1)***

Note: Award ***(M1)*** for $\alpha = \mathrm{A\hat{P}T} - \mathrm{B\hat{P}T}$, ***(A1)*** for a correct $\mathrm{A\hat{P}T}$ and ***(A1)*** for a correct $\mathrm{B\hat{P}T}$.

OR

$\alpha = \arctan 2 - \arctan\frac{10}{7}$ $(= 63.434\ldots^\circ - 55.008\ldots^\circ)$ ***(M1)(A1)(A1)***

Note: Award ***(M1)*** for $\alpha = \mathrm{P\hat{B}T} - \mathrm{P\hat{A}T}$, ***(A1)*** for a correct $\mathrm{P\hat{B}T}$ and ***(A1)*** for a correct $\mathrm{P\hat{A}T}$.

OR

$\alpha = \arccos\left(\frac{125 + 149 - 4}{2 \times \sqrt{125} \times \sqrt{149}}\right)$ ***(M1)(A1)(A1)***

Note: Award ***(M1)*** for use of cosine rule, ***(A1)*** for a correct numerator and ***(A1)*** for a correct denominator.

THEN

$= 8.43^\circ$ ***A1***

(b) **EITHER**

$$\tan\alpha = \frac{\frac{7}{x} - \frac{5}{x}}{1 + \left(\frac{7}{x}\right)\left(\frac{5}{x}\right)}$$ ***M1A1A1***

Note: Award ***M1*** for use of $\tan(A - B)$, ***A1*** for a correct numerator and ***A1*** for a correct denominator.

$$= \frac{\frac{2}{x}}{1 + \frac{35}{x^2}}$$ ***M1***

OR

$$\tan\alpha = \frac{\frac{x}{5} - \frac{x}{7}}{1 + \left(\frac{x}{5}\right)\left(\frac{x}{7}\right)}$$ ***M1A1A1***

Note: Award ***M1*** for use of $\tan(A - B)$, ***A1*** for a correct numerator and ***A1*** for a correct denominator.

$$= \frac{\frac{2x}{35}}{1 + \frac{x^2}{35}}$$ ***M1***

OR

$$\cos\alpha = \frac{x^2 + 35}{\sqrt{(x^2 + 25)(x^2 + 49)}}$$ ***M1A1***

Note: Award ***M1*** for either use of the cosine rule or use of $\cos(A - B)$.

$$\sin\alpha = \frac{2x}{\sqrt{(x^2 + 25)(x^2 + 49)}}$$ ***A1***

$$\tan\alpha = \frac{\frac{2x}{\sqrt{(x^2 + 25)(x^2 + 49)}}}{\frac{x^2 + 35}{\sqrt{(x^2 + 25)(x^2 + 49)}}}$$ ***M1***

THEN

$$\tan\alpha = \frac{2x}{x^2 + 35}$$ ***AG***

(c) (i) $\frac{\mathrm{d}}{\mathrm{d}x}(\tan\alpha) = \frac{2\left(x^2+35\right)-(2x)(2x)}{\left(x^2+35\right)^2} \left(= \frac{70-2x^2}{\left(x^2+35\right)^2}\right)$

Note: Award ***M1*** for attempting product or quotient rule differentiation, ***A1*** for a correct numerator and ***A1*** for a correct denominator.

(ii) **METHOD 1**

EITHER

$\frac{\mathrm{d}}{\mathrm{d}x}(\tan\alpha) = 0 \Rightarrow 70 - 2x^2 = 0$

$x = \sqrt{35}(\mathrm{m})\left(= 5.9161\ldots(\mathrm{m})\right)$

$\tan\alpha = \frac{1}{\sqrt{35}}(= 0.16903\ldots)$

OR

attempting to locate the stationary point on the graph of

$\tan\alpha = \frac{2x}{x^2+35}$

$x = 5.9161\ldots\ (\mathrm{m})\left(= \sqrt{35}\,(\mathrm{m})\right)$

$\tan\alpha = 0.16903\ldots\left(= \frac{1}{\sqrt{35}}\right)$

THEN

$\alpha = 9.59^\circ$

METHOD 2

EITHER

$\alpha = \arctan\left(\frac{2x}{x^2+35}\right) \Rightarrow \frac{d\alpha}{dx} = \frac{70-2x^2}{\left(x^2+35\right)^2+4x^2}$ ***M1***

$\frac{d\alpha}{dx} = 0 \Rightarrow x = \sqrt{35}(\text{m})\left(= 5.9161\ldots(\text{m})\right)$ ***A1***

OR

attempting to locate the stationary point on the graph of

$\alpha = \arctan\left(\frac{2x}{x^2+35}\right)$ ***(M1)***

$x = 5.9161\ldots(\text{m})\ \left(= \sqrt{35}\,(\text{m})\right)$ ***A1***

THEN

$\alpha = 0.1674\ldots\left(= \arctan\frac{1}{\sqrt{35}}\right)$ ***(A1)***

$= 9.59^\circ$ ***A1***

(iii) $\frac{d^2}{dx^2}(\tan\alpha) = \frac{\left(x^2+35\right)^2(-4x)-(2)(2x)\left(x^2+35\right)\left(70-2x^2\right)}{\left(x^2+35\right)^4}\ \left(=\frac{4x\left(x^2-105\right)}{\left(x^2+35\right)^3}\right)$

M1A1

substituting $x = \sqrt{35}\ (= 5.9161\ldots)$ into $\frac{d^2}{dx^2}(\tan\alpha)$ ***M1***

$\frac{d^2}{dx^2}(\tan\alpha) < 0\ (= -0.004829\ldots)$ and so $\alpha = 9.59^\circ$ is the maximum value of α ***R1***

α never exceeds 10° ***AG***

(d) attempting to solve $\frac{2x}{x^2+35} \geq \tan 7^\circ$

Note: Award ***(M1)*** for attempting to solve $\frac{2x}{x^2+35} = \tan 7^\circ$.

$x = 2.55$ and $x = 13.7$

$2.55 \leq x \leq 13.7\ (\text{m})$

22 – (MAT/21_HL_Summer_2016_Q12) – *Differentiation*

(a) $a^2 = 5 - 1$

$a = 2$

(b) $2y\dfrac{dy}{dx} - \left(2x\dfrac{dy}{dx} + 2y\right) = -e^x$

Note: Award ***M1*** for an attempt at implicit differentiation, ***A1*** for each part.

$$\frac{dy}{dx} = \frac{2y - e^x}{2(y - x)}$$

(c) at $x = 0$, $\dfrac{dy}{dx} = \dfrac{3}{4}$

finding the negative reciprocal of a number

gradient of normal is $-\dfrac{4}{3}$

$y = -\dfrac{4}{3}x + 2$

(d) substituting linear expression

$\left(-\dfrac{4}{}x + 2\right)^2 - 2x\left(-\dfrac{4}{3}x + 2\right) + e^x - 5 = 0$ or equivalent

$x = 1.56$

$y = -0.0779$

$(1.56\ ,\ -0.0779)$

(e) $\dfrac{dv}{dx} = 3y^2\dfrac{dy}{dx}$

$\dfrac{dv}{dx} = 3 \times 4 \times \dfrac{3}{4} = 9$

23 – (MAT/22_HL_Summer_2016_Q12) – *Differentiation, Integration*

(a) (i)

$$\frac{1}{4\left(\frac{e^x+e^{-x}}{2}\right)-2\left(\frac{e^x-e^{-x}}{2}\right)}$$

$$=\frac{1}{2\left(e^x+e^{-x}\right)-\left(e^x-e^{-x}\right)}$$

$$=\frac{1}{e^x+3e^{-x}}$$

$$=\frac{e^x}{e^{2x}+3}$$

(ii) $u=e^x \Rightarrow du=e^x dx$

$$\int\frac{e^x}{e^{2x}+3}dx=\int\frac{1}{u^2+3}du$$

(when $x=0$, $u=1$ and when $x=\ln 3$, $u=3$)

$$\int_1^3\frac{1}{u^2+3}du=\left[\frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)\right]_1^3$$

$$\left(=\left[\frac{1}{\sqrt{3}}\arctan\left(\frac{e^x}{\sqrt{3}}\right)\right]_0^{\ln 3}\right)$$

$$=\frac{\pi\sqrt{3}}{9}-\frac{\pi\sqrt{3}}{18}$$

$$=\frac{\pi\sqrt{3}}{18}$$

(b) (i) $(n+1)e^{2x} - 2ke^{x} + (n-1) = 0$

$$e^{x} = \frac{2k \pm \sqrt{4k^2 - 4\left(n^2 - 1\right)}}{2(n+1)}$$

$$x = \ln\left(\frac{k \pm \sqrt{k^2 - n^2 + 1}}{n+1}\right)$$

(ii) for two real solutions, we require $k > \sqrt{k^2 - n^2 + 1}$

and we also require $k^2 - n^2 + 1 > 0$

$k^2 > n^2 - 1$

$\Rightarrow k > \sqrt{n^2 - 1}$ $(k \in \mathbb{R}^+)$

(c) (i) **METHOD 1**

$$t(x) = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

$$t'(x) = \frac{\left(e^{x} + e^{-x}\right)^2 - \left(e^{x} - e^{-x}\right)^2}{\left(e^{x} + e^{-x}\right)^2}$$

$$t'(x) = \frac{\left(\frac{e^{x} + e^{-x}}{2}\right)^2 - \left(\frac{e^{x} - e^{-x}}{2}\right)^2}{\left(\frac{e^{x} + e^{-x}}{2}\right)^2}$$

$$= \frac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$$

METHOD 2

$$t'(x) = \frac{f(x)g'(x) - g(x)f'(x)}{[f(x)]^2}$$

$g'(x) = f(x)$ and $f'(x) = g(x)$

$$= \frac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$$

METHOD 3

$t(x) = \left(e^x - e^{-x}\right)\left(e^x + e^{-x}\right)^{-1}$

$t'(x) = 1 - \dfrac{\left(e^x - e^{-x}\right)^2}{\left(e^x + e^{-x}\right)^2}$

$= 1 - \dfrac{[g(x)]^2}{[f(x)]^2}$

$= \dfrac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$

METHOD 4

$t'(x) = \dfrac{g'(x)}{f(x)} - \dfrac{g(x)f'(x)}{[f(x)]^2}$

$g'(x) = f(x)$ and $f'(x) = g(x)$ gives $t'(x) = 1 - \dfrac{[g(x)]^2}{[f(x)]^2}$

$= \dfrac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$

(ii) **METHOD 1**

$[f(x)]^2 > [g(x)]^2$ (or equivalent)

$[f(x)]^2 > 0$

hence $t'(x) > 0$, $x \in \mathbb{R}$

METHOD 2

$[f(x)]^2 - [g(x)]^2 = 1$ (or equivalent) ***M1A1***

$[f(x)]^2 > 0$ ***R1***

hence $t'(x) > 0$, $x \in \mathbb{R}$ ***AG***

Note: Award as above for use of either $f(x) = \dfrac{e^x + e^{-x}}{2}$ and $g(x) = \dfrac{e^x - e^{-x}}{2}$ or $e^x + e^{-x}$ and $e^x - e^{-x}$.

METHOD 3

$$t'(x) = \frac{4}{\left(e^x + e^{-x}\right)^2}$$

$\left(e^x + e^{-x}\right)^2 > 0$ ***M1A1***

$\dfrac{4}{\left(e^x + e^{-x}\right)^2} > 0$ ***R1***

hence $t'(x) > 0$, $x \in \mathbb{R}$ ***AG***

24 – (MAT/20_HL_Winter_2016_Q6) – *Differentiation*

METHOD 1

substituting for x and attempting to solve for y (or vice versa) ***(M1)***

$y = (\pm)\ 0.11821\ldots$ ***(A1)***

EITHER

$145x + 143y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad \left(\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{145x}{143y}\right)$ ***M1A1***

OR

$145x\frac{\mathrm{d}x}{\mathrm{d}t} + 143y\frac{\mathrm{d}y}{\mathrm{d}t} = 0$ ***M1A1***

THEN

attempting to find $\frac{\mathrm{d}y}{\mathrm{d}t} \quad \left(\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{145\left(3.2\times10^{-3}\right)}{143\left((\pm)\ 0.11821\ldots\right)}\times\left(7.75\times10^{-5}\right)\right)$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}t} = \pm 2.13\times10^{-6}$ ***A1***

Note: Award all marks except the final ***A1*** to candidates who do not consider $\pm$.

METHOD 2

$y = (\pm)\sqrt{\frac{1-72.5x^2}{71.5}}$ ***M1A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = (\pm)0.0274\ldots$ ***(M1)(A1)***

$\frac{\mathrm{d}y}{\mathrm{d}t} = (\pm)0.0274\ldots\times7.75\times10^{-5}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}t} = \pm 2.13\times10^{-6}$ ***A1***

Note: Award all marks except the final ***A1*** to candidates who do not consider $\pm$.

[6 marks]

25 – (MAT/20_HL_Winter_2016_Q10) – *Functions - Roots, Differentiation, Integration*

(a) attempting to solve either $2e^x - 1 = 0$ or $2e^x - 1 \neq 0$ for x ***(M1)***

$D = \mathbb{R} \setminus \{-\ln 2\}$ (or equivalent *eg* $x \neq -\ln 2$) ***A1***

Note: Accept $D = \mathbb{R} \setminus \{-0.693\}$ or equivalent *eg* $x \neq -0.693$.

[2 marks]

(b) considering $\lim\limits_{x \to -\ln 2} f(x)$ ***(M1)***

$x = -\ln 2 \ (x = -0.693)$ ***A1***

considering one of $\lim\limits_{x \to -\infty} f(x)$ or $\lim\limits_{x \to +\infty} f(x)$ ***M1***

$\lim\limits_{x \to -\infty} f(x) = -2 \Rightarrow y = -2$ ***A1***

$\lim\limits_{x \to +\infty} f(x) = -\frac{1}{2} \Rightarrow y = -\frac{1}{2}$ ***A1***

Note: Award ***A0A0*** for $y = -2$ and $y = -\frac{1}{2}$ stated without any justification.

[5 marks]

(c) $f'(x) = \dfrac{-e^x\left(2e^x - 1\right) - 2e^x\left(2 - e^x\right)}{\left(2e^x - 1\right)^2}$ ***M1A1A1***

$= -\dfrac{3e^x}{\left(2e^x - 1\right)^2}$ ***AG***

[3 marks]

(d) $f'(x) < 0$ (for all $x \in D$) $\Rightarrow f$ is (strictly) decreasing ***R1***

Note: Award ***R1*** for a statement such as $f'(x) \neq 0$ and so the graph of f has no turning points.

one branch is above the upper horizontal asymptote and the other branch is below the lower horizontal asymptote ***R1***

f has an inverse ***AG***

$-\infty < x < -2 \cup -\frac{1}{2} < x < \infty$ ***A2***

Note: Award ***A2*** if the domain of the inverse is seen in either part (d) or in part (e).

[4 marks]

(e) $x = \dfrac{2 - e^y}{2e^y - 1}$ ***M1***

Note: Award ***M1*** for interchanging x and y (can be done at a later stage).

$2xe^y - x = 2 - e^y$ ***M1***

$e^y(2x+1) = x + 2$ ***A1***

$f^{-1}(x) = \ln\left(\dfrac{x+2}{2x+1}\right)$ $\left(f^{-1}(x) = \ln(x+2) - \ln(2x+1)\right)$ ***A1***

[4 marks]

(f) use of $V = \pi\int_a^b x^2\,dy$ ***(M1)***

$= \pi\int_0^1 \left(\ln\left(\dfrac{y+2}{2y+1}\right)\right)^2 dy$ ***(A1)(A1)***

Note: Award ***(A1)*** for the correct integrand and ***(A1)*** for the limits.

$= 0.331$ ***A1***

[4 marks]

26 – (MAT/21_HL_Summer_2017_Q2) – *Differentiation*

(a) $y + x\dfrac{dy}{dx} - \dfrac{1}{y}\dfrac{dy}{dx} = 0$ ***M1A1A1***

Note: Award ***A1*** for the first two terms, ***A1*** for the third term and the 0.

$\dfrac{dy}{dx} = \dfrac{y^2}{1 - xy}$ ***A1***

Note: Accept $\dfrac{-y^2}{\ln y}$.

Note: Accept $\dfrac{-y}{x - \dfrac{1}{y}}$.

[4 marks]

(b) $m_T = \dfrac{e^2}{1 - e \times \dfrac{2}{e}}$ ***(M1)***

$m_T = -e^2$ ***(A1)***

$y - e = -e^2x + 2e$

$-e^2x - y + 3e = 0$ or equivalent ***A1***

Note: Accept $y = -7.39x + 8.15$.

[3 marks]

27 – (MAT/22_HL_Summer_2017_Q2) – *Differentiation, Integration*

(a) **METHOD 1**

$4x^2 + y^2 = 7$

$8x + 2y\frac{dy}{dx} = 0$ ***(M1)(A1)***

$\frac{dy}{dx} = -\frac{4x}{y}$

Note: Award ***M1A1*** for finding $\frac{dy}{dx} = -2.309\ldots$ using any alternative method.

hence gradient of normal $= \frac{y}{4x}$ ***(M1)***

hence gradient of normal at $\left(1, \sqrt{3}\right)$ is $\frac{\sqrt{3}}{4}(= 0.433)$ ***(A1)***

hence equation of normal is $y - \sqrt{3} = \frac{\sqrt{3}}{4}(x - 1)$ ***(M1)A1***

$\left(y = \frac{\sqrt{3}}{4}x + \frac{3\sqrt{3}}{4}\right)$ $\left(y = 0.433x + 1.30\right)$

METHOD 2

$4x^2 + y^2 = 7$

$y = \sqrt{7 - 4x^2}$ ***(M1)***

$\frac{dy}{dx} = -\frac{4x}{\sqrt{7 - 4x^2}}$ ***(A1)***

Note: Award ***M1A1*** for finding $\frac{dy}{dx} = -2.309\ldots$ using any alternative method.

hence gradient of normal $= \frac{\sqrt{7 - 4x^2}}{4x}$ ***(M1)***

hence gradient of normal at $\left(1, \sqrt{3}\right)$ is $\frac{\sqrt{3}}{4} (= 0.433)$ ***(A1)***

hence equation of normal is $y - \sqrt{3} = \frac{\sqrt{3}}{4}(x - 1)$ ***(M1)A1***

$\left(y = \frac{\sqrt{3}}{4}x + \frac{3\sqrt{3}}{4}\right)\left(y = 0.433x + 1.30\right)$

[6 marks]

(b) Use of $V = \pi \int_0^{\frac{\sqrt{7}}{2}} y^2 dx$

$V = \pi \int_0^{\frac{\sqrt{7}}{2}} \left(7 - 4x^2\right) dx$ ***(M1)(A1)***

Note: Condone absence of limits or incorrect limits for ***M*** mark.
Do not condone absence of or multiples of π.

$= 19.4 \left(= \frac{7\sqrt{7}\pi}{3}\right)$ ***A1***

[3 marks]

28 – (MAT/21_HL_Summer_2017_Q8) – *Differentiation*

(a) area of segment $= \frac{1}{2} \times 0.5^2 \times (\theta - \sin\theta)$ ***M1A1***

$V =$ area of segment $\times 10$

$V = \frac{5}{4}(\theta - \sin\theta)$ ***A1***

[3 marks]

(b) **METHOD 1**

$\frac{dV}{dt} = \frac{5}{4}(1 - \cos\theta)\frac{d\theta}{dt}$ ***M1A1***

$0.0008 = \frac{5}{4}\left(1 - \cos\frac{\pi}{3}\right)\frac{d\theta}{dt}$ ***(M1)***

$\frac{d\theta}{dt} = 0.00128 \text{ (rad } s^{-1})$ ***A1***

METHOD 2

$\frac{d\theta}{dt} = \frac{d\theta}{dV} \times \frac{dV}{dt}$ ***(M1)***

$\frac{dV}{d\theta} = \frac{5}{4}(1 - \cos\theta)$ ***A1***

$\frac{d\theta}{dt} = \frac{4 \times 0.0008}{5\left(1 - \cos\frac{\pi}{3}\right)}$ ***(M1)***

$\frac{d\theta}{dt} = 0.00128 \left(\frac{4}{3125}\right) \text{(rad } s^{-1})$ ***A1***

[4 marks]

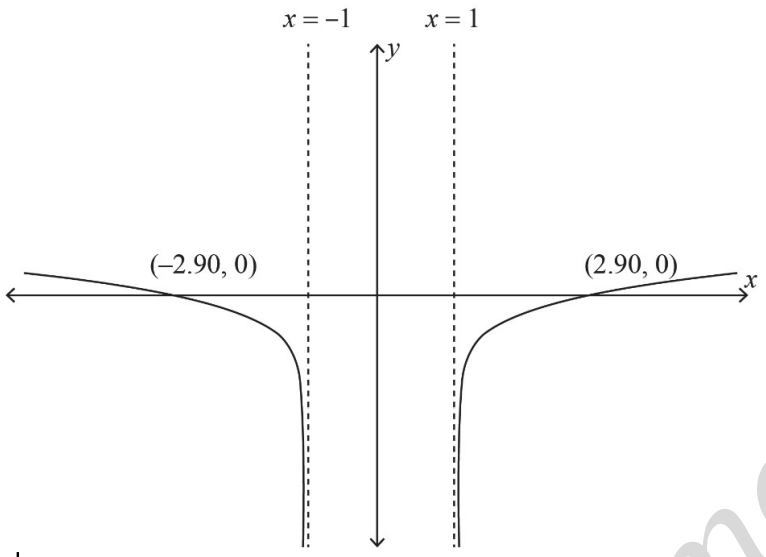
29 – (MAT/21_HL_Summer_2017_Q12) – *Functions - Roots, Graphs, Differentiation*

(a) $x^2 - 1 > 0$ ***(M1)***

$x < -1$ or $x > 1$ ***A1***

[2 marks]

(b)



shape ***A1***

$x = 1$ and $x = -1$ ***A1***

x-intercepts ***A1***

[3 marks]

(c) **EITHER**

f is symmetrical about the y-axis ***R1***

OR

$f(-x) = f(x)$ ***R1***

[1 mark]

(d) **EITHER**

f is not one-to-one function ***R1***

OR

horizontal line cuts twice ***R1***

Note: Accept any equivalent correct statement.

[1 mark]

(e) $x = -1 + \ln\left(\sqrt{y^2 - 1}\right)$ ***M1***

$e^{2x+2} = y^2 - 1$ ***M1***

$g^{-1}(x) = \sqrt{e^{2x+2} + 1},\ x \in \mathbb{R}$ ***A1A1***

[4 marks]

(f) $g'(x) = \dfrac{1}{\sqrt{x^2 - 1}} \times \dfrac{2x}{2\sqrt{x^2 - 1}}$ ***M1A1***

$g'(x) = \dfrac{x}{x^2 - 1}$ ***A1***

[3 marks]

(g) (i) $g'(x) = \dfrac{x}{x^2 - 1} = 0 \Rightarrow x = 0$ ***M1***

which is not in the domain of g (hence no solutions to $g'(x) = 0$) ***R1***

(ii) $\left(g^{-1}\right)'(x) = \dfrac{e^{2x+2}}{\sqrt{e^{2x+2} + 1}}$ ***M1***

as $e^{2x+2} > 0 \Rightarrow \left(g^{-1}\right)'(x) > 0$ so no solutions to $\left(g^{-1}\right)'(x) = 0$ ***R1***

Note: Accept: equation $e^{2x+2} = 0$ has no solutions.

[4 marks]

30 – (MAT/20_HL_Winter_2017_Q10) – *Differentiation, Integration*

(a) (i) attempt to use quotient rule or product rule ***M1***

$$f'(x) = \frac{\sin x\left(\frac{1}{2}x^{-\frac{1}{2}}\right) - \sqrt{x}\cos x}{\sin^2 x}\left(= \frac{1}{2\sqrt{x}\sin x} - \frac{\sqrt{x}\cos x}{\sin^2 x}\right)$$ ***A1A1***

Note: Award ***A1*** for $\frac{1}{2\sqrt{x}\sin x}$ or equivalent and ***A1*** for $-\frac{\sqrt{x}\cos x}{\sin^2 x}$ or equivalent.

setting $f'(x) = 0$ ***M1***

$\frac{\sin x}{2\sqrt{x}} - \sqrt{x}\cos x = 0$

$\frac{\sin x}{2\sqrt{x}} = \sqrt{x}\cos x$ or equivalent ***A1***

$\tan x = 2x$ ***AG***

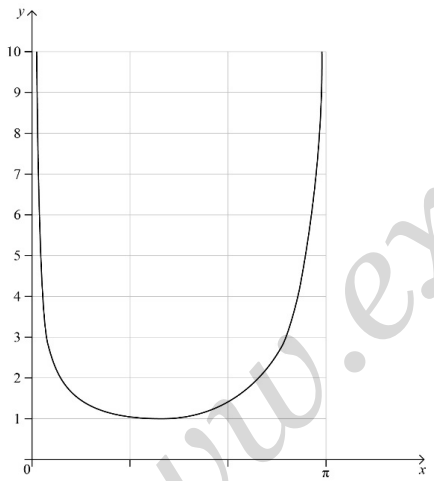
(ii) $x = 1.17$

$0 < x \le 1.17$ ***A1A1***

Note: Award ***A1*** for $0 < x$ and ***A1*** for $x \le 1.17$. Accept $x < 1.17$.

[7 marks]

(b)



concave up curve over correct domain with one minimum point above the *x*-axis. ***A1***

approaches $x = 0$ asymptotically ***A1***

approaches $x = \pi$ asymptotically ***A1***

Note: For the final ***A1*** an asymptote must be seen, and π must be seen on the x-axis or in an equation.

[3 marks]

(c) $f'(x)\left(=\dfrac{\sin x\left(\frac{1}{2}x^{-\frac{1}{2}}\right)-\sqrt{x}\cos x}{\sin^2 x}\right)=1$ ***(A1)***

attempt to solve for x ***(M1)***

$x=1.96$ ***A1***

$y=f(1.96...)$

$=1.51$ ***A1***

[4 marks]

(d) $V=\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\dfrac{x\,\mathrm{d}x}{\sin^2 x}$ ***(M1)(A1)***

Note: ***M1*** is for an integral of the correct squared function (with or without limits and/or π).

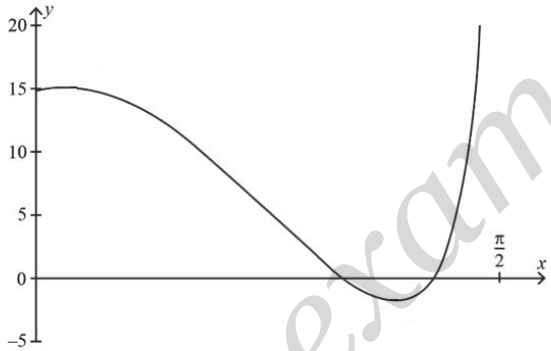
$=2.68(=0.852\pi)$ ***A1***

[3 marks]

31 – (MAT/20_HL_Winter_2017_Q11) – *Trigonometry, Differentiation*

(a) (i) $f'(x)=4\sin x\cos x+14\cos 2x+\sec^2 x$ (or equivalent) ***(M1)A1***

(ii)



A1A1A1A1

Note: Award ***A1*** for correct behaviour at $x=0$, ***A1*** for correct domain and correct behaviour for $x\to\frac{\pi}{2}$, ***A1*** for two clear intersections with x-axis and minimum point, ***A1*** for clear maximum point.

(iii) $x = 0.0736$ ***A1***

$x = 1.13$ ***A1***

[8 marks]

(b) (i) attempt to write $\sin x$ in terms of u only ***(M1)***

$\sin x = \dfrac{u}{\sqrt{1+u^2}}$ ***A1***

(ii) $\cos x = \dfrac{1}{\sqrt{1+u^2}}$ ***(A1)***

attempt to use $\sin 2x = 2\sin x\cos x \left(= 2\dfrac{u}{\sqrt{1+u^2}}\dfrac{1}{\sqrt{1+u^2}} \right)$ ***(M1)***

$\sin 2x = \dfrac{2u}{1+u^2}$ ***A1***

(iii) $2\sin^2 x + 7\sin 2x + \tan x - 9 = 0$

$\dfrac{2u^2}{1+u^2} + \dfrac{14u}{1+u^2} + u - 9\ (= 0)$ ***M1***

$\dfrac{2u^2 + 14u + u\left(1+u^2\right) - 9\left(1+u^2\right)}{1+u^2} = 0$ (or equivalent) ***A1***

$u^3 - 7u^2 + 15u - 9 = 0$ ***AG***

[7 marks]

(c) $u = 1$ or $u = 3$ ***(M1)***

$x = \arctan(1)$ ***A1***

$x = \arctan(3)$ ***A1***

Note: Only accept answers given the required form.

[3 marks]

32 – (MAT/21_HL_Summer_2018_Q9) – *Differentiation, Integration*

(a) differentiating implicitly: ***M1***

$2xy + x^2 \frac{dy}{dx} = -4y^3 \frac{dy}{dx}$ ***A1A1***

Note: Award ***A1*** for each side.

if $\frac{dy}{dx} = 0$ then either $x = 0$ or $y = 0$ ***M1A1***

$x = 0 \Rightarrow$ two solutions for $y\left(y = \pm\sqrt[4]{5}\right)$ ***R1***

$y = 0$ not possible (as $0 \neq 5$) ***R1***

hence exactly two points ***AG***

Note: For a solution that only refers to the graph giving two solutions at $x = 0$ and no solutions for $y = 0$ award ***R1*** only.

[7 marks]

(b) at $(2, 1)$ $4 + 4\frac{dy}{dx} = -4\frac{dy}{dx}$ ***M1***

$\frac{dy}{dx} = -\frac{1}{2}$ ***(A1)***

gradient of normal is 2 ***M1***

$1 = 4 + c$ ***(M1)***

equation of normal is $y = 2x - 3$ ***A1***

[5 marks]

(c) substituting ***(M1)***

$x^2(2x - 3) = 5 - (2x - 3)^4$ or $\left(\frac{y+3}{2}\right)^2 y = 5 - y^4$ ***(A1)***

$x = 0.724$ ***A1***

[3 marks]

(d) recognition of two volumes ***(M1)***

volume 1 $= \pi\int_1^{\sqrt[4]{5}} \frac{5-y^4}{y}\,\mathrm{d}y (= 1.01\pi = 3.178\ldots)$ ***M1A1A1***

Note: Award ***M1*** for attempt to use $\pi\int x^2\mathrm{d}y$, ***A1*** for limits, ***A1*** for $\frac{5-y^4}{y}$. Condone omission of π at this stage.

volume 2

EITHER

$= \frac{1}{3}\pi \times 2^2 \times 4 (= 16.75\ldots)$ ***(M1)(A1)***

OR

$= \pi\int_{-3}^{1}\left(\frac{y+3}{2}\right)^2 \mathrm{d}y (= \frac{16\pi}{3} = 16.75\ldots)$ ***(M1)(A1)***

THEN

total volume $= 19.9$ ***A1***

[7 marks]

33 – (MAT/22_HL_Summer_2018_Q11) – *Differentiation*

(a) attempt at implicit differentiation ***M1***

$1 + \frac{dy}{dx} + (y + x\frac{dy}{dx})\sin(xy) = 0$ ***A1M1A1***

Note: Award ***A1*** for first two terms. Award ***M1*** for an attempt at chain rule ***A1*** for last term.

$(1 + x\sin(xy))\frac{dy}{dx} = -1 - y\sin(xy)$ or equivalent ***A1***

$\frac{dy}{dx} = -\left(\frac{1 + y\sin(xy)}{1 + x\sin(xy)}\right)$ ***AG***

[5 marks]

(b) (i) **EITHER**

when $xy = -\frac{\pi}{2}$, $\cos xy = 0$ ***M1***

$\Rightarrow x + y = 0$ ***(A1)***

OR

$x - \frac{\pi}{2x} - \cos\left(\frac{-\pi}{2}\right) = 0$ or equivalent ***M1***

$x - \frac{\pi}{2x} = 0$ ***(A1)***

THEN

therefore $x^2 = \frac{\pi}{2}\left(x = \pm\sqrt{\frac{\pi}{2}}\right)(x = \pm 1.25)$ ***A1***

$\mathrm{P}\left(\sqrt{\frac{\pi}{2}}, -\sqrt{\frac{\pi}{2}}\right), \mathrm{Q}\left(-\sqrt{\frac{\pi}{2}}, \sqrt{\frac{\pi}{2}}\right)$ or $P(1.25, -1.25)$, $Q(-1.25, 1.25)$ ***A1***

(ii) $m_1 = -\left(\dfrac{1-\sqrt{\frac{\pi}{2}}\times -1}{1+\sqrt{\frac{\pi}{2}}\times -1}\right)$ ***M1A1***

$m_2 = -\left(\dfrac{1+\sqrt{\frac{\pi}{2}}\times -1}{1-\sqrt{\frac{\pi}{2}}\times -1}\right)$ ***A1***

$m_1 m_2 = 1$ ***AG***

Note: Award ***M1A0A0*** if decimal approximations are used.

Note: No ***FT*** applies.

[7 marks]

(c) equate derivative to -1 ***M1***

$(y-x)\sin(xy) = 0$ ***(A1)***

$y = x,\ \sin(xy) = 0$ ***R1***

in the first case, attempt to solve $2x = \cos\left(x^2\right)$ ***M1***

$(0.486, 0.486)$ ***A1***

in the second case, $\sin(xy) = 0 \Rightarrow xy = 0$ and $x + y = 1$ ***(M1)***

$(0,1), (1,0)$ ***A1***

[7 marks]

34 – (MAT/20_HL_Winter_2018_Q5) – *Differentiation*

METHOD 1

$\frac{f(x+h)-f(x)}{h}$

$= \frac{\left(3(x+h)^3-(x+h)\right)-\left(3x^3-x\right)}{h}$ ***M1***

$= \frac{3\left(x^3+3x^2h+3xh^2+h^3\right)-x-h-3x^3+x}{h}$ ***(A1)***

$= \frac{9x^2h+9xh^2+3h^3-h}{h}$ ***A1***

cancelling h ***M1***

$= 9x^2+9xh+3h^2-1$

then $\lim_{h\to 0}\left(9x^2+9xh+3h^2-1\right)$

$= 9x^2-1$ ***A1***

Note: Final ***A1*** dependent on all previous marks.

METHOD 2

$\frac{f(x+h)-f(x)}{h}$

$= \frac{\left(3(x+h)^3-(x+h)\right)-\left(3x^3-x\right)}{h}$ ***M1***

$= \frac{3\left((x+h)^3-x^3\right)+\left(x-(x+h)\right)}{h}$ ***(A1)***

$= \frac{3h\left((x+h)^2+x(x+h)+x^2\right)-h}{h}$ ***A1***

cancelling h ***M1***

$= 3\left((x+h)^2+x(x+h)+x^2\right)-1$

then $\lim_{h\to 0}\left(3\left((x+h)^2+x(x+h)+x^2\right)-1\right)$

$= 9x^2-1$ ***A1***

Note: Final ***A1*** dependent on all previous marks.

[5 marks]

35 – (MAT/20_HL_Winter_2018_Q9) – *Functions - Roots, Differentiation*

(a) **METHOD 1**

$$f'(x) = \frac{\frac{2(x-3)}{x} - (2\ln x + 1)}{(x-3)^2} \left(= \frac{2(x-3) - x(2\ln x + 1)}{x(x-3)^2} \right)$$ ***(M1)A1A1A1***

Note: Award ***M1*** for attempt at quotient rule, ***A1A1*** for numerator and ***A1*** for denominator.

METHOD 2

$$f(x) = (2\ln x + 1)(x-3)^{-1}$$ ***(A1)***

$$f'(x) = \left(\frac{2}{x}\right)(x-3)^{-1} - (2\ln x + 1)(x-3)^{-2} \left(= \frac{2(x-3) - x(2\ln x + 1)}{x(x-3)^2} \right)$$ ***(M1)A1A1***

Note: Award ***M1*** for attempt at product rule, ***A1*** for first term, ***A1*** for second term.

[4 marks]

(b) finding turning point of $y = f'(x)$ or finding root of $y = f''(x)$ ***(M1)***

$x = 0.899$ ***A1***

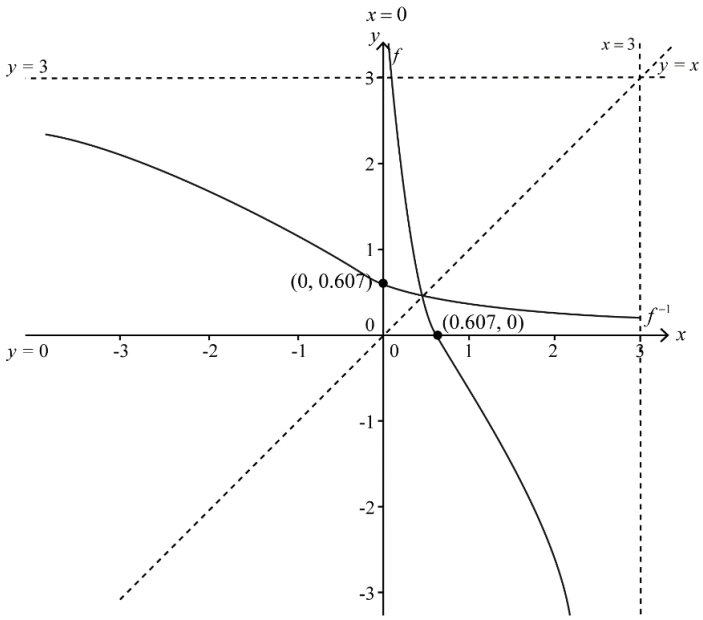
$y = f(0.899048\ldots) = -0.375$ ***(M1)A1***

$(0.899, -0.375)$

Note: Do not accept $x = 0.9$. Accept y-coordinates rounding to -0.37 or -0.375 but not -0.38.

[4 marks]

(c)



(i) smooth curve over the correct domain which does not cross the y-axis and is concave down for $x > 1$ ***A1***

x-intercept at 0.607 ***A1***

equations of asymptotes given as $x = 0$ and $x = 3$ (the latter must be drawn) ***A1A1***

[4 marks]

(ii) attempt to reflect graph of f in $y = x$ ***(M1)***

smooth curve over the correct domain which does not cross the x-axis and is concave down for $y > 1$ ***A1***

y-intercept at 0.607 ***A1***

equations of asymptotes given as $y = 0$ and $y = 3$ (the latter must be drawn) ***A1***

Note: For ***FT*** from (i) to (ii) award max ***M1A0A1A0***.

[4 marks]

(d) solve $f(x) = f^{-1}(x)$ or $f(x) = x$ to get $x = 0.372$ ***(M1)A1***

$0 < x < 0.372$ ***A1***

Note: Do not award ***FT*** marks.

[3 marks]

36 – (MAT/21_HL_Summer_2019_Q1) – *Differentiation*

METHOD 1

equation of tangent is $y = 22.167...\,x - 14.778...$ **OR** $y \quad 7.389... = 22.167...\,(x \quad 1)$ ***(M1)(A1)***

meets the x-axis when $y = 0$

$x = 0.667$

meets x -axis at $(0.667, 0)\left(= \left(\frac{2}{3}, 0\right)\right)$ ***A1A1***

Note: Award ***A1*** for $x = \frac{2}{3}$ or $x = 0.667$ seen and ***A1*** for coordinates $(x, 0)$ given.

37 – (MAT/20_HL_Winter_2019_Q11) – *Differentiation, Integration*

(a) (i) valid attempt to differentiate implicitly ***(M1)***

$4x = 3\sin^2 y \cos y \frac{dy}{dx}$ ***A1A1***

$\frac{dy}{dx} = \frac{4x}{3\sin^2 y \cos y}$ ***A1***

(ii) at $\left(\frac{1}{4}, \frac{5\pi}{6}\right)$, $\frac{dy}{dx} = \frac{4x}{3\sin^2 y \cos y} = \frac{1}{3\left(\frac{1}{2}\right)^2\left(-\frac{\sqrt{3}}{2}\right)}$ ***(M1)***

$\Rightarrow \frac{dy}{dx} = -\frac{8}{3\sqrt{3}}(= -1.54)$ ***A1***

hence equation of tangent is

$y - \frac{5\pi}{6} = -1.54\left(x - \frac{1}{4}\right)$ **OR** $y = -1.54x + 3.00$ ***(M1)A1***

Note: Accept $y = -1.54x + 3$.

[8 marks]

(b) $x = \sqrt{\frac{1}{2}\sin^3 y}$ ***(M1)***

$\int_0^{\pi} \sqrt{\frac{1}{2}\sin^3 y}\, dy$ ***(A1)***

$= 1.24$ ***A1***

[3 marks]

(c) use of volume $= \int \pi x^2 \mathrm{d}y$ ***(M1)***

$= \int_0^{\pi} \frac{1}{2}\pi \sin^3 y \, \mathrm{d}y$ ***A1***

$= \frac{1}{2}\pi \int_0^{\pi} \left(\sin y - \sin y \cos^2 y\right) \mathrm{d}y$

Note: Condone absence of limits up to this point.

reasonable attempt to integrate ***(M1)***

$= \frac{1}{2}\pi \left[-\cos y + \frac{1}{3}\cos^3 y\right]_0^{\pi}$ ***A1A1***

Note: Award ***A1*** for correct limits (not to be awarded if previous ***M1*** has not been awarded) and ***A1*** for correct integrand.

$= \frac{1}{2}\pi\left(1 - \frac{1}{3}\right) - \frac{1}{2}\pi\left(-1 + \frac{1}{3}\right)$ ***A1***

$= \frac{2\pi}{3}$ ***AG***

Note: Do not accept decimal answer equivalent to $\frac{2\pi}{3}$.

[6 marks]

38 – (MAT/20_HL_Winter_2020_Q8) – *Differentiation*

(a) valid attempt to use chain rule or quotient rule ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-10\mathrm{e}^{0.5x}}{\left(3 - 2\mathrm{e}^{0.5x}\right)^2}$ OR $\frac{\mathrm{d}y}{\mathrm{d}x} = -10\mathrm{e}^{0.5x}\left(3 - 2\mathrm{e}^{0.5x}\right)^{-2}$ ***A1A1***

[3 marks]

Note: Award ***A1*** for numerator and ***A1*** for denominator, or ***A1*** for each part if the second alternative given.

(b) valid attempt to use chain rule $\left(\text{eg } \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}\right)$ ***(M1)***

$\frac{\mathrm{d}x}{\mathrm{d}t} = -0.1 \div \frac{-10\mathrm{e}^{-2}}{\left(3 - 2\mathrm{e}^{-2}\right)^2}$ $\left(= -0.1 \div -0.181676...\right)$ or equivalent ***(A1)***

$= 0.550428...$

$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.550 \ (\mathrm{ms}^{-1})$ ***A1***

[3 marks]

39 – (MathAA/21_HL_Summer_2021_Q9) – *Differentiation, Kinematics, Relative Velocity*

(a) $\tan\theta = \dfrac{50}{y-x}$ OR $\cot\theta = \dfrac{y-x}{50}$ ***A1***

$y = x + 50\cot\theta$ ***AG***

Note: $y - x$ may be identified as a length on a diagram, and not written explicitly.

[1 mark]

(b) attempt to differentiate with respect to t ***(M1)***

$\dfrac{dy}{dt} = \dfrac{dx}{dt} - 50(\operatorname{cosec}\theta)^2\dfrac{d\theta}{dt}$ ***A1***

attempt to set speed of B equal to double the speed of A ***(M1)***

$2\dfrac{dx}{dt} = \dfrac{dx}{dt} - 50(\operatorname{cosec}\theta)^2\dfrac{d\theta}{dt}$

$\dfrac{dx}{dt} = -50(\operatorname{cosec}\theta)^2\dfrac{d\theta}{dt}$ ***A1***

$\theta = \arctan 5 (= 1.373\ldots = 78.69\ldots°)$ OR $\operatorname{cosec}^2\theta = 1 + \cot^2\theta = 1 + \left(\dfrac{1}{5}\right)^2 = \dfrac{26}{25}$ ***(A1)***

Note: This ***A1*** can be awarded independently of previous marks.

$\dfrac{dx}{dt} = -50\left(\dfrac{26}{25}\right) \times -0.1$

So the speed of boat A is 5.2 (ms^{-1}) ***A1***

Note: Accept 5.20 from the use of inexact values.

[6 marks]

40 – (MathAA/21_HL_Summer_2021_Q10) – *Exponentials - Logarithms, Differentiation*

(a) attempt to find the point of intersection of the graph of f and the line $y = x$ ***(M1)***

$x = 5.56619...$

$= 5.57$ ***A1***

[2 marks]

(b) $f'(x) = -45e^{-0.5x}$ ***A1***

attempt to set the gradient of f equal to -1 ***(M1)***

$-45e^{-0.5x} = -1$

Q has coordinates $(2\ln 45, 2)$ (accept $(-2\ln\frac{1}{45}, 2)$) ***A1A1***

Note: Award ***A1*** for each value, even if the answer is not given as a coordinate pair.

Do not accept $\frac{\ln\frac{1}{45}}{-0.5}$ or $\frac{\ln 45}{0.5}$ as a final value for x. Do not accept 2.0 or 2.00 as a final value for y.

[4 marks]

(c) attempt to substitute coordinates of Q (in any order) into an appropriate equation ***(M1)***

$y - 2 = -(x - 2\ln 45)$ OR $2 = -2\ln 45 + c$ ***A1***

equation of L is $y = -x + 2\ln 45 + 2$ ***AG***

[2 marks]

(d) (i) $x = \ln 45 + 1 \left(= 4.81\right)$ ***A1***

(ii) appropriate method to find the sum of two areas using integrals of the difference of two functions ***(M1)***

Note: Allow absent or incorrect limits.

$$\int_{4.806...}^{5.566...} \left(x - \left(-x + 2\ln 45 + 2\right)\right) dx + \int_{5.566...}^{7.613...} \left(90e^{-0.5x} - \left(-x + 2\ln 45 + 2\right)\right) dx$$ ***(A1)(A1)***

Note: Award ***A1*** for one correct integral expression including correct limits and integrand.
Award ***A1*** for a second correct integral expression including correct limits and integrand.

$= 1.51965...$
$= 1.52$ ***A1***

[5 marks]

(e) by symmetry 2×1.52 ***(M1)***
$= 3.03930...$
$= 3.04$ ***A1***

Note: Accept any answer that rounds to 3.0 (but do not accept 3).

[2 marks]

41 – (MathAA/22_HL_Summer_2021_Q11) – *Differentiation*

(a) attempt to use $V = \pi\int_a^b \left(f(x)\right)^2 \mathrm{d}x$ ***(M1)***

$$V = \pi\int_0^{\ln 16}\left(\frac{k\mathrm{e}^{\frac{x}{2}}}{1+\mathrm{e}^x}\right)^2 \mathrm{d}x \quad \left(V = k^2\pi\int_0^{\ln 16}\frac{\mathrm{e}^x}{\left(1+\mathrm{e}^x\right)^2}\mathrm{d}x\right)$$

EITHER

applying integration by recognition ***(M1)***

$= k^2\pi\left[-\frac{1}{1+\mathrm{e}^x}\right]_0^{\ln 16}$ ***A3***

OR

$u = 1+\mathrm{e}^x \Rightarrow \mathrm{d}u = \mathrm{e}^x\mathrm{d}x$ ***(A1)***

attempt to express the integral in terms of u ***(M1)***

when $x = 0$, $u = 2$ and when $x = \ln 16$, $u = 17$

$V = k^2\pi\int_2^{17}\frac{1}{u^2}\mathrm{d}u$ ***(A1)***

$= k^2\pi\left[-\frac{1}{u}\right]_2^{17}$ ***A1***

OR

$u = \mathrm{e}^x \Rightarrow \mathrm{d}u = \mathrm{e}^x\mathrm{d}x$ ***(A1)***

attempt to express the integral in terms of u ***(M1)***

when $x = 0$, $u = 1$ and when $x = \ln 16$, $u = 16$

$V = k^2\pi\int_1^{16}\frac{1}{(1+u)^2}\mathrm{d}u$ ***(A1)***

$= k^2\pi\left[-\frac{1}{1+u}\right]_1^{16}$ ***A1***

Note: Accept equivalent working with indefinite integrals and original limits for x.

THEN

$= k^2\pi\left(\frac{1}{2}-\frac{1}{17}\right)$ ***A1***

so the volume of the solid formed is $\frac{15k^2\pi}{34}$ cubic units ***AG***

Note: Award ***(M1)(A0)(M0)(A0)(A0)(A1)*** when $\frac{15}{34}$ is obtained from GDC

[6 marks]

(b) a valid algebraic or graphical attempt to find k ***(M1)***

$$k^2 = \frac{300 \times 34}{15\pi}$$

$$k = 14.7 \left(= 2\sqrt{\frac{170}{\pi}} = \sqrt{\frac{680}{\pi}} \right) \text{ (as } k \in \mathbb{R}^+\text{)}$$ ***A1***

Note: Candidates may use their GDC numerical solve feature.

[2 marks]

(c) (i) attempting to find $\text{OA} = f(0) = \frac{k}{2}$

with $k = 14.712... \left(= 2\sqrt{\frac{170}{\pi}} = \sqrt{\frac{680}{\pi}} \right)$ ***(M1)***

$$\text{OA} = 7.36 \left(= \sqrt{\frac{170}{\pi}} \right)$$ ***A1***

(ii) attempting to find $\text{BC} = f(\ln 16) = \frac{4k}{17}$

with $k = 14.712... \left(= 2\sqrt{\frac{170}{\pi}} = \sqrt{\frac{680}{\pi}} \right)$ ***(M1)***

$$\text{BC} = 3.46 \left(= \frac{8}{17}\sqrt{\frac{170}{\pi}} = \frac{8\sqrt{10}}{\sqrt{17\pi}} \right)$$ ***A1***

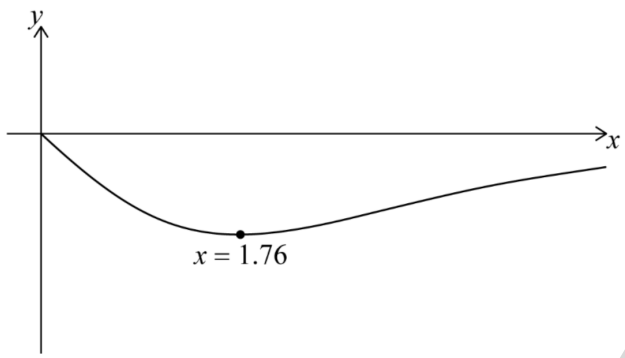
[4 marks]

(d) (i) **EITHER**

recognising to graph $y = f'(x)$ ***(M1)***

Note: Award ***M1*** for attempting to use quotient rule or product rule differentiation.

$$f'(x) = \frac{k\mathrm{e}^{\frac{x}{2}}\left(1-\mathrm{e}^{x}\right)}{2\left(1+\mathrm{e}^{x}\right)^{2}}$$



for $x > 0$ graph decreasing to the local minimum ***A1***

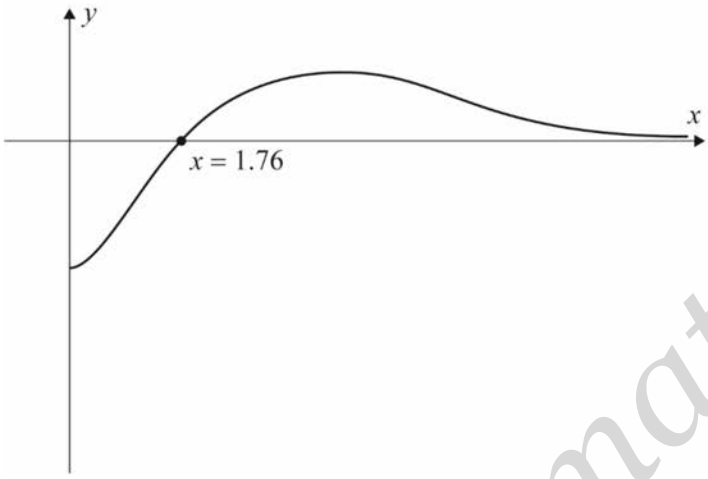
before increasing towards the x-axis ***A1***

OR

recognising to graph $y = f''(x)$ ***(M1)***

Note: Award ***M1*** for attempting to use quotient rule or product rule differentiation.

$$f''(x) = \frac{ke^{\frac{x}{2}}\left(e^{2x} - 6e^{x} + 1\right)}{4\left(1 + e^{x}\right)^{3}}$$



for $x > 0$, graph increasing towards and beyond the x-intercept ***A1***

recognising $f''(x) = 0$ for maximum rate ***(A1)***

THEN

$x = 1.76 \ \left(= \ln\left(2\sqrt{2} + 3\right)\right)$ ***A1***

Note: Only award ***A*** marks if either graph is seen.

(ii) attempting to find $f(1.76...)$ ***(M1)***

the cross-sectional radius at this point is $5.20 \ \left(\sqrt{\frac{85}{\pi}}\right)$ (cm) ***A1***

[6 marks]

42 – (MathAA/22_HL_Summer_2021_Q12) – *Algebra-functions - Roots, Differentiation*

(a) **EITHER**

$$f(-x) = \arcsin\left(\frac{(-x)^2 - 1}{(-x)^2 + 1}\right) = \arcsin\left(\frac{x^2 - 1}{x^2 + 1}\right) = f(x)$$ ***R1***

OR

a sketch graph of $y = f(x)$ with line symmetry in the y-axis indicated ***R1***

THEN

so $f(x)$ is an even function ***AG***

[1 mark]

(b) as $x \to \pm\infty$, $f(x) \to \arcsin 1 \left(\to \frac{\pi}{2}\right)$ ***A1***

so the horizontal asymptote is $y = \frac{\pi}{2}$ ***A1***

[2 marks]

(c) (i) attempting to use the quotient rule to find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2-1}{x^2+1}\right)$ ***M1***

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2-1}{x^2+1}\right)=\frac{2x\left(x^2+1\right)-2x\left(x^2-1\right)}{\left(x^2+1\right)^2}\quad\left(=\frac{4x}{\left(x^2+1\right)^2}\right)$$ ***A1***

attempting to use the chain rule to find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\arcsin\left(\frac{x^2-1}{x^2+1}\right)\right)$ ***M1***

let $u=\frac{x^2-1}{x^2+1}$ and so $y=\arcsin u$ and $\frac{\mathrm{d}y}{\mathrm{d}u}=\frac{1}{\sqrt{1-u^2}}$

$$f'(x)=\frac{1}{\sqrt{1-\left(\frac{x^2-1}{x^2+1}\right)^2}}\times\frac{4x}{\left(x^2+1\right)^2}$$ ***M1***

$$=\frac{4x}{\sqrt{\left(x^2+1\right)^2-\left(x^2-1\right)^2}}\times\frac{1}{\left(x^2+1\right)}$$ ***A1***

$$=\frac{4x}{\sqrt{4x^2}}\times\frac{1}{\left(x^2+1\right)}$$ ***A1***

$$=\frac{2x}{\sqrt{x^2}\left(x^2+1\right)}$$ ***AG***

(ii) $f'(x)=\frac{2x}{|x|\left(x^2+1\right)}$

EITHER

for $x<0$, $|x|=-x$ ***(A1)***

so $f'(x)=-\frac{2}{x^2+1}$ ***A1***

OR

$|x|>0$ and $x^2+1>0$ ***A1***

$2x<0,\ x<0$ ***A1***

THEN

$f'(x)<0$ ***R1***

Note: Award ***R1*** for stating that in $f'(x)$, the numerator is negative, and the denominator is positive.

so f is decreasing for $x<0$ ***AG***

Note: Do not accept a graphical solution.

[9 marks]

(d) $x = \arcsin\left(\frac{y^2-1}{y^2+1}\right)$ ***M1***

$\sin x = \frac{y^2-1}{y^2+1} \Rightarrow y^2 \sin x + \sin x = y^2 - 1$ ***A1***

$y^2 = \frac{1+\sin x}{1-\sin x}$ ***A1***

domain of g is $x \in \mathbb{R}, x \geq 0$ and so the range of g^{-1} must be $y \in \mathbb{R}, y \geq 0$

hence the positive root is taken (or the negative root is rejected) ***R1***

Note: The ***R1*** is dependent on the above ***A1***.

so $\left(g^{-1}(x) =\right)\sqrt{\frac{1+\sin x}{1-\sin x}}$ ***A1***

Note: The final ***A1*** is not dependent on ***R1*** mark.

[5 marks]

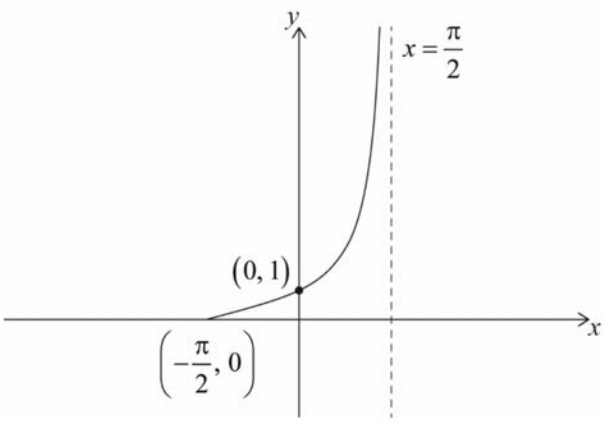
(e) domain is $-\frac{\pi}{2} \leq x < \frac{\pi}{2}$ ***A1***

Note: Accept correct alternative notations, for example, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right[$ or $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Accept $[-1.57, 1.57[$ if correct to 3 s.f.

[1 mark]

(f)



A1A1A1

Note: ***A1*** for correct domain and correct range and y-intercept at $y = 1$

A1 for asymptotic behaviour $x \to \frac{\pi}{2}$

A1 for $x = \frac{\pi}{2}$

Coordinates are not required.
Do not accept $x = 1.57$ or other inexact values.

[3 marks]

43 – (MathAA/20_HL_Winter_2021_Q8) – *Differentiation*

(a) **METHOD 1**

attempts to differentiate implicitly including at least one application of the product rule ***(M1)***

$u = xy$, $v = \ln(xy)$, $\frac{du}{dx} = x\frac{dy}{dx} + y$, $\frac{dv}{dx} = \left(x\frac{dy}{dx} + y\right)\frac{1}{xy}$

$$\frac{dy}{dx} = 1 - \left[\frac{xy}{xy}\left(x\frac{dy}{dx} + y\right) + \left(x\frac{dy}{dx} + y\right)\ln(xy)\right]$$ ***A1***

Note: Award ***(M1)A1*** for implicitly differentiating $y = x(1 - y\ln(xy))$ and obtaining

$$\frac{dy}{dx} = 1 - \left[\frac{xy}{xy}\left(x\frac{dy}{dx} + y\right) + x\frac{dy}{dx}\ln(xy) + y\ln(xy)\right].$$

$$\frac{dy}{dx} = 1 - \left[\left(x\frac{dy}{dx} + y\right) + \left(x\frac{dy}{dx} + y\right)\ln(xy)\right]$$

$$\frac{dy}{dx} = 1 - \left(x\frac{dy}{dx} + y\right)(1 + \ln(xy))$$ ***A1***

$$\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right)(1 + \ln(xy)) = 1$$ ***AG***

METHOD 2

$y = x - xy\ln x - xy\ln y$

attempts to differentiate implicitly including at least one application of the product rule ***(M1)***

$$\frac{dy}{dx} = 1 - \left(\frac{xy}{x} + \left(x\frac{dy}{dx} + y\right)\ln x\right) - \left(\frac{xy}{y}\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right)\ln y\right)$$ ***A1***

or equivalent to the above, for example

$$\frac{dy}{dx} = 1 - \left(x\ln x\frac{dy}{dx} + (1 + \ln x)y\right) - \left(y\ln y + x\left(\ln y\frac{dy}{dx} + \frac{dy}{dx}\right)\right)$$

$$\frac{dy}{dx} = 1 - x\frac{dy}{dx}(\ln x + \ln y + 1) - y(\ln x + \ln y + 1)$$ ***A1***

or equivalent to the above, for example

$$\frac{dy}{dx} = 1 - x\frac{dy}{dx}(\ln(xy) + 1) - y(\ln(xy) + 1)$$

$$\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right)(1 + \ln(xy)) = 1$$ ***AG***

METHOD 3

attempt to differentiate implicitly including at least one application of the product rule ***M1***

$u = x\ln(xy),\ v = y,\ \frac{du}{dx} = \ln(xy) + \left(x\frac{dy}{dx} + y\right)\frac{x}{xy},\ \frac{dv}{dx} = \frac{dy}{dx}$

$\frac{dy}{dx} = 1 - \left(x\frac{dy}{dx}\ln(xy) + y\ln(xy) + \frac{xy}{xy}\left(x\frac{dy}{dx} + y\right)\right)$ ***A1***

$\frac{dy}{dx} = 1 - x\frac{dy}{dx}(\ln(xy) + 1) - y(\ln(xy) + 1)$ ***A1***

$\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right)(1 + \ln(xy)) = 1$ ***AG***

METHOD 4

lets $w = xy$ and attempts to find $\frac{dy}{dx}$ where $y = x - w\ln w$ ***M1***

$\frac{dy}{dx} = 1 - \left(\frac{dw}{dx} + \frac{dw}{dx}\ln w\right)\ \left(= 1 - \frac{dw}{dx}(1 + \ln w)\right)$ ***A1***

$\frac{dw}{dx} = x\frac{dy}{dx} + y$ ***A1***

$\frac{dy}{dx} = 1 - \left(x\frac{dy}{dx} + y + \left(x\frac{dy}{dx} + y\right)\ln(xy)\right)\ \left(= 1 - \left(x\frac{dy}{dx} + y\right)(1 + \ln(xy))\right)$

$\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right)(1 + \ln(xy)) = 1$ ***AG***

[3 marks]

(b) **METHOD 1**

substitutes $x=1$ into $y=x-xy\ln(xy)$ ***(M1)***

$y=1-y\ln y \Rightarrow y=1$ ***A1***

substitutes $x=1$ and their non-zero value of y into $\frac{dy}{dx}+\left(x\frac{dy}{dx}+y\right)(1+\ln(xy))=1$ ***(M1)***

$2\frac{dy}{dx}=0\ \left(\frac{dy}{dx}=0\right)$ ***A1***

equation of the tangent is $y=1$ ***A1***

METHOD 2

substitutes $x=1$ into $\frac{dy}{dx}+\left(x\frac{dy}{dx}+y\right)(1+\ln(xy))=1$ ***(M1)***

$\frac{dy}{dx}+\left(\frac{dy}{dx}+y\right)(1+\ln(y))=1$

EITHER

correctly substitutes $\ln y=\frac{1-y}{y}$ into $\frac{dy}{dx}+\left(\frac{dy}{dx}+y\right)(1+\ln(y))=1$ ***A1***

$\frac{dy}{dx}\left(1+\frac{1}{y}\right)=0 \Rightarrow \frac{dy}{dx}=0\ (y=1)$ ***A1***

OR

correctly substitutes $y+y\ln y=1$ into $\frac{dy}{dx}+\left(\frac{dy}{dx}+y\right)(1+\ln(y))=1$ ***A1***

$\frac{dy}{dx}(2+\ln y)=0 \Rightarrow \frac{dy}{dx}=0\ (y=1)$ ***A1***

THEN

substitutes $x=1$ into $y=x-xy\ln(xy)$ ***(M1)***

$y=1-y\ln y \Rightarrow y=1$

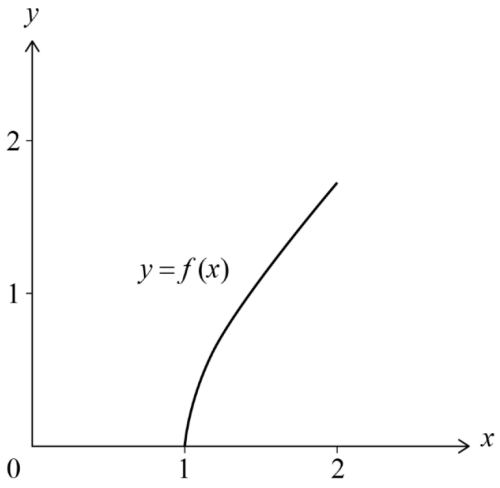
equation of the tangent is $y=1$ ***A1***

[5 marks]

Total [8 marks]

44 – (MathAA/21_HL_Summer_2022_Q10) – *Algebra-functions - Roots, Differentiation, Integration*

(a)



correct shape (concave down) within the given domain $1 \le x \le 2$ ***A1***

$(1,0)$ and $(2,\sqrt{3})(=(2,1.73))$ ***A1***

Note: The coordinates of endpoints may be seen on the graph or marked on the axes.

(b) (i) interchanging x and y (seen anywhere) ***M1***

$x = \sqrt{y^2 - 1}$

$x^2 = y^2 - 1$ ***A1***

$y = \sqrt{x^2 + 1}$ ***A1***

$f^{-1}(x) = \sqrt{x^2 + 1}$ ***AG***

(ii) $0 \le x \le \sqrt{3}$ OR domain $[0,\sqrt{3}](=[0,1.73])$ ***A1***

$1 \le y \le 2$ OR $1 \le f^{-1}(x) \le 2$ OR range $[1,2]$ ***A1***

[5 marks]

(c) (i) attempt to substitute $x=\sqrt{y^2+1}$ into the correct volume formula ***(M1)***

$V=\pi\int_0^h\left(\sqrt{y^2+1}\right)^2\,\mathrm{d}y\ \left(=\pi\int_0^h\left(y^2+1\right)\,\mathrm{d}y\right)$ ***A1***

$=\pi\left[\frac{1}{3}y^3+y\right]_0^h$ ***A1***

$=\pi\left(\frac{1}{3}h^3+h\right)$ ***AG***

Note: Award marks as appropriate for correct work using a different variable e.g.

$\pi\int_0^h\left(\sqrt{x^2+1}\right)^2\,\mathrm{d}x$

(ii) attempt to substitute $h=\sqrt{3}\ (=1.732...)$ into V ***(M1)***

$V=10.8828...$

$V=10.9$ (m³) $\left(=2\sqrt{3}\pi\right)$ (m³) ***A1***

[5 marks]

(d) **METHOD 1**

time $=\frac{10.8828...}{0.4}\left(=\frac{2\sqrt{3}\pi}{0.4}\right)$ ***(M1)***

$=27.207...$

$=27.2\left(=5\sqrt{3}\pi\right)(s)$ ***A1***

[2 marks]

(e) attempt to find the height of the tank when $V = 5.4414...\ \left(= \sqrt{3}\pi\right)$ ***(M1)***

$\pi\left(\frac{1}{3}h^3 + h\right) = 5.4414...\ \left(= \sqrt{3}\pi\right)$

$h = 1.1818...$ ***(A1)***

attempt to use the chain rule or differentiate $V = \pi\left(\frac{1}{3}h^3 + h\right)$ with respect to t ***(M1)***

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{\pi(h^2+1)} \times \frac{\mathrm{d}V}{\mathrm{d}t}$ OR $\frac{\mathrm{d}V}{\mathrm{d}t} = \pi\left(h^2+1\right)\frac{\mathrm{d}h}{\mathrm{d}t}$ ***(A1)***

attempt to substitute **their** h and $\frac{\mathrm{d}V}{\mathrm{d}t} = 0.4$ ***(M1)***

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{0.4}{\pi\left(1.1818...^2+1\right)} = 0.053124...$

$= 0.0531$ (m s^{-1}) ***A1***

[6 marks]

45 – (MathAA/22_HL_Summer_2022_Q10) – *Trigonometry, Differentiation*

(a) (i) 32 (cm) ***A1***

(ii) $h_A(0) = \sin(6) + 27$ ***(M1)***

$= 26.7205...$

$= 26.7$ (cm) ***A1***

[3 marks]

(b) attempts to solve $h_A(t) = h_B(t)$ for t ***(M1)***

$t = 4.0074..., 4.7034..., 5.88332...$

$t = 4.01, 4.70, 5.88$ (weeks) ***A2***

[3 marks]

(c) $h_A(t) - h_B(t) = \sin(2t+6) + t - 5$ ***A1***

EITHER

for $t > 6$, $t - 5 > 1$ ***A1***

and as $\sin(2t+6) \geq -1 \Rightarrow h_A(t) - h_B(t) > 0$ ***R1***

OR

the minimum value of $\sin(2t+6) = -1$ ***R1***

so for $t > 6$, $h_A(t) - h_B(t) = t - 6 > 0$ ***A1***

THEN

hence for $t > 6$, Plant A was always taller than Plant B ***AG***

[3 marks]

(d) recognises that $h_A'(t)$ and $h_B'(t)$ are required ***(M1)***

attempts to solve $h_A'(t) = h_B'(t)$ for t ***(M1)***

$t = 1.18879...$ and $2.23598...$ OR $4.33038...$ and $5.37758...$ OR $7.47197...$ and $8.51917...$ ***(A1)***

Note: Award full marks for $t = \frac{4\pi}{3} - 3, \frac{5\pi}{3} - 3, \left(\frac{7\pi}{3} - 3, \frac{8\pi}{3} - 3, \frac{10\pi}{3} - 3, \frac{11\pi}{3} - 3\right)$.
Award subsequent marks for correct use of these exact values.

$1.18879... < t < 2.23598...$ OR $4.33038... < t < 5.37758...$ OR
$7.47197... < t < 8.51917...$ ***(A1)***

attempts to calculate the total amount of time ***(M1)***

$3(2.2359... - 1.1887...)$ $\left(= 3\left(\left(\frac{5\pi}{3} - 3\right) - \left(\frac{4\pi}{3} - 3\right)\right)\right)$

$= 3.14$ $(= \pi)$ (weeks) ***A1***

[6 marks]

46 – (MathAA/20_HL_Winter_2022_Q2) – *Differentiation, Integration*

(a) $(0.708519...,0.639580...)$

$(0.709,0.640)$ $(x=0.709,\ y=0.640)$ ***A1A1***

[2 marks]

(b) $1.09885...$

$x=1.10$ (accept $(1.10,0)$) ***A1***

[1 mark]

(c) **METHOD 1**

$\int_0^2 |f(x)|dx$ ***(A1)***

$4.61117...$

area = 4.61 ***A2***

METHOD 2

$-\int_{1.09885...}^2 f(x)dx$ OR $\int_{1.09885...}^2 |f(x)|dx$ OR 4.17527... ***(A1)***

$\int_0^{1.09885...} f(x)dx - \int_{1.09885...}^2 f(x)dx$ OR $0.435901... + 4.17527...$ ***(A1)***

$4.61117...$

area = 4.61 ***A1***

[3 marks]

47 – (MathAA/20_HL_Winter_2022_Q8) – *Differentiation*

EITHER

$\left(\frac{\mathrm{d}V}{\mathrm{d}h}=\right)10\pi h-\pi h^2$ ***(A1)***

Note: This ***A1*** may be implied by the value $\frac{\mathrm{d}V}{\mathrm{d}h}=76.5616...$.

attempt to use chain rule to find a relationship between $\frac{\mathrm{d}h}{\mathrm{d}t}$, $\frac{\mathrm{d}V}{\mathrm{d}t}$ and $\frac{\mathrm{d}V}{\mathrm{d}h}$ ***(M1)***

$$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mathrm{d}h}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}\left(=\frac{1}{\left(\frac{\mathrm{d}V}{\mathrm{d}h}\right)}\times\frac{\mathrm{d}V}{\mathrm{d}t}\right)$$

OR

attempt to differentiate $V=5\pi h^2-\frac{1}{3}\pi h^3$ throughout with respect to t ***(M1)***

$\frac{\mathrm{d}V}{\mathrm{d}t}=10\pi h\frac{\mathrm{d}h}{\mathrm{d}t}-\pi h^2\frac{\mathrm{d}h}{\mathrm{d}t}$ ***(A1)***

THEN

$\left(10\pi h - \pi h^2\right)\frac{dh}{dt} = 2$ OR $\frac{dh}{dt} = \frac{2}{10\pi h - \pi h^2}$ ***(A1)***

Note: Award this ***A1*** if the correct expression is seen with their h already substituted.

attempt to solve $200 = 5\pi h^2 - \frac{1}{3}\pi h^3$ ***(M1)***

$h = 4.20648...$ ***(A1)***

Note: This ***(M1)(A1)*** can be awarded independently of all previous marks, and may be implied by the value $\frac{dV}{dh} = 76.5616...$
Ignore extra values of h -3.24 and 14.0.

$\frac{dh}{dt} = 0.0261227...$

$\frac{dh}{dt} = 0.0261\left(\text{cms}^{-1}\right)$ ***A1***

[6 marks]

48 – (MathAA/20_HL_Winter_2022_Q11) – *Algebra-functions - Roots, Differentiation*

(a) attempt to use product rule ***(M1)***

$f''(x) = 3e^{2x} + 2e^{2x}(3x-4)\left(= e^{2x}(6x-5)\right)$ ***A2***

Note: Award ***A1*** for 2 out of 3 of $3e^{2x}$, $6xe^{2x}$ and $-8e^{2x}$ seen or implied.

[3 marks]

(b) $f'(x) = 1$ ***(M1)***

$x = 0.86299...$

$x = 0.863$ ***A1***

$y = -7.92719...$

$y = -7.93$ ***A1***

$(0.863, -7.93)$

[3 marks]

(c) x-intercept is at $\frac{4}{3}(1.33)$ ***(A1)***

attempt to use formula for volume of revolution ***(M1)***

Note: Award ***(M1)*** for an integral involving π and $(f(x))^2$. Condone use of 2π and incorrect or absent limits.

$\pi\int_0^{\frac{4}{3}}\left(e^{2x}(3x-4)\right)^2 dx$ ***(A1)***

Note: This ***(A1)*** can be awarded if the dx is omitted.

$= 164.849...$

$= 165$ ***A1***

[4 marks]

(d) (i) attempt to compose functions in the correct order ***(M1)***

$(f \circ g)(0) = f(g(0)) = f(1)$

$= -7.38905...$

$= -7.39 (= -e^{-2})$ ***A1***

(ii) attempt to use the chain rule ***(M1)***

$(f \circ g)'(0) = f'(g(0))g'(0)$

Note: For this ***(M1)*** to be awarded, multiplication of two derivatives should be seen or implied.

$= 2f'(1)(= 2 \times 7.38905...)$ ***(A1)***

$= 14.7781...$

$= 14.8 (= 2e^{-2})$ ***A1***

[5 marks]

1 – (MAT/21_HL_Summer_2012_Q8) – *Integration*

$x = r - \frac{r}{h}y$ or $x = \frac{r}{h}(h - y)$ (or equivalent)

$\int \pi x^2 \, dy$

$= \pi \int_0^h \left(r - \frac{r}{h}y \right)^2 dy$

Note: Award ***M1*** for $\int x^2 \, dy$ and ***A1*** for correct expression.

Accept $\pi \int_0^h \left(\frac{r}{h}y - r \right)^2 dy$ and $\pi \int_0^h \left(\pm \left(r - \frac{r}{h}x \right) \right)^2 dx$

$= \pi \int_0^h \left(r^2 - \frac{2r^2}{h}y + \frac{r^2}{h^2}y^2 \right) dy$

Note: Accept substitution method and apply markscheme to corresponding steps.

$= \pi \left[r^2 y - \frac{r^2 y^2}{h} + \frac{r^2 y^3}{3h^2} \right]_0^h$

Note: Award ***M1*** for attempted integration of any quadratic trinomial.

$= \pi \left(r^2 h - r^2 h + \frac{1}{3} r^2 h \right)$

Note: Award ***M1*** for attempted substitution of limits in a trinomial.

$= \frac{1}{3} \pi r^2 h$

2 – (MAT/20_HL_Winter_2012_Q8) – *Integration*

$x = \sin t$, $\mathrm{d}x = \cos t\,\mathrm{d}t$

$$\int \frac{x^3}{\sqrt{1-x^2}}\mathrm{d}x = \int \frac{\sin^3 t}{\sqrt{1-\sin^2 t}}\cos t\ \mathrm{d}t$$

$$= \int \sin^3 t\,\mathrm{d}t$$

$$= \int \sin^2 t\ \sin t\,\mathrm{d}t$$

$$= \int \left(1-\cos^2 t\right)\sin t\,\mathrm{d}t$$

$$= \int \sin t\,\mathrm{d}t - \int \cos^2 t \sin t\,\mathrm{d}t$$

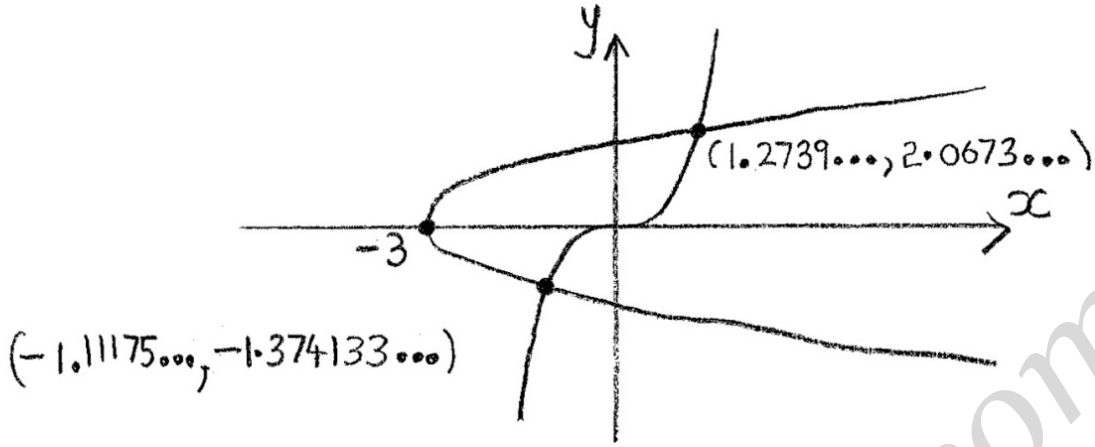
$$= -\cos t + \frac{\cos^3 t}{3} + C$$

$$= -\sqrt{1-x^2} + \frac{1}{3}\left(\sqrt{1-x^2}\right)^3 + C$$

$$\left(= -\sqrt{1-x^2}\left(1-\frac{1}{3}(1-x^2)\right) + C\right)$$

$$\left(= -\frac{1}{3}\sqrt{1-x^2}\,(2+x^2) + C\right)$$

3 – (MAT/20_HL_Winter_2012_Q9) – *Integration*



intersection points

Note: Only either the x-coordinate or the y-coordinate is needed.

EITHER

$x = y^2 - 3 \Rightarrow y = \pm\sqrt{x+3}$ $\quad$ $\left(\text{accept } y = \sqrt{x+3}\right)$

$A = \int_{-3}^{-1.111...} 2\sqrt{x+3}\,\mathrm{d}x + \int_{-1.111...}^{1.2739...} \sqrt{x+3} - x^3\,\mathrm{d}x$

$= 3.4595... + 3.8841...$

$= 7.34$ (3sf)

OR

$y = x^3 \Rightarrow x = \sqrt[3]{y}$

$A = \int_{-1.374...}^{2.067...} \sqrt[3]{y} - \left(y^2 - 3\right)\mathrm{d}y$

$= 7.34$ (3sf)

4 – (MAT/21_HL_Summer_2013_Q4) – *Integration*

volume $= \pi\int x^2\mathrm{d}y$

$x = \arcsin y + 1$

volume $= \pi\int_0^1 (\arcsin y + 1)^2\,\mathrm{d}y$

Note: *A1* is for the limits, provided a correct integration of y.

$= 2.608993\ldots\pi = 8.20$

5 – (MAT/22_HL_Summer_2013_Q4) – *Integration*

(a) $\int x\sec^2 x\,\mathrm{d}x = x\tan x - \int 1\times\tan x\,\mathrm{d}x$

$= x\tan x + \ln|\cos x|\ (+c)\ \ (= x\tan x - \ln|\sec x|(+c)\ )$

(b) attempting to solve an appropriate equation *eg* $m\tan m + \ln(\cos m) = 0.5$

$m = 0.822$

6 – (MAT/22_HL_Summer_2013_Q12) – *Differentiation, Integration*

(a) (i) **METHOD 1**

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\sin x + \cos x$

$y\frac{\mathrm{d}y}{\mathrm{d}x} = (\cos x + \sin x)(-\sin x + \cos x)$

$= \cos^2 x - \sin^2 x$

$= \cos 2x$

METHOD 2

$y^2 = (\sin x + \cos x)^2$

$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2(\cos x + \sin x)(\cos x - \sin x)$

$y\frac{\mathrm{d}y}{\mathrm{d}x} = \cos^2 x - \sin^2 x$

$= \cos 2x$

(ii) attempting to separate variables $\int y\,\mathrm{d}y = \int \cos 2x\,\mathrm{d}x$

$\frac{1}{2}y^2 = \frac{1}{2}\sin 2x + C$

Note: Award ***A1*** for a correct LHS and ***A1*** for a correct RHS.

$y = \pm(\sin 2x + A)^{\frac{1}{2}}$

(iii) $\sin 2x + A \equiv (\cos x + \sin x)^2$

$(\cos x + \sin x)^2 = \cos^2 x + 2\sin x\cos x + \sin^2 x$

use of $\sin 2x \equiv 2\sin x\cos x$.

$A = 1$

(b) (i) substituting $x=\frac{\pi}{4}$ and $y=2$ into $y=(\sin 2x+A)^{\frac{1}{2}}$

so $g(x)=(\sin 2x+3)^{\frac{1}{2}}$.

range g is $\left[\sqrt{2}, 2\right]$

Note: Accept $[1.41, 2]$. Award ***A1*** for each correct endpoint and ***A1*** for the correct closed interval.

(ii) $\int_0^{\frac{\pi}{2}}(\sin 2x+3)^{\frac{1}{2}}\,\mathrm{d}x$

$=2.99$

(iii) $\pi\int_0^{\frac{\pi}{2}}(\sin 2x+3)\,\mathrm{d}x-\pi(1)\left(\frac{\pi}{2}\right)$ (or equivalent)

Note: Award ***(M1)(A1)(A1)*** for $\pi\int_0^{\frac{\pi}{2}}(\sin 2x+2)\mathrm{d}x$

$=17.946-4.935\ \left(=\frac{\pi}{2}(3\pi+2)-\pi\left(\frac{\pi}{2}\right)\right)$

$=13.0$

Note: Award ***A1*** for $\pi(\pi+1)$.

7 – (MAT/20_HL_Winter_2013_Q10) – *Integration*

EITHER

$$\frac{\mathrm{d}x}{\mathrm{d}u} = 2\sec^2 u$$

$$\int \frac{2\sec^2 u \ \mathrm{d}u}{4\tan^2 u\sqrt{4+4\tan^2 u}}$$

$$= \int \frac{2\sec^2 u \ \mathrm{d}u}{4\tan^2 u \times 2\sec u} \qquad \left(= \int \frac{\mathrm{d}u}{4\sin^2 u\sqrt{\tan^2 u+1}} \text{ or } = \int \frac{2\sec^2 u \ \mathrm{d}u}{4\tan^2 u\sqrt{4\sec^2 u}}\right)$$

OR

$$u = \arctan\frac{x}{2}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2}{x^2+4}$$

$$\int \frac{\sqrt{4\tan^2 u + 4}\,\mathrm{d}u}{2\times 4\tan^2 u}$$

$$\int \frac{2\sec u \ \mathrm{d}u}{2\times 4\tan^2 u}$$

THEN

$$= \frac{1}{4}\int \frac{\sec u \ \mathrm{d}u}{\tan^2 u}$$

$$= \frac{1}{4}\int \operatorname{cosec} u \cot u \ \mathrm{d}u \quad \left(= \frac{1}{4}\int \frac{\cos u}{\sin^2 u}\mathrm{d}u\right)$$

$$= -\frac{1}{4}\operatorname{cosec} u \ \ (+C) \quad \left(= -\frac{1}{4\sin u} \ (+C)\right)$$

use of either $\tan u = \frac{x}{2}$ or an appropriate trigonometric identity

either $\sin u = \frac{x}{\sqrt{x^2+4}}$ or $\operatorname{cosec} u = \frac{\sqrt{x^2+4}}{x}$ (or equivalent)

$$= \frac{-\sqrt{x^2+4}}{4x}(+C)$$

8 – (MAT/20_HL_Winter_2013_Q13) – *Functions - Roots, Exponentials - Logarithms, Integration*

(a) (i) either counterexample or sketch or
recognising that $y = k$ $(k > 1)$ intersects the graph of $y = f(x)$ twice
function is not 1–1 (does not obey horizontal line test)
so f^{-1} does not exist

(ii) $f'(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right)$

$f'(\ln 3) = \frac{4}{3}$ $(= 1.33)$

$m = -\frac{3}{4}$

$f(\ln 3) = \frac{5}{3}$ $(= 1.67)$

EITHER

$$\frac{y - \frac{5}{3}}{x - \ln 3} = -\frac{3}{4}$$

$$4y - \frac{20}{3} = -3x + 3\ln 3$$

OR

$$\frac{5}{3} = -\frac{3}{4}\ln 3 + c$$

$$c = \frac{5}{3} + \frac{3}{4}\ln 3$$

$$y = -\frac{3}{4}x + \frac{5}{3} + \frac{3}{4}\ln 3$$

$$12y = -9x + 20 + 9\ln 3$$

THEN

$$9x + 12y - 9\ln 3 - 20 = 0$$

(iii) The tangent at $(a, f(a))$ has equation $y - f(a) = f'(a)(x - a)$.

$f'(a) = \frac{f(a)}{a}$ (or equivalent)

$e^{a} - e^{-a} = \frac{e^{a} + e^{-a}}{a}$ (or equivalent)

attempting to solve for a

$a = \pm 1.20$

(b) (i) $2y = e^x + e^{-x}$

$e^{2x} - 2ye^x + 1 = 0$ ***M1A1***

Note: Award ***M1*** for either attempting to rearrange or interchanging x and y.

$e^x = \dfrac{2y \pm \sqrt{4y^2 - 4}}{2}$ ***A1***

$e^x = y \pm \sqrt{y^2 - 1}$

$x = \ln\left(y \pm \sqrt{y^2 - 1}\right)$ ***A1***

$f^{-1}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$ ***A1***

Note: Award ***A1*** for correct notation and for stating the positive "branch".

(ii) $V = \pi\int_1^5 \left(\ln\left(y + \sqrt{y^2 - 1}\right)\right)^2 \, dy$ ***(M1)(A1)***

Note: Award ***M1*** for attempting to use $V = \pi\int_c^d x^2 dy$.

$= 37.1 \ (\text{units}^3)$ ***A1***

9 – (MAT/22_HL_Summer_2014_Q3) – *Integration*

(a) $x_A = 2.87$

$x_B = 6.78$

(b) $\int_{2.87172K}^{6.7768K} 1 - 2\sin x - x^2 e^{-x} \, dx$

$= 6.76$

10 – (MAT/21_HL_Summer_2014_Q5) – *Integration*

(a) $\frac{\pi}{2}(1.57), \frac{3\pi}{2}(4.71)$

hence the coordinates are $\left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$

(b) (i) $\pi\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(x^2 - (x + 2\cos x)^2\right) \mathrm{d}x$

Note: Award *A1* for $x^2 - (x + 2\cos x)^2$, *A1* for correct limits and *A1* for π.

(ii) $6\pi^2 (= 59.2)$

11 - (MAT/21_HL_Summer_2014_Q6) - *Integration*

(a) **METHOD 1**

sketch showing where the lines cross or zeros of $y = x(x+2)^6 - x$
$x = 0$
$x = -1$ and $x = -3$
the solution is $-3 < x < -1$ or $x > 0$

Note: Do not award either final *A1* mark if strict inequalities are not given.

METHOD 2

separating into two cases $x > 0$ and $x < 0$
if $x > 0$ then $(x+2)^6 > 1 \Rightarrow$ always true
if $x < 0$ then $(x+2)^6 < 1 \Rightarrow -3 < x < -1$
so the solution is $-3 < x < -1$ or $x > 0$

Note: Do not award either final *A1* mark if strict inequalities are not given.

METHOD 3

$f(x) = x^7 + 12x^6 + 60x^5 + 160x^4 + 240x^3 + 192x^2 + 64x$
solutions to $x^7 + 12x^6 + 60x^5 + 160x^4 + 240x^3 + 192x^2 + 63x = 0$ are
$x = 0$, $x = -1$ and $x = -3$
so the solution is $-3 < x < -1$ or $x > 0$

Note: Do not award either final *A1* mark if strict inequalities are not given.

METHOD 4

$f(x) = x$ when $x(x+2)^6 = x$
either $x = 0$ or $(x+2)^6 = 1$
if $(x+2)^6 = 1$ then $x + 2 = \pm 1$ so $x = -1$ or $x = -3$
the solution is $-3 < x < -1$ or $x > 0$

(b) **METHOD 1** (by substitution)

substituting $u = x + 2$

$\mathrm{d}u = \mathrm{d}x$

$\int (u-2)u^6 \mathrm{d}u$

$= \frac{1}{8}u^8 - \frac{2}{7}u^7 (+c)$

$= \frac{1}{8}(x+2)^8 - \frac{2}{7}(x+2)^7 (+c)$

METHOD 2 (by parts)

$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1,\ \frac{\mathrm{d}v}{\mathrm{d}x} = (x+2)^6 \Rightarrow v = \frac{1}{7}(x+2)^7$

$\int x(x+2)^6 \,\mathrm{d}x = \frac{1}{7}x(x+2)^7 - \frac{1}{7}\int (x+2)^7 \,\mathrm{d}x$

$= \frac{1}{7}x(x+2)^7 - \frac{1}{56}(x+2)^8 (+c)$

METHOD 3 (by expansion)

$\int f(x)\mathrm{d}x = \int \left(x^7 + 12x^6 + 60x^5 + 160x^4 + 240x^3 + 192x^2 + 64x\right)\mathrm{d}x$

$= \frac{1}{8}x^8 + \frac{12}{7}x^7 + 10x^6 + 32x^5 + 60x^4 + 64x^3 + 32x^2 (+c)$ are

12 – (MAT/20_HL_Winter_2014_Q13) – *Differentiation, Integration*

(a) attempting to use $V = \pi\int_a^b x^2 \mathrm{d}y$

attempting to express x^2 in terms of y *ie* $x^2 = 4(y+16)$

for $y = h$, $V = 4\pi\int_0^h y+16\,\mathrm{d}y$

$V = 4\pi\left(\frac{h^2}{2} + 16h\right)$

(b) (i) **METHOD 1**

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t}$

$\frac{\mathrm{d}V}{\mathrm{d}h} = 4\pi(h+16)$

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{4\pi(h+16)} \times \frac{-250\sqrt{h}}{\pi(h+16)}$

Note: Award ***M1*** for substitution into $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t}$.

$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{250\sqrt{h}}{4\pi^2(h+16)^2}$

METHOD 2

$\frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi(h+16)\frac{\mathrm{d}h}{\mathrm{d}t}$ (implicit differentiation)

$\frac{-250\sqrt{h}}{\pi(h+16)} = 4\pi(h+16)\frac{\mathrm{d}h}{\mathrm{d}t}$ (or equivalent)

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{4\pi(h+16)} \times \frac{-250\sqrt{h}}{\pi(h+16)}$

$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{250\sqrt{h}}{4\pi^2(h+16)^2}$

(ii) $\frac{\mathrm{d}t}{\mathrm{d}h} = -\frac{4\pi^2(h+16)^2}{250\sqrt{h}}$

$t = \int -\frac{4\pi^2(h+16)^2}{250\sqrt{h}}\,\mathrm{d}h$

$t = \int -\frac{4\pi^2(h^2+32h+256)}{250\sqrt{h}}\,\mathrm{d}h$

$t = \frac{-4\pi^2}{250}\int\left(h^{\frac{3}{2}} + 32h^{\frac{1}{2}} + 256h^{-\frac{1}{2}}\right)\mathrm{d}h$

(iii) **METHOD 1**

$$t=\frac{-4\pi^2}{250}\int_{48}^{0}\left(h^{\frac{3}{2}}+32h^{\frac{1}{2}}+256h^{-\frac{1}{2}}\right)\mathrm{d}h$$ ***(M1)***

$t=2688.756\ldots$ (s) ***(A1)***

45 minutes (correct to the nearest minute) ***A1***

METHOD 2

$$t=\frac{-4\pi^2}{250}\left(\frac{2}{5}h^{\frac{5}{2}}+\frac{64}{3}h^{\frac{3}{2}}+512h^{\frac{1}{2}}\right)+c$$

when $t=0, h=48 \Rightarrow c=2688.756\ldots\left(c=\frac{4\pi^2}{250}\left(\frac{2}{5}\times 48^{\frac{5}{2}}+\frac{64}{3}\times 48^{\frac{3}{2}}+512\times 48^{\frac{1}{2}}\right)\right)$

when $h=0$, $t=2688.756\ldots\left(t=\frac{4\pi^2}{250}\left(\frac{2}{5}\times 48^{\frac{5}{2}}+\frac{64}{3}\times 48^{\frac{3}{2}}+512\times 48^{\frac{1}{2}}\right)\right)$(s)

45 minutes (correct to the nearest minute)

(c) **EITHER**

the depth stabilises when $\frac{\mathrm{d}V}{\mathrm{d}t}=0$ *ie* $8.5-\frac{250\sqrt{h}}{\pi(h+16)}=0$ ***R1***

attempting to solve $8.5-\frac{250\sqrt{h}}{\pi(h+16)}=0$ for h ***(M1)***

OR

the depth stabilises when $\frac{\mathrm{d}h}{\mathrm{d}t}=0$ *ie* $\frac{1}{4\pi(h+16)}\left(8.5-\frac{250\sqrt{h}}{\pi(h+16)}\right)=0$ ***R1***

attempting to solve $\frac{1}{4\pi(h+16)}\left(8.5-\frac{250\sqrt{h}}{\pi(h+16)}\right)=0$ for h ***(M1)***

THEN

$h=5.06$ (cm) ***A1***

13 – (MAT/21_HL_Summer_2015_Q1) – *Integration*

$$\int_{-1}^{1}\pi\left(e^{-x^2}\right)^2 dx \qquad \left(\int_{-1}^{1}\pi e^{-2x^2}\,dx \text{ or } \int_{0}^{1}2\pi e^{-2x^2}\,dx\right)$$

Note: Award ***M1*** for integral involving the function given; ***A1*** for correct limits; ***A1*** for π and $\left(e^{-x^2}\right)^2$

$= 3.758249... = 3.76$

14 – (MAT/22_HL_Summer_2015_Q6) – *Integration*

(a) **EITHER**

$y = \ln(x-a)+b = \ln(5x+10)$

$y = \ln(x-a)+\ln c = \ln(5x+10)$

$y = \ln\left(c(x-a)\right) = \ln(5x+10)$

OR

$y = \ln(5x+10) = \ln\left(5(x+2)\right)$

$y = \ln(5)+\ln(x+2)$

THEN

$a = -2,\ b = \ln 5$

Note: Accept graphical approaches.

Note: Accept $a = 2,\ b = 1.61$

(b) $V = \pi\int_{e}^{2e}\left[\ln(5x+10)\right]^2 dx$

$= 99.2$

15 – (MAT/20_HL_Winter_2015_Q5) – *Integration*

(a) (i) area $= \int_2^4 \sqrt{y-2}\,\mathrm{d}y$

(ii) $= 1.886$ (4 sf only)

(b) volume $= \pi\int_2^4 (y-2)\,\mathrm{d}y$

$= \pi\left[\frac{y^2}{2} - 2y\right]_2^4$

$= 2\pi$ (exact only)

16 – (MAT/22_HL_Summer_2016_Q12) – *Differentiation, Integration*

(a) (i) $\dfrac{1}{4\left(\frac{e^x+e^{-x}}{2}\right) - 2\left(\frac{e^x-e^{-x}}{2}\right)}$

$= \dfrac{1}{2\left(e^x+e^{-x}\right) - \left(e^x-e^{-x}\right)}$

$= \dfrac{1}{e^x+3e^{-x}}$

$= \dfrac{e^x}{e^{2x}+3}$

(ii) $u = e^x \Rightarrow \mathrm{d}u = e^x\mathrm{d}x$

$\int \frac{e^x}{e^{2x}+3}\mathrm{d}x = \int \frac{1}{u^2+3}\mathrm{d}u$

(when $x = 0$, $u = 1$ and when $x = \ln 3$, $u = 3$)

$\int_1^3 \frac{1}{u^2+3}\mathrm{d}u = \left[\frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)\right]_1^3$

$\left(= \left[\frac{1}{\sqrt{3}}\arctan\left(\frac{e^x}{\sqrt{3}}\right)\right]_0^{\ln 3}\right)$

$= \frac{\pi\sqrt{3}}{9} - \frac{\pi\sqrt{3}}{18}$

$= \frac{\pi\sqrt{3}}{18}$

(b) (i) $(n+1)e^{2x} - 2ke^{x} + (n-1) = 0$

$$e^{x} = \frac{2k \pm \sqrt{4k^2 - 4\left(n^2 - 1\right)}}{2(n+1)}$$

$$x = \ln\left(\frac{k \pm \sqrt{k^2 - n^2 + 1}}{n+1}\right)$$

(ii) for two real solutions, we require $k > \sqrt{k^2 - n^2 + 1}$

and we also require $k^2 - n^2 + 1 > 0$

$k^2 > n^2 - 1$

$\Rightarrow k > \sqrt{n^2 - 1}$ $(k \in \mathbb{R}^+)$

(c) (i) **METHOD 1**

$$t(x) = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

$$t'(x) = \frac{\left(e^{x} + e^{-x}\right)^2 - \left(e^{x} - e^{-x}\right)^2}{\left(e^{x} + e^{-x}\right)^2}$$

$$t'(x) = \frac{\left(\frac{e^{x} + e^{-x}}{2}\right)^2 - \left(\frac{e^{x} - e^{-x}}{2}\right)^2}{\left(\frac{e^{x} + e^{-x}}{2}\right)^2}$$

$$= \frac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$$

METHOD 2

$$t'(x) = \frac{f(x)g'(x) - g(x)f'(x)}{[f(x)]^2}$$

$g'(x) = f(x)$ and $f'(x) = g(x)$

$$= \frac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$$

METHOD 3

$$t(x) = \left(e^x - e^{-x}\right)\left(e^x + e^{-x}\right)^{-1}$$

$$t'(x) = 1 - \frac{\left(e^x - e^{-x}\right)^2}{\left(e^x + e^{-x}\right)^2}$$

$$= 1 - \frac{[g(x)]^2}{[f(x)]^2}$$

$$= \frac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$$

METHOD 4

$$t'(x) = \frac{g'(x)}{f(x)} - \frac{g(x)f'(x)}{[f(x)]^2}$$

$g'(x) = f(x)$ and $f'(x) = g(x)$ gives $t'(x) = 1 - \frac{[g(x)]^2}{[f(x)]^2}$

$$= \frac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$$

(ii) **METHOD 1**

$[f(x)]^2 > [g(x)]^2$ (or equivalent)

$[f(x)]^2 > 0$

hence $t'(x) > 0$, $x \in \mathbb{R}$

METHOD 2

$[f(x)]^2 - [g(x)]^2 = 1$ (or equivalent) ***M1A1***

$[f(x)]^2 > 0$ ***R1***

hence $t'(x) > 0$, $x \in \mathbb{R}$ ***AG***

Note: Award as above for use of either $f(x) = \dfrac{e^x + e^{-x}}{2}$ and $g(x) = \dfrac{e^x - e^{-x}}{2}$ or $e^x + e^{-x}$ and $e^x - e^{-x}$.

METHOD 3

$$t'(x) = \frac{4}{\left(e^x + e^{-x}\right)^2}$$

$\left(e^x + e^{-x}\right)^2 > 0$ ***M1A1***

$\dfrac{4}{\left(e^x + e^{-x}\right)^2} > 0$ ***R1***

hence $t'(x) > 0$, $x \in \mathbb{R}$ ***AG***

17 – (MAT/20_HL_Winter_2016_Q10) – *Functions - Roots, Differentiation, Integration*

(a) attempting to solve either $2e^x - 1 = 0$ or $2e^x - 1 \neq 0$ for x ***(M1)***

$D = \mathbb{R} \setminus \{-\ln 2\}$ (or equivalent *eg* $x \neq -\ln 2$) ***A1***

Note: Accept $D = \mathbb{R} \setminus \{-0.693\}$ or equivalent *eg* $x \neq -0.693$.

[2 marks]

(b) considering $\lim\limits_{x \to -\ln 2} f(x)$ ***(M1)***

$x = -\ln 2 \ (x = -0.693)$ ***A1***

considering one of $\lim\limits_{x \to -\infty} f(x)$ or $\lim\limits_{x \to +\infty} f(x)$ ***M1***

$\lim\limits_{x \to -\infty} f(x) = -2 \Rightarrow y = -2$ ***A1***

$\lim\limits_{x \to +\infty} f(x) = -\frac{1}{2} \Rightarrow y = -\frac{1}{2}$ ***A1***

Note: Award ***A0A0*** for $y = -2$ and $y = -\frac{1}{2}$ stated without any justification.

[5 marks]

(c) $f'(x) = \dfrac{-e^x\left(2e^x - 1\right) - 2e^x\left(2 - e^x\right)}{\left(2e^x - 1\right)^2}$ ***M1A1A1***

$= -\dfrac{3e^x}{\left(2e^x - 1\right)^2}$ ***AG***

[3 marks]

(d) $f'(x) < 0$ (for all $x \in D$) $\Rightarrow f$ is (strictly) decreasing ***R1***

Note: Award ***R1*** for a statement such as $f'(x) \neq 0$ and so the graph of f has no turning points.

one branch is above the upper horizontal asymptote and the other branch is below the lower horizontal asymptote ***R1***

f has an inverse ***AG***

$-\infty < x < -2 \cup -\frac{1}{2} < x < \infty$ ***A2***

Note: Award ***A2*** if the domain of the inverse is seen in either part (d) or in part (e).

[4 marks]

(e) $x = \dfrac{2-e^y}{2e^y - 1}$ ***M1***

Note: Award ***M1*** for interchanging x and y (can be done at a later stage).

$2xe^y - x = 2 - e^y$ ***M1***

$e^y(2x+1) = x+2$ ***A1***

$f^{-1}(x) = \ln\left(\dfrac{x+2}{2x+1}\right)$ $\left(f^{-1}(x) = \ln(x+2) - \ln(2x+1)\right)$ ***A1***

[4 marks]

(f) use of $V = \pi\int_a^b x^2\,dy$ ***(M1)***

$= \pi\int_0^1 \left(\ln\left(\dfrac{y+2}{2y+1}\right)\right)^2 dy$ ***(A1)(A1)***

Note: Award ***(A1)*** for the correct integrand and ***(A1)*** for the limits.

$= 0.331$ ***A1***

[4 marks]

18 – (MAT/22_HL_Summer_2017_Q2) – *Differentiation, Integration*

(a) **METHOD 1**

$4x^2 + y^2 = 7$

$8x + 2y\dfrac{dy}{dx} = 0$ ***(M1)(A1)***

$\dfrac{dy}{dx} = -\dfrac{4x}{y}$

Note: Award ***M1A1*** for finding $\dfrac{dy}{dx} = -2.309\ldots$ using any alternative method.

hence gradient of normal $= \dfrac{y}{4x}$ ***(M1)***

hence gradient of normal at $\left(1, \sqrt{3}\right)$ is $\dfrac{\sqrt{3}}{4}(= 0.433)$ ***(A1)***

hence equation of normal is $y - \sqrt{3} = \dfrac{\sqrt{3}}{4}(x-1)$ ***(M1)A1***

$\left(y = \dfrac{\sqrt{3}}{4}x + \dfrac{3\sqrt{3}}{4}\right)$ $\left(y = 0.433x + 1.30\right)$

METHOD 2

$4x^2 + y^2 = 7$

$y = \sqrt{7 - 4x^2}$ ***(M1)***

$\frac{dy}{dx} = -\frac{4x}{\sqrt{7 - 4x^2}}$ ***(A1)***

Note: Award ***M1A1*** for finding $\frac{dy}{dx} = -2.309\ldots$ using any alternative method.

hence gradient of normal $= \frac{\sqrt{7 - 4x^2}}{4x}$ ***(M1)***

hence gradient of normal at $\left(1, \sqrt{3}\right)$ is $\frac{\sqrt{3}}{4} (= 0.433)$ ***(A1)***

hence equation of normal is $y - \sqrt{3} = \frac{\sqrt{3}}{4}(x - 1)$ ***(M1)A1***

$\left(y = \frac{\sqrt{3}}{4}x + \frac{3\sqrt{3}}{4}\right)\left(y = 0.433x + 1.30\right)$

[6 marks]

(b) Use of $V = \pi \int_0^{\frac{\sqrt{7}}{2}} y^2 dx$

$V = \pi \int_0^{\frac{\sqrt{7}}{2}} \left(7 - 4x^2\right) dx$ ***(M1)(A1)***

Note: Condone absence of limits or incorrect limits for ***M*** mark.
Do not condone absence of or multiples of π.

$= 19.4 \left(= \frac{7\sqrt{7}\pi}{3}\right)$ ***A1***

[3 marks]

19 – (MAT/21_HL_Summer_2017_Q4) – *Integration*

(a) **METHOD 1**

$2\arcsin(x-1)-\frac{\pi}{4}=\frac{\pi}{4}$ ***(M1)***

$x=1+\frac{1}{\sqrt{2}}(=1.707\ldots)$ ***(A1)***

$\int_0^{1+\frac{1}{\sqrt{2}}}\frac{\pi}{4}-\left(2\arcsin(x-1)-\frac{\pi}{4}\right)\mathrm{d}x$ ***M1A1***

Note: Award ***M1*** for an attempt to find the difference between two functions, ***A1*** for all correct.

METHOD 2

when $x=0,\ y=\frac{-5\pi}{4}\ (=-3.93)$ ***A1***

$x=1+\sin\left(\frac{4y+\pi}{8}\right)$ ***M1A1***

Note: Award ***M1*** for an attempt to find the inverse function.

$\int_{\frac{-5\pi}{4}}^{\frac{\pi}{4}}\left(1+\sin\left(\frac{4y+\pi}{8}\right)\right)\mathrm{d}y$ ***A1***

METHOD 3

$\left|\int_0^{1.38\ldots}\left(2\arcsin(x-1)-\frac{\pi}{4}\right)\mathrm{d}x\right|+\int_0^{1.71\ldots}\frac{\pi}{4}\mathrm{d}x-\int_{1.38\ldots}^{1.71\ldots}\left(2\arcsin(x-1)-\frac{\pi}{4}\right)\mathrm{d}x$ ***M1A1A1A1***

Note: Award ***M1*** for considering the area below the x-axis and above the x-axis and ***A1*** for each correct integral.

[4 marks]

(b) area $=3.30$ (square units) ***A2***

[2 marks]

20 – (MAT/20_HL_Winter_2017_Q8) – *Integration*

EITHER

$x^2 = 2\sec\theta$

$2x\dfrac{dx}{d\theta} = 2\sec\theta\tan\theta$ ***M1A1***

$\displaystyle\int \frac{dx}{x\sqrt{x^4-4}}$

$= \displaystyle\int \frac{\sec\theta\tan\theta d\theta}{2\sec\theta\sqrt{4\sec^2\theta-4}}$ ***M1A1***

OR

$x = \sqrt{2}(\sec\theta)^{\frac{1}{2}}\left(=\sqrt{2}(\cos\theta)^{-\frac{1}{2}}\right)$

$\dfrac{dx}{d\theta} = \dfrac{\sqrt{2}}{2}(\sec\theta)^{\frac{1}{2}}\tan\theta\left(=\dfrac{\sqrt{2}}{2}(\cos\theta)^{-\frac{3}{2}}\sin\theta\right)$ ***M1A1***

$\displaystyle\int \frac{dx}{x\sqrt{x^4-4}}$

$= \displaystyle\int \frac{\sqrt{2}(\sec\theta)^{\frac{1}{2}}\tan\theta d\theta}{2\sqrt{2}(\sec\theta)^{\frac{1}{2}}\sqrt{4\sec^2\theta-4}}\left(=\int \frac{\sqrt{2}(\cos\theta)^{-\frac{3}{2}}\sin\theta d\theta}{2\sqrt{2}(\cos\theta)^{-\frac{1}{2}}\sqrt{4\sec^2\theta-4}}\right)$ ***M1A1***

THEN

$= \dfrac{1}{2}\displaystyle\int \frac{\tan\theta d\theta}{2\tan\theta}$ ***(M1)***

$= \dfrac{1}{4}\displaystyle\int d\theta$

$= \dfrac{\theta}{4} + c$ ***A1***

$x^2 = 2\sec\theta \Rightarrow \cos\theta = \dfrac{2}{x^2}$ ***M1***

Note: This ***M1*** may be seen anywhere, including a sketch of an appropriate triangle.

so $\dfrac{\theta}{4} + c = \dfrac{1}{4}\arccos\left(\dfrac{2}{x^2}\right) + c$ ***AG***

[7 marks]

21 – (MAT/20_HL_Winter_2017_Q10) – *Differentiation, Integration*

(a) (i) attempt to use quotient rule or product rule ***M1***

$$f'(x) = \frac{\sin x\left(\frac{1}{2}x^{-\frac{1}{2}}\right) - \sqrt{x}\cos x}{\sin^2 x}\left(= \frac{1}{2\sqrt{x}\sin x} - \frac{\sqrt{x}\cos x}{\sin^2 x}\right)$$ ***A1A1***

Note: Award ***A1*** for $\frac{1}{2\sqrt{x}\sin x}$ or equivalent and ***A1*** for $-\frac{\sqrt{x}\cos x}{\sin^2 x}$ or equivalent.

setting $f'(x) = 0$ ***M1***

$\frac{\sin x}{2\sqrt{x}} - \sqrt{x}\cos x = 0$

$\frac{\sin x}{2\sqrt{x}} = \sqrt{x}\cos x$ or equivalent ***A1***

$\tan x = 2x$ ***AG***

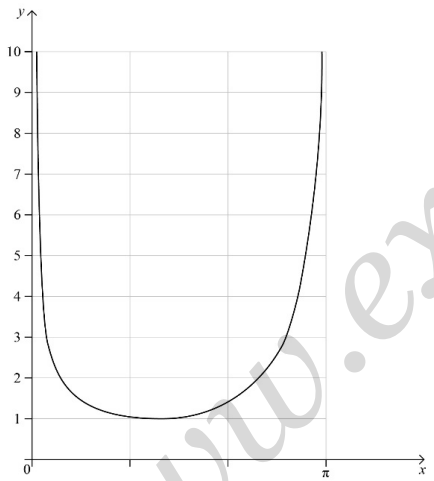
(ii) $x = 1.17$

$0 < x \le 1.17$ ***A1A1***

Note: Award ***A1*** for $0 < x$ and ***A1*** for $x \le 1.17$. Accept $x < 1.17$.

[7 marks]

(b)



concave up curve over correct domain with one minimum point above the *x*-axis. ***A1***

approaches $x = 0$ asymptotically ***A1***

approaches $x = \pi$ asymptotically ***A1***

Note: For the final ***A1*** an asymptote must be seen, and π must be seen on the x-axis or in an equation.

[3 marks]

(c) $f'(x)\left(=\dfrac{\sin x\left(\frac{1}{2}x^{-\frac{1}{2}}\right)-\sqrt{x}\cos x}{\sin^2 x}\right)=1$ ***(A1)***

attempt to solve for x ***(M1)***

$x=1.96$ ***A1***

$y=f(1.96...)$

$=1.51$ ***A1***

[4 marks]

(d) $V=\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\dfrac{x\,\mathrm{d}x}{\sin^2 x}$ ***(M1)(A1)***

Note: ***M1*** is for an integral of the correct squared function (with or without limits and/or π).

$=2.68(=0.852\pi)$ ***A1***

[3 marks]

22 – (MAT/21_HL_Summer_2018_Q5) – *Integration*

(a) $2x^3-3x+1=Ax(x^2+1)+Bx+C$

$A=2$, $C=1$, ***A1***

$A+B=-3\Rightarrow B=-5$ ***A1***

[2 marks]

(b) $\int\dfrac{2x^3-3x+1}{x^2+1}\mathrm{d}x=\int\left(2x-\dfrac{5x}{x^2+1}+\dfrac{1}{x^2+1}\right)\mathrm{d}x$ ***M1M1***

Note: Award ***M1*** for dividing by (x^2+1) to get $2x$, ***M1*** for separating the $5x$ and 1.

$=x^2-\dfrac{5}{2}\ln(x^2+1)+\arctan x(+c)$ ***(M1)A1A1***

Note: Award ***(M1)A1*** for integrating $\dfrac{5x}{x^2+1}$, ***A1*** for the other two terms.

[5 marks]

23 – (MAT/21_HL_Summer_2018_Q9) – *Differentiation, Integration*

(a) differentiating implicitly: ***M1***

$2xy + x^2 \frac{dy}{dx} = -4y^3 \frac{dy}{dx}$ ***A1A1***

Note: Award ***A1*** for each side.

if $\frac{dy}{dx} = 0$ then either $x = 0$ or $y = 0$ ***M1A1***

$x = 0 \Rightarrow$ two solutions for $y\left(y = \pm\sqrt[4]{5}\right)$ ***R1***

$y = 0$ not possible (as $0 \neq 5$) ***R1***

hence exactly two points ***AG***

Note: For a solution that only refers to the graph giving two solutions at $x = 0$ and no solutions for $y = 0$ award ***R1*** only.

[7 marks]

(b) at $(2, 1)$ $4 + 4\frac{dy}{dx} = -4\frac{dy}{dx}$ ***M1***

$\frac{dy}{dx} = -\frac{1}{2}$ ***(A1)***

gradient of normal is 2 ***M1***

$1 = 4 + c$ ***(M1)***

equation of normal is $y = 2x - 3$ ***A1***

[5 marks]

(c) substituting ***(M1)***

$x^2(2x - 3) = 5 - (2x - 3)^4$ or $\left(\frac{y+3}{2}\right)^2 y = 5 - y^4$ ***(A1)***

$x = 0.724$ ***A1***

[3 marks]

(d) recognition of two volumes ***(M1)***

volume 1 $= \pi\int_1^{\sqrt[4]{5}} \frac{5-y^4}{y}\,\mathrm{d}y (= 1.01\pi = 3.178\ldots)$ ***M1A1A1***

Note: Award ***M1*** for attempt to use $\pi\int x^2\mathrm{d}y$, ***A1*** for limits, ***A1*** for $\frac{5-y^4}{y}$. Condone omission of π at this stage.

volume 2

EITHER

$= \frac{1}{3}\pi \times 2^2 \times 4 (= 16.75\ldots)$ ***(M1)(A1)***

OR

$= \pi\int_{-3}^{1}\left(\frac{y+3}{2}\right)^2 \mathrm{d}y \left(= \frac{16\pi}{3} = 16.75\ldots\right)$ ***(M1)(A1)***

THEN

total volume $= 19.9$ ***A1***

[7 marks]

24 – (MAT/20_HL_Winter_2018_Q2) – *Integration*

$f'(x) = \int\left(15\sqrt{x} + \frac{1}{(x+1)^2}\right)\mathrm{d}x = 10x^{\frac{3}{2}} - \frac{1}{x+1}\ (+c)$ ***(M1)A1A1***

Note: ***A1*** for first term, ***A1*** for second term. Withhold one ***A1*** if extra terms are seen.

$f(x) = \int\left(10x^{\frac{3}{2}} - \frac{1}{x+1} + c\right)\mathrm{d}x = 4x^{\frac{5}{2}} - \ln(x+1) + cx + d$ ***A1***

Note: Allow FT from incorrect $f'(x)$ if it is of the form $f'(x) = Ax^{\frac{3}{2}} + \frac{B}{x+1} + c$.

Accept $\ln|x+1|$.

attempt to use at least one boundary condition in their $f(x)$ ***(M1)***

$x = 0, y = -4$

$\Rightarrow d = -4$ ***A1***

$x = 1, y = 0$

$\Rightarrow 0 = 4 - \ln 2 + c - 4$

$\Rightarrow c = \ln 2 (= 0.693)$ ***A1***

$f(x) = 4x^{\frac{5}{2}} - \ln(x+1) + x\ln 2 - 4$

[7 marks]

25 – (MAT/21_HL_Summer_2019_Q5) – *Integration*

METHOD 1

write as $\int 1\times(\ln x)^2 dx$ ***(M1)***

$= x(\ln x)^2 - \int x\times\frac{2(\ln x)}{x}dx\left(= x(\ln x)^2 - \int 2\ln x\right)$ ***M1A1***

$= x(\ln x)^2 - 2x\ln x + \int 2dx$ ***(M1)(A1)***

$= x(\ln x)^2 - 2x\ln x + 2x + c$ ***A1***

26 – (MAT/21_HL_Summer_2019_Q7) – *Integration*

volume $= \pi\int_0^9\left(y^{\frac{1}{2}}+1\right)^2 dy - \pi\int_1^9 (y-1)\,dy$ ***(M1)(M1)(M1)(A1)(A1)***

Note: Award ***(M1)*** for use of formula for rotating about y-axis, ***(M1)*** for finding at least one inverse, ***(M1)*** for subtracting volumes, ***(A1)(A1)***for each correct expression, including limits.

$= 268.6\ldots - 100.5\ldots(85.5\pi - 32\pi)$

$= 168 (= 53.5\pi)$ ***A2***

27 – (MAT/20_HL_Winter_2019_Q11) – *Differentiation, Integration*

(a) (i) valid attempt to differentiate implicitly ***(M1)***

$4x = 3\sin^2 y\cos y\frac{dy}{dx}$ ***A1A1***

$\frac{dy}{dx} = \frac{4x}{3\sin^2 y\cos y}$ ***A1***

(ii) at $\left(\frac{1}{4}, \frac{5\pi}{6}\right)$, $\frac{dy}{dx} = \frac{4x}{3\sin^2 y\cos y} = \frac{1}{3\left(\frac{1}{2}\right)^2\left(-\frac{\sqrt{3}}{2}\right)}$ ***(M1)***

$\Rightarrow \frac{dy}{dx} = -\frac{8}{3\sqrt{3}}(= -1.54)$ ***A1***

hence equation of tangent is

$y - \frac{5\pi}{6} = -1.54\left(x - \frac{1}{4}\right)$ **OR** $y = -1.54x + 3.00$ ***(M1)A1***

Note: Accept $y = -1.54x + 3$.

[8 marks]

(b) $x = \sqrt{\frac{1}{2}\sin^3 y}$ ***(M1)***

$\int_0^{\pi}\sqrt{\frac{1}{2}\sin^3 y}\,dy$ ***(A1)***

$= 1.24$ ***A1***

[3 marks]

(c) use of volume $= \int \pi x^2 \mathrm{d}y$ ***(M1)***

$= \int_0^{\pi} \frac{1}{2}\pi \sin^3 y \, \mathrm{d}y$ ***A1***

$= \frac{1}{2}\pi \int_0^{\pi} \left(\sin y - \sin y \cos^2 y\right) \mathrm{d}y$

Note: Condone absence of limits up to this point.

reasonable attempt to integrate ***(M1)***

$= \frac{1}{2}\pi \left[-\cos y + \frac{1}{3}\cos^3 y\right]_0^{\pi}$ ***A1A1***

Note: Award ***A1*** for correct limits (not to be awarded if previous ***M1*** has not been awarded) and ***A1*** for correct integrand.

$= \frac{1}{2}\pi\left(1 - \frac{1}{3}\right) - \frac{1}{2}\pi\left(-1 + \frac{1}{3}\right)$ ***A1***

$= \frac{2\pi}{3}$ ***AG***

Note: Do not accept decimal answer equivalent to $\frac{2\pi}{3}$.

[6 marks]

28 – (MAT/20_HL_Winter_2020_Q3) – *Trigonometry, Integration*

(a) the principal axis is $\frac{5+(-1)}{2} (= 2)$

so $p = 2$ ***A1***

the amplitude is $\frac{5-(-1)}{2} (= 3)$

so $q = 3$ ***A1***

EITHER

one period is $2\left(-\frac{3\pi}{4}-\left(-\frac{9\pi}{4}\right)\right)$ ***(M1)***

$=3\pi$

$\Rightarrow \frac{2\pi}{r}=3\pi$

OR

Substituting a point eg $-1=2+\sin\left(-\frac{3\pi}{4}r\right)$

$\sin\left(-\frac{3\pi}{4}r\right)=-1\Rightarrow -\frac{3\pi}{4}r=\ldots-\frac{5\pi}{2},-\frac{\pi}{2},\frac{3\pi}{2},\ldots$

Choice of correct solution $-\frac{3\pi}{4}r=-\frac{\pi}{2}$ ***(M1)***

THEN

$\Rightarrow r=\frac{2}{3}$ ***A1***

$\left(\Rightarrow y=2+3\sin\left(\frac{2x}{3}\right)\right)$

Note: q and r can be both given as negatives for full marks

[4 marks]

(b) roots are $x=-1.09459\ldots, x=-3.617797\ldots$ ***(A1)***

$\int_{-3.617797\ldots}^{-1.09459\ldots}\left(2+3\sin\left(\frac{2x}{3}\right)\right)\mathrm{d}x$ ***(M1)***

$=-1.66\left(=-1.66179\ldots\right)$ ***(A1)***

so area $=1.66$ (units^2) ***A1***

[4 marks]

29 – (MathAA/20_HL_Winter_2021_Q10) – *Algebra-functions - Roots, Graphs, Integration*

(a) (i)

Note: In part (a), penalise once only, if correct values are given instead of correct coordinates.

attempts to solve $x^2 - x - 12 = 0$ ***(M1)***

$(-3, 0)$ and $(4, 0)$ ***A1***

(ii) $\left(0, \frac{4}{5}\right)$ ***A1***

[3 marks]

(b) $x = \frac{15}{2}$ ***A1***

Note: Award ***A0*** for $x \neq \frac{15}{2}$.

Award ***A1*** in part (b), if $x = \frac{15}{2}$ is seen on their graph in part (d).

[1 mark]

(c) **METHOD 1**

$(ax+b)(2x-15) \equiv x^2 - x - 12$

attempts to expand $(ax+b)(2x-15)$ ***(M1)***

$2ax^2 - 15ax + 2bx - 15b \equiv x^2 - x - 12$

$a = \frac{1}{2}$ ***A1***

equates coefficients of x ***(M1)***

$-1 = -\frac{15}{2} + 2b$

$b = \frac{13}{4}$ ***A1***

$\left(y = \frac{x}{2} + \frac{13}{4}\right)$

METHOD 2

attempts division on $\frac{x^2-x-12}{2x-15}$ ***M1***

$\frac{x}{2}+\frac{13}{4}+...$ ***M1***

$a=\frac{1}{2}$ ***A1***

$b=\frac{13}{4}$ ***A1***

$\left(y=\frac{x}{2}+\frac{13}{4}\right)$

METHOD 3

$a=\frac{1}{2}$ ***A1***

$\frac{x^2-x-12}{2x-15}\equiv\frac{x}{2}+b+\frac{c}{2x-15}$ ***M1***

$x^2-x-12\equiv\frac{(2x-15)x}{2}+(2x-15)b+c$

equates coefficients of x : ***(M1)***

$-1=-\frac{15}{2}+2b$

$b=\frac{13}{4}$ ***A1***

$\left(y=\frac{x}{2}+\frac{13}{4}\right)$

METHOD 4

attempts division on $\frac{x^2-x-12}{2x-15}$ ***M1***

$\frac{x^2-x-12}{2x-15}=\frac{x}{2}+\frac{\frac{13x}{2}-12}{2x-15}$

$a=\frac{1}{2}$ ***A1***

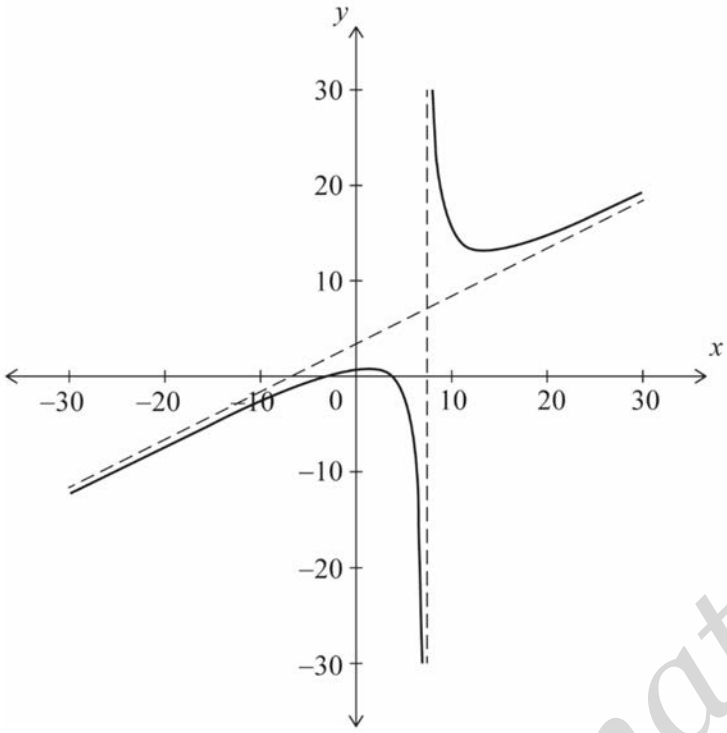
$\frac{\frac{13x}{2}-12}{2x-15}=\frac{13}{4}+...$ ***M1***

$b=\frac{13}{4}$ ***A1***

$\left(y=\frac{x}{2}+\frac{13}{4}\right)$

[4 marks]

(d)



two branches with approximately correct shape (for $-30 \le x \le 30$) ***A1***

their vertical and oblique asymptotes in approximately correct positions with both branches showing correct asymptotic behaviour to these asymptotes ***A1***

their axes intercepts in approximately the correct positions ***A1***

Note: Points of intersection with the axes and the equations of asymptotes are not required to be labelled.

[3 marks]

(e) (i) attempts to split into partial fractions: ***(M1)***

$$\frac{2x-15}{(x+3)(x-4)} \equiv \frac{A}{x+3} + \frac{B}{x-4}$$

$$2x-15 \equiv A(x-4)+B(x+3)$$

$A=3$ ***A1***

$B=-1$ ***A1***

$$\left(\frac{3}{x+3}-\frac{1}{x-4}\right)$$

(ii) $\int_0^3 \left(\frac{3}{x+3}-\frac{1}{x-4}\right) \mathrm{d}x$

attempts to integrate and obtains two terms involving 'ln' ***(M1)***

$=\left[3\ln|x+3|-\ln|x-4|\right]_0^3$ ***A1***

$=3\ln 6-\ln 1-3\ln 3+\ln 4$ ***A1***

$=3\ln 2+\ln 4 \ (=\ln 8+\ln 4)$

$=\ln 32 \ (=5\ln 2)$ ***A1***

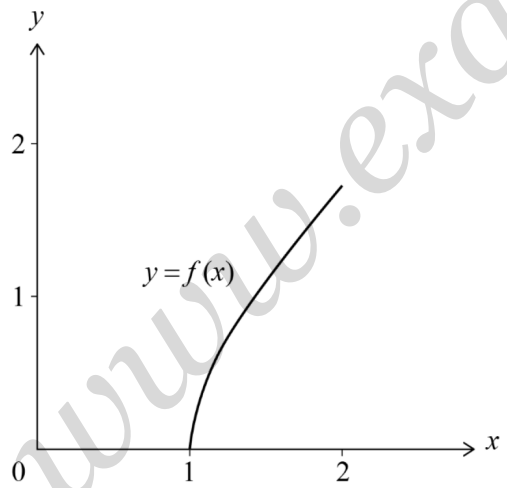
Note: The final ***A1*** is dependent on the previous two ***A*** marks.

[7 marks]

Total [18 marks]

30 – (MathAA/21_HL_Summer_2022_Q10) – *Algebra-functions - Roots, Differentiation, Integration*

(a)



correct shape (concave down) within the given domain $1 \le x \le 2$ ***A1***

$(1,0)$ and $(2,\sqrt{3})(=(2,1.73))$ ***A1***

Note: The coordinates of endpoints may be seen on the graph or marked on the axes.

(b) (i) interchanging x and y (seen anywhere) ***M1***

$x = \sqrt{y^2 - 1}$

$x^2 = y^2 - 1$ ***A1***

$y = \sqrt{x^2 + 1}$ ***A1***

$f^{-1}(x) = \sqrt{x^2 + 1}$ ***AG***

(ii) $0 \le x \le \sqrt{3}$ OR domain $\left[0, \sqrt{3}\right] \left(= [0, 1.73]\right)$ ***A1***

$1 \le y \le 2$ OR $1 \le f^{-1}(x) \le 2$ OR range $[1, 2]$ ***A1***

[5 marks]

(c) (i) attempt to substitute $x = \sqrt{y^2 + 1}$ into the correct volume formula ***(M1)***

$V = \pi \int_0^h \left(\sqrt{y^2 + 1}\right)^2 \mathrm{d}y \quad \left(= \pi \int_0^h \left(y^2 + 1\right) \mathrm{d}y\right)$ ***A1***

$= \pi \left[\frac{1}{3} y^3 + y\right]_0^h$ ***A1***

$= \pi \left(\frac{1}{3} h^3 + h\right)$ ***AG***

Note: Award marks as appropriate for correct work using a different variable e.g.

$\pi \int_0^h \left(\sqrt{x^2 + 1}\right)^2 \mathrm{d}x$

(ii) attempt to substitute $h = \sqrt{3}$ $(= 1.732...)$ into V ***(M1)***

$V = 10.8828...$

$V = 10.9$ (m³) $\left(= 2\sqrt{3}\pi\right)$ (m³) ***A1***

[5 marks]

(d) **METHOD 1**

time $= \frac{10.8828...}{0.4}\left(= \frac{2\sqrt{3}\pi}{0.4}\right)$ ***(M1)***

$= 27.207...$

$= 27.2\left(= 5\sqrt{3}\pi\right)(s)$ ***A1***

[2 marks]

(e) attempt to find the height of the tank when $V = 5.4414...\ \left(= \sqrt{3}\pi\right)$ ***(M1)***

$\pi\left(\frac{1}{3}h^3 + h\right) = 5.4414...\ \left(= \sqrt{3}\pi\right)$

$h = 1.1818...$ ***(A1)***

attempt to use the chain rule or differentiate $V = \pi\left(\frac{1}{3}h^3 + h\right)$ with respect to t ***(M1)***

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{\pi(h^2+1)} \times \frac{dV}{dt}$ OR $\frac{dV}{dt} = \pi\left(h^2+1\right)\frac{dh}{dt}$ ***(A1)***

attempt to substitute **their** h and $\frac{dV}{dt} = 0.4$ ***(M1)***

$\frac{dh}{dt} = \frac{0.4}{\pi\left(1.1818...^2+1\right)} = 0.053124...$

$= 0.0531$ (m s^{-1}) ***A1***

[6 marks]

31 – (MathAA/22_HL_Summer_2022_Q2) – *Integration*

METHOD 1

recognises that $g(x)=\int\left(3x^2+5e^x\right)\,dx$ ***(M1)***

$g(x)=x^3+5e^x\,(+C)$ ***(A1)(A1)***

Note: Award ***A1*** for each integrated term.

substitutes $x=0$ and $y=4$ into their integrated function (must involve $+C$) ***(M1)***

$4=0+5+C\Rightarrow C=-1$

$g(x)=x^3+5e^x-1$ ***A1***

METHOD 2

attempts to write both sides in the form of a definite integral ***(M1)***

$\int_{11}^{x} g'(t)\,dt=\int_{11}^{x}\left(3t^2+5e^t\right)\,dt$ ***(A1)***

$g(x)-4=x^3+5e^x-5e^0$ ***(A1)(A1)***

Note: Award ***A1*** for $g(x)-4$ and ***A1*** for $x^3+5e^x-5e^0$.

$g(x)=x^3+5e^x-1$ ***A1***

[5 marks]

32 – (MathAA/22_HL_Summer_2022_Q6) – *Integration*

attempts to express x^2 in terms of y ***(M1)***

$V = \pi\int_h^4 36\left(1-\frac{(y-4)^2}{16}\right)\mathrm{d}y$ ***A1***

Note: Correct limits are required.

Attempts to solve $\pi\int_h^4 36\left(1-\frac{(y-4)^2}{16}\right)\mathrm{d}y = 285$ for h ***(M1)***

Note: Award ***M1*** for attempting to solve $36\pi\left(\frac{h^3}{48}-\frac{h^2}{4}+\frac{8}{3}\right) = 285$ or equivalent for h.

$h = 0.7926...$

$h = 0.793$ (cm) ***A2***

[5 marks]

33 – (MathAA/20_HL_Winter_2022_Q2) – *Differentiation, Integration*

(a) $(0.708519...,0.639580...)$

$(0.709,0.640)$ $(x=0.709,\ y=0.640)$ ***A1A1***

[2 marks]

(b) $1.09885...$

$x=1.10$ (accept $(1.10,0)$) ***A1***

[1 mark]

(c) **METHOD 1**

$\int_0^2 |f(x)|dx$ ***(A1)***

$4.61117...$

area = 4.61 ***A2***

METHOD 2

$-\int_{1.09885...}^2 f(x)dx$ OR $\int_{1.09885...}^2 |f(x)|dx$ OR 4.17527... ***(A1)***

$\int_0^{1.09885...} f(x)dx - \int_{1.09885...}^2 f(x)dx$ OR $0.435901... + 4.17527...$ ***(A1)***

$4.61117...$

area = 4.61 ***A1***

[3 marks]

1 – (MathAA/21_HL_Summer_2021_Q12) – *Differential Equations*

(a) $\frac{1}{x(k-x)} \equiv \frac{a}{x} + \frac{b}{k-x}$

$a(k-x) + bx = 1$ ***(A1)***

attempt to compare coefficients OR substitute $x = k$ and $x = 0$ and solve ***(M1)***

$a = \frac{1}{k}$ and $b = \frac{1}{k}$ ***A1***

$f'(x) = \frac{1}{kx} + \frac{1}{k(k-x)}$

[3 marks]

(b) attempt to integrate their $\frac{a}{x} + \frac{b}{k-x}$ ***(M1)***

$f(x) = \frac{1}{k}\int\left(\frac{1}{x} + \frac{1}{k-x}\right)\mathrm{d}x$

$= \frac{1}{k}\left(\ln|x| - \ln|k-x|\right)(+c)\left(= \frac{1}{k}\ln\left|\frac{x}{k-x}\right|(+c)\right)$ ***A1A1***

[3 marks]

Note: Award ***A1*** for each correct term. Award ***A1A0*** for a correct answer without modulus signs. Condone the absence of $+c$.

(c) attempt to separate variables and integrate both sides ***M1***

$$5k\int \frac{1}{P(k-P)}\,\mathrm{d}P = \int 1\,\mathrm{d}t$$

$$5\left(\ln P - \ln(k-P)\right) = t + c$$ ***A1***

Note: There are variations on this which should be accepted, such as $\frac{1}{k}\left(\ln P - \ln(k-P)\right) = \frac{1}{5k}t + c$. Subsequent marks for these variations should be awarded as appropriate.

EITHER

attempt to substitute $t = 0$, $P = 1200$ into an equation involving c ***M1***

$$c = 5\left(\ln 1200 - \ln(k-1200)\right)\left(= 5\ln\left(\frac{1200}{k-1200}\right)\right)$$ ***A1***

$$5\left(\ln P - \ln(k-P)\right) = t + 5\left(\ln 1200 - \ln(k-1200)\right)$$ ***A1***

$$\ln\left(\frac{P(k-1200)}{1200(k-P)}\right) = \frac{t}{5}$$

$$\frac{P(k-1200)}{1200(k-P)} = \mathrm{e}^{\frac{t}{5}}$$ ***A1***

OR

$$\ln\left(\frac{P}{k-P}\right) = \frac{t+c}{5}$$

$$\frac{P}{k-P} = A\mathrm{e}^{\frac{t}{5}}$$ ***A1***

attempt to substitute $t = 0$, $P = 1200$ ***M1***

$$\frac{1200}{k-1200} = A$$ ***A1***

$$\frac{P}{k-P} = \frac{1200\mathrm{e}^{\frac{t}{5}}}{k-1200}$$ ***A1***

THEN

attempt to rearrange and isolate P ***M1***

$Pk - 1200P = 1200ke^{\frac{t}{5}} - 1200Pe^{\frac{t}{5}}$ OR $Pke^{\frac{t}{5}} - 1200Pe^{\frac{t}{5}} = 1200k - 1200P$

OR $\frac{k}{P} - 1 = \frac{k-1200}{1200e^{\frac{t}{5}}}$

$P\left(k - 1200 + 1200e^{\frac{t}{5}}\right) = 1200ke^{\frac{t}{5}}$ OR $P\left(ke^{-\frac{t}{5}} - 1200e^{-\frac{t}{5}} + 1200\right) = 1200k$ ***A1***

$P = \frac{1200k}{(k-1200)e^{\frac{t}{5}} + 1200}$ ***AG***

[8 marks]

(d) attempt to substitute $t = 10$, $P = 2400$ ***(M1)***

$2400 = \frac{1200k}{(k-1200)e^{2} + 1200}$ ***(A1)***

$k = 2845.34...$

$k = 2845$ ***A1***

Note: Award ***(M1)(A1)A0*** for any other value of k which rounds to 2850

[3 marks]

(e) attempt to find the maximum of the first derivative graph OR zero of the second derivative graph OR that $P = \frac{k}{2}(= 1422.67...)$ ***(M1)***

$t = 1.57814...$

$= 1.58$ (days) ***A2***

Note: Accept any value which rounds to 1.6.

[3 marks]

2 – (exam-mate QID138653) – *What Is Science* [0]

C

3 – (exam-mate QID138654) – *What Is Science* [0]

B

4 – (MathAA/21_HL_Summer_2022_Q12) – *Differential Equations*

(a) attempt to use Euler's method ***(M1)***

$x_{n+1} = x_n + 0.1;\ \ y_{n+1} = y_n + 0.1 \times \frac{dy}{dx}$, where $\frac{dy}{dx} = \frac{y^2 - 2x^2}{x^2}$

correct intermediate y-values ***(A1)(A1)***

$3.7, 4.63140..., 5.92098..., 7.79542...$

Note: ***A1*** for any two correct y-values seen

$y = 10.6958...$

$y = 10.7$ ***A1***

Note: For the final ***A1,*** the value 10.7 must be the last value in a table or a list, or be given as a final answer, not just embedded in a table which has further lines.

[4 marks]

(b) $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ ***(A1)***

replacing y with vx and $\frac{dy}{dx}$ with $v + x\frac{dv}{dx}$ ***M1***

$x^2\frac{dy}{dx} = y^2 - 2x^2 \Rightarrow x^2\left(v + x\frac{dv}{dx}\right) = v^2x^2 - 2x^2$ ***A1***

$v + x\frac{dv}{dx} = v^2 - 2$ (since $x > 0$)

$x\frac{dv}{dx} = v^2 - v - 2$ ***AG***

[3 marks]

(c) (i) attempt to separate variables v and x ***(M1)***

$$\int \frac{\mathrm{d}v}{v^2 - v - 2} = \int \frac{\mathrm{d}x}{x}$$

$$\int \frac{\mathrm{d}v}{(v-2)(v+1)} = \int \frac{\mathrm{d}x}{x}$$ ***(A1)***

attempt to express in partial fraction form ***M1***

$$\frac{1}{(v-2)(v+1)} \equiv \frac{A}{v-2} + \frac{B}{v+1}$$

$$\frac{1}{(v-2)(v+1)} = \frac{1}{3}\left(\frac{1}{v-2} - \frac{1}{v+1}\right)$$ ***A1***

$$\frac{1}{3}\int\left(\frac{1}{v-2} - \frac{1}{v+1}\right)\mathrm{d}v = \int \frac{\mathrm{d}x}{x}$$

$$\frac{1}{3}\left(\ln|v-2| - \ln|v+1|\right) = \ln|x|\,(+c)$$ ***A1***

Note: Condone absence of modulus signs throughout.

EITHER

attempt to find c using $x = 1, y = 3, v = 3$ ***M1***

$$c = \frac{1}{3}\ln\frac{1}{4}$$

$$\frac{1}{3}\left(\ln|v-2| - \ln|v+1|\right) = \ln|x| + \frac{1}{3}\ln\frac{1}{4}$$

expressing both sides as a single logarithm ***(M1)***

$$\ln\left|\frac{v-2}{v+1}\right| = \ln\left(\frac{|x|^3}{4}\right)$$

OR

expressing both sides as a single logarithm ***(M1)***

$$\ln\left|\frac{v-2}{v+1}\right| = \ln\left(A|x|^3\right)$$

attempt to find A using $x=1, y=3, v=3$ ***M1***

$$A = \frac{1}{4}$$

THEN

$$\left|\frac{v-2}{v+1}\right| = \frac{1}{4}x^3 \text{ (since } x>0\text{)}$$

substitute $v = \frac{y}{x}$ (seen anywhere) ***M1***

$$\frac{\frac{y}{x}-2}{\frac{y}{x}+1} = \frac{1}{4}x^3 \text{ (since } y>2x\text{)}$$

$$\left(\Rightarrow \frac{y-2x}{y+x} = \frac{1}{4}x^3\right)$$

attempt to make y the subject ***M1***

$$y - \frac{x^3 y}{4} = 2x + \frac{x^4}{4}$$ ***A1***

$$y = \frac{8x+x^4}{4-x^3}$$ ***AG***

(ii) actual value at $y(1.5) = 27.3$ ***A1***

(iii) gradient changes rapidly (during the interval considered) OR

the curve has a vertical asymptote at $x = \sqrt[3]{4}\left(=1.5874...\right)$ ***R1***

[12 marks]

5 – (MathAA/22_HL_Summer_2022_Q12) – *Differential Equations*

(a) rate of growth (change) of the (marsupial) population (with respect to time) ***A1***

[1 mark]

Note: Do not accept growth (change) in the (marsupials) population per year.

(b) **METHOD 1**

attempts implicit differentiation on $\frac{\mathrm{d}P}{\mathrm{d}t} = kP - \frac{kP^2}{N}$ by expanding $kP\left(1 - \frac{P}{N}\right)$ ***(M1)***

$\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = k\frac{\mathrm{d}P}{\mathrm{d}t} - 2\frac{kP}{N}\frac{\mathrm{d}P}{\mathrm{d}t}$ ***A1A1***

$= k\frac{\mathrm{d}P}{\mathrm{d}t}\left(1 - \frac{2P}{N}\right)$ ***A1***

$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{N}\right)$ and so $\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = k^2P\left(1 - \frac{P}{N}\right)\left(1 - \frac{2P}{N}\right)$ ***AG***

METHOD 2

attempts implicit differentiation (product rule) on $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{N}\right)$ ***M1***

$\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = k\frac{\mathrm{d}P}{\mathrm{d}t}\left(1 - \frac{P}{N}\right) + kP\left(-\left(\frac{1}{N}\right)\frac{\mathrm{d}P}{\mathrm{d}t}\right)$ ***A1***

substitutes $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{N}\right)$ into their $\frac{\mathrm{d}^2P}{\mathrm{d}t^2}$ ***M1***

$\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = k\left(kP\left(1 - \frac{P}{N}\right)\right)\left(1 - \frac{P}{N}\right) + kP\left(-\left(\frac{1}{N}\right)kP\left(1 - \frac{P}{N}\right)\right)$

$= k^2P\left(1 - \frac{P}{N}\right)^2 - k^2P\left(1 - \frac{P}{N}\right)\left(\frac{P}{N}\right)$

$= k^2P\left(1 - \frac{P}{N}\right)\left(1 - \frac{P}{N} - \frac{P}{N}\right)$ ***A1***

so $\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = k^2P\left(1 - \frac{P}{N}\right)\left(1 - \frac{2P}{N}\right)$ ***AG***

[4 marks]

(c) $\frac{d^2P}{dt^2} = 0 \Rightarrow k^2P\left(1-\frac{P}{N}\right)\left(1-\frac{2P}{N}\right) = 0$ ***(M1)***

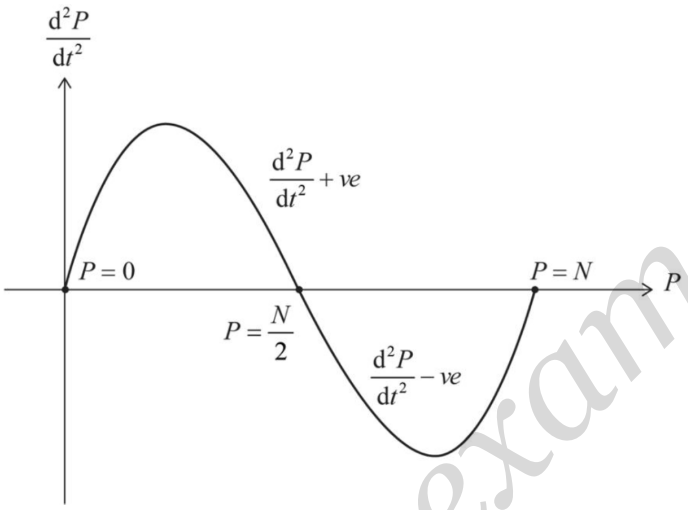
$P = 0, \frac{N}{2}, N$ ***A2***

Note: Award ***A1*** for $P = \frac{N}{2}$ only.

uses the second derivative to show that concavity changes at $P = \frac{N}{2}$ or the first derivative to show a local maximum at $P = \frac{N}{2}$ ***M1***

EITHER

a clearly labelled correct sketch of $\frac{d^2P}{dt^2}$ versus P showing $P = \frac{N}{2}$ corresponding to a local maximum point for $\frac{dP}{dt}$ ***R1***



OR

a correct and clearly labelled sign diagram (table) showing $P = \frac{N}{2}$ corresponding to a local maximum point for $\frac{dP}{dt}$ ***R1***

OR

for example, $\frac{d^2P}{dt^2} = \frac{3k^2N}{32}(>0)$ with $P = \frac{N}{4}$ and $\frac{d^2P}{dt^2} = -\frac{3k^2N}{32}(<0)$ with $P = \frac{3N}{4}$

showing $P = \frac{N}{2}$ corresponds to a local maximum point for $\frac{dP}{dt}$ ***R1***

so the population is increasing at its maximum rate when $P = \frac{N}{2}$ ***AG***

[5 marks]

(d) substitutes $P = \frac{N}{2}$ into $\frac{dP}{dt}$ ***(M1)***

$$\frac{dP}{dt} = k\left(\frac{N}{2}\right)\left(1 - \frac{\frac{N}{2}}{N}\right)$$

the maximum value of $\frac{dP}{dt}$ is $\frac{kN}{4}$ ***A1***

[2 marks]

(e) **METHOD 1**

attempts to separate variables ***M1***

$$\int \frac{N}{P(N-P)}\,\mathrm{d}P = \int k\ \mathrm{d}t$$

attempts to write $\frac{N}{P(N-P)}$ in partial fractions form ***M1***

$$\frac{N}{P(N-P)} \equiv \frac{A}{P} + \frac{B}{(N-P)} \Rightarrow N \equiv A(N-P) + BP$$

$A = 1,\ B = 1$ ***A1***

$$\frac{N}{P(N-P)} \equiv \frac{1}{P} + \frac{1}{(N-P)}$$

$$\int \left(\frac{1}{P} + \frac{1}{(N-P)}\right)\mathrm{d}P = \int k\ \mathrm{d}t$$

$\Rightarrow \ln P - \ln(N-P) = kt(+C)$ ***A1A1***

Note: Award ***A1*** for $-\ln(N-P)$ and ***A1*** for $\ln P$ and $kt(+C)$. Absolute value signs are not required.

attempts to find C in terms of N and P_0 ***M1***

when $t = 0$, $P = P_0$ and so $C = \ln P_0 - \ln(N - P_0)$

$$kt = \ln\left(\frac{P}{N-P}\right) - \ln\left(\frac{P_0}{N-P_0}\right) \left(= \ln\left(\frac{\frac{P}{N-P}}{\frac{P_0}{N-P_0}}\right)\right)$$ ***A1***

so $kt = \ln \frac{P}{P_0}\left(\frac{N-P_0}{N-P}\right)$ ***AG***

[7 marks]

METHOD 2

attempts to separate variables ***M1***

$$\int \frac{1}{P\left(1-\frac{P}{N}\right)}\,\mathrm{d}P = \int k\ \mathrm{d}t$$

attempts to write $\frac{1}{P\left(1-\frac{P}{N}\right)}$ in partial fractions form ***M1***

$$\frac{1}{P\left(1-\frac{P}{N}\right)} \equiv \frac{A}{P} + \frac{B}{1-\frac{P}{N}} \Rightarrow 1 \equiv A\left(1-\frac{P}{N}\right) + BP$$

$A = 1,\ B = \frac{1}{N}$ ***A1***

$$\frac{1}{P\left(1-\frac{P}{N}\right)} \equiv \frac{1}{P} + \frac{1}{N\left(1-\frac{P}{N}\right)}$$

$$\int \frac{1}{P} + \frac{1}{N\left(1-\frac{P}{N}\right)}\,\mathrm{d}P = \int k\ \mathrm{d}t$$

$\Rightarrow \ln P - \ln\left(1-\frac{P}{N}\right) = kt\,(+C)$ ***A1A1***

Note: Award ***A1*** for $-\ln\left(1-\frac{P}{N}\right)$ and ***A1*** for $\ln P$ and $kt\,(+C)$. Absolute value signs are not required.

$$\ln\left(\frac{P}{1-\frac{P}{N}}\right)=kt+C \Rightarrow \ln\left(\frac{NP}{N-P}\right)=kt+C$$

attempts to find C in terms of N and P_0 ***M1***

when $t=0$, $P=P_0$ and so $C=\ln\left(\frac{NP_0}{N-P_0}\right)$

$$kt=\ln\left(\frac{NP}{N-P}\right)-\ln\left(\frac{NP_0}{N-P_0}\right)\left(=\ln\left(\frac{\frac{P}{N-P}}{\frac{P_0}{N-P_0}}\right)\right)$$ ***A1***

$$kt=\ln\frac{P}{P_0}\left(\frac{N-P_0}{N-P}\right)$$ ***AG***

[7 marks]

1 - (MAT/22_HL_Summer_2012_Q12) - *Kinematics*

(a) $\frac{dv}{dt} = -v^2 - 1$

attempt to separate the variables

$\int \frac{1}{1+v^2} dv = \int -1 dt$

$\arctan v = -t + k$

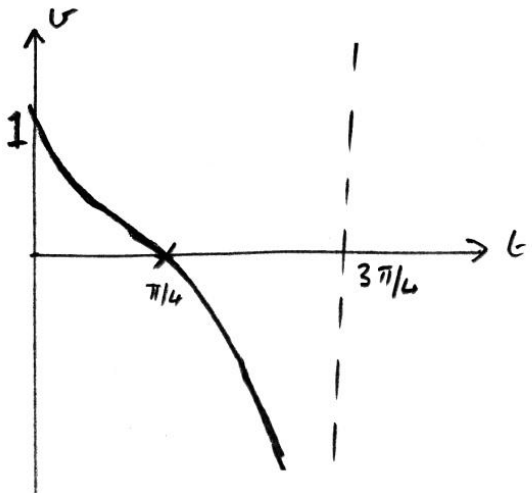
Note: Do not penalize the lack of constant at this stage.

when $t = 0,\ v = 1$

$\Rightarrow k = \arctan 1 = \left(\frac{\pi}{4}\right) = \left(45^\circ\right)$

$\Rightarrow v = \tan\left(\frac{\pi}{4} - t\right)$

(b)



Note:	Award *A1* for general shape, *A1* for asymptote, *A1* for correct t and v intercept.

Note:	Do not penalise if a larger domain is used.

(c) (i) $T = \frac{\pi}{4}$

(ii) area under curve $= \int_0^{\frac{\pi}{4}} \tan\left(\frac{\pi}{4} - t\right) \mathrm{d}t$

$= 0.347 \left(= \frac{1}{2}\ln 2 \right)$

(d) $v = \tan\left(\frac{\pi}{4} - t\right)$

$s = \int \tan\left(\frac{\pi}{4} - t\right) \mathrm{d}t$

$\int \frac{\sin\left(\frac{\pi}{4} - t\right)}{\cos\left(\frac{\pi}{4} - t\right)} \mathrm{d}t$

$= \ln\cos\left(\frac{\pi}{4} - t\right) + k$

when $t = 0,\ s = 0$

$k = -\ln\cos\frac{\pi}{4}$

$s = \ln\cos\left(\frac{\pi}{4} - t\right) - \ln\cos\frac{\pi}{4} \quad \left(= \ln\left[\sqrt{2}\cos\left(\frac{\pi}{4} - t\right)\right]\right)$

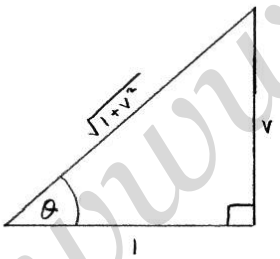
(e) **METHOD 1**

$\frac{\pi}{4} - t = \arctan v$

$t = \frac{\pi}{4} - \arctan v$

$s = \ln\left[\sqrt{2}\cos\left(\frac{\pi}{4} - \frac{\pi}{4} + \arctan v\right)\right]$

$s = \ln\left[\sqrt{2}\cos(\arctan v)\right]$



$s = \ln\left[\sqrt{2}\cos\left(\arccos\frac{1}{\sqrt{1+v^2}}\right)\right]$

$= \ln\frac{\sqrt{2}}{\sqrt{1+v^2}}$

$= \frac{1}{2}\ln\frac{2}{1+v^2}$

METHOD 2

$$s = \ln\cos\left(\frac{\pi}{4} - t\right) - \ln\cos\frac{\pi}{4}$$

$$= -\ln\sec\left(\frac{\pi}{4} - t\right) - \ln\cos\frac{\pi}{4}$$

$$= -\ln\sqrt{1 + \tan^2\left(\frac{\pi}{4} - t\right)} - \ln\cos\frac{\pi}{4}$$

$$= -\ln\sqrt{1 + v^2} - \ln\cos\frac{\pi}{4}$$

$$= \ln\frac{1}{\sqrt{1 + v^2}} + \ln\sqrt{2}$$

$$= \frac{1}{2}\ln\frac{2}{1 + v^2}$$

METHOD 3

$$v\frac{dv}{ds} = -v^2 - 1$$

$$\int\frac{v}{v^2 + 1}dv = -\int 1ds$$

$$\frac{1}{2}\ln(v^2 + 1) = -s + k$$

when $s = 0$, $t = 0 \Rightarrow v = 1$

$$\Rightarrow k = \frac{1}{2}\ln 2$$

$$\Rightarrow s = \frac{1}{2}\ln\frac{2}{1 + v^2}$$

2 – (MAT/20_HL_Winter_2012_Q6) – *Kinematics*

$$2s\frac{\mathrm{d}s}{\mathrm{d}t}+\frac{\mathrm{d}s}{\mathrm{d}t}-2=0$$

$$v=\frac{\mathrm{d}s}{\mathrm{d}t}=\frac{2}{2s+1}$$

EITHER

$$a=\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{\mathrm{d}v}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t}$$

$$\frac{\mathrm{d}v}{\mathrm{d}s}=\frac{-4}{(2s+1)^2}$$

$$a=\frac{-4}{(2s+1)^2}\frac{\mathrm{d}s}{\mathrm{d}t}$$

OR

$$2\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2+2s\frac{\mathrm{d}^2s}{\mathrm{d}t^2}+\frac{\mathrm{d}^2s}{\mathrm{d}t^2}=0$$

$$\underbrace{\frac{\mathrm{d}^2s}{\mathrm{d}t^2}}_{a}=\frac{-2\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2}{2s+1}$$

THEN

$$a=\frac{-8}{(2s+1)^3}$$

3 – (MAT/22_HL_Summer_2013_Q10) – *Kinematics*

METHOD 1

$\frac{dv}{dt} = \frac{1}{40}(60 - v)$

attempting to separate variables $\int \frac{dv}{60 - v} = \int \frac{dt}{40}$

$-\ln(60 - v) = \frac{t}{40} + c$

$c = -\ln 60$ (or equivalent)

attempting to solve for v when $t = 30$

$v = 60 - 60e^{-\frac{3}{4}}$

$v = 31.7\ (\text{m s}^{-1})$

METHOD 2

$\frac{dv}{dt} = \frac{1}{40}(60 - v)$

$\frac{dt}{dv} = \frac{40}{60 - v}$ (or equivalent)

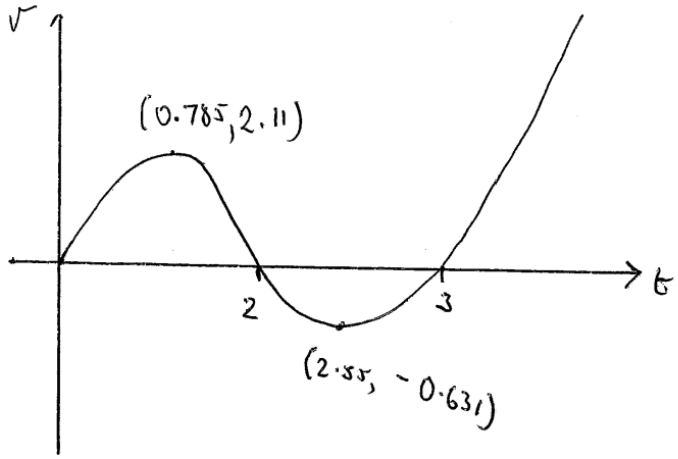
$\int_0^{v_f} \frac{40}{60 - v}\, dv = 30$ where v_f is the velocity of the car after 30 seconds.

attempting to solve $\int_0^{v_f} \frac{40}{60 - v}\, dv = 30$ for v_f

$v = 31.7\ (\text{m s}^{-1})$

4 – (MAT/21_HL_Summer_2013_Q12) – *Kinematics*

(a)



Note: Award ***A1*** for general shape, ***A1*** for correct maximum and minimum, ***A1*** for intercepts.

Note: Follow through applies to (b) and (c).

(b) $0 \le t < 0.785, \left(\text{or } 0 \le t < \frac{5-\sqrt{7}}{3} \right)$

(allow $t < 0.785$)

and $t > 2.55 \left(\text{or } t > \frac{5+\sqrt{7}}{3} \right)$

(c) $0 \le t < 0.785, \left(\text{or } 0 \le t < \frac{5-\sqrt{7}}{3} \right)$

(allow $t < 0.785$)

$2 < t < 2.55, \left(\text{or } 2 < t < \frac{5+\sqrt{7}}{3} \right)$

$t > 3$

(d) position of A: $x_A = \int t^3 - 5t^2 + 6t \, \mathrm{d}t$

$x_A = \frac{1}{4}t^4 - \frac{5}{3}t^3 + 3t^2 \quad (+c)$

when $t = 0$, $x_A = 0$ so $c = 0$

(e) $\frac{\mathrm{d}v_B}{\mathrm{d}t} = -2v_B \Rightarrow \int \frac{1}{v_B}\mathrm{d}v_B = \int -2\mathrm{d}t$

$\ln|v_B| = -2t + c$

$v_B = Ae^{-2t}$

$v_B = -20$ when $t = 0$ so $v_B = -20e^{-2t}$

(f) $x_B = 10e^{-2t}(+c)$

$x_B = 20$ when $t = 0$ so $x_B = 10e^{-2t} + 10$

meet when $\frac{1}{4}t^4 - \frac{5}{3}t^3 + 3t^2 = 10e^{-2t} + 10$

$t = 4.41(290\ldots)$

5 – (MAT/20_HL_Winter_2013_Q12) – *Kinematics*

(a) (i) **METHOD 1**

$v = \int 3\cos\frac{t}{4}\,\mathrm{d}t$

$= 12\sin\frac{t}{4} + c$

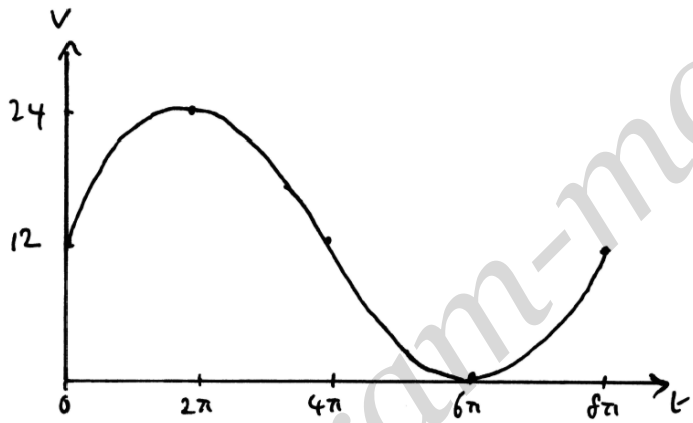
$t = 0,\ v = 12 \Rightarrow v = 12\sin\frac{t}{4} + 12$

METHOD 2

$v - 12 = \int_0^t 3\cos\frac{t}{4}\,\mathrm{d}t$

$v = 12\sin\frac{t}{4} + 12$

(ii)



Note: Award *A1* for shape and domain $0 \le t \le 8\pi$.
Award *A1* for $(0,\ 12)$ and $(6\pi,\ 0)$ $\big((18.8,\ 0)\big)$.
Award *A1* for $(2\pi,\ 24)$ $\big((6.28,\ 24)\big)$.

(iii) **METHOD 1**

$\int_0^{6\pi}\left(12\sin\frac{t}{4} + 12\right)\mathrm{d}t$

$= 274\ (\mathrm{m})\ \big(= 72\pi + 48\ (\mathrm{m})\big)$

METHOD 2

$s = \int 12\sin\frac{t}{4} + 12\ \mathrm{d}t$

$= -48\cos\frac{t}{4} + 12t + c$

When $t = 0$, $s = 0$ and so $c = 48$.
When $t = 6\pi$, $s = 274\ (\mathrm{m})\ \big(= 72\pi + 48\ (\mathrm{m})\big)$.

(b) (i) **METHOD 1**

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\left(v^2 + 4\right)$$

$$\int \frac{\mathrm{d}v}{v^2 + 4} = -\int \mathrm{d}t$$

$$\frac{1}{2}\arctan\left(\frac{v}{2}\right) = -t + c$$

EITHER

$$t = 0,\ v = 2 \Rightarrow c = \frac{\pi}{8}$$

$$\arctan\left(\frac{v}{2}\right) = \frac{\pi}{4} - 2t$$

OR

$$v = 2\tan\left(2c - 2t\right)$$

$$t = 0,\ v = 2 \Rightarrow \tan 2c = 1 \text{ and so } c = \frac{\pi}{8}$$

THEN

$$v = 2\tan\left(\frac{\pi}{4} - 2t\right)$$

$$v = 2\tan\left(\frac{\pi - 8t}{4}\right)$$

METHOD 2

$$\frac{dv}{dt} = -4\sec^2\left(\frac{\pi - 8t}{4}\right)$$

Substituting $v = 2\tan\left(\frac{\pi - 8t}{4}\right)$ into $\frac{dv}{dt} = -\left(v^2 + 4\right)$:

$$\frac{dv}{dt} = -\left(4\tan^2\left(\frac{\pi - 8t}{4}\right) + 4\right)$$

$$= -4\left(\tan^2\left(\frac{\pi - 8t}{4}\right) + 1\right)$$

$$= -4\sec^2\left(\frac{\pi - 8t}{4}\right)$$

Verifying that $v = 2$ when $t = 0$.

(ii) **METHOD 1**

$$v\frac{\mathrm{d}v}{\mathrm{d}s} = -\left(v^2+4\right)$$

$$\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}s} = -\frac{\left(v^2+4\right)}{v}$$

METHOD 2

$$\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{\mathrm{d}v}{\mathrm{d}t}\times\frac{\mathrm{d}t}{\mathrm{d}s}$$

$$\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}s} = -\frac{\left(v^2+4\right)}{v}$$

(iii) **METHOD 1**

When $v=0$, $t=\frac{\pi}{8}$ $(t=0.392...)$.

$$s=\int_0^{\frac{\pi}{8}} 2\tan\left(\frac{\pi-8t}{4}\right)\mathrm{d}t$$

$$s=0.347 \text{ (m)} \left(s=\frac{1}{2}\ln 2 \text{ (m)}\right)$$

METHOD 2

$$\int \frac{v}{4+v^2}\mathrm{d}v = -\int \mathrm{d}s$$

EITHER

$$\frac{1}{2}\ln\left(v^2+4\right) = -s+c \text{ (or equivalent)}$$

$$v=2,\ s=0 \Rightarrow c=\frac{1}{2}\ln 8$$

$$s=-\frac{1}{2}\ln\left(v^2+4\right)+\frac{1}{2}\ln 8 \left(s=\frac{1}{2}\ln\left(\frac{8}{v^2+4}\right)\right)$$

$$v=0 \Rightarrow s=\frac{1}{2}\ln 2 \text{ (m)} \left(s=0.347 \text{ (m)}\right)$$

OR

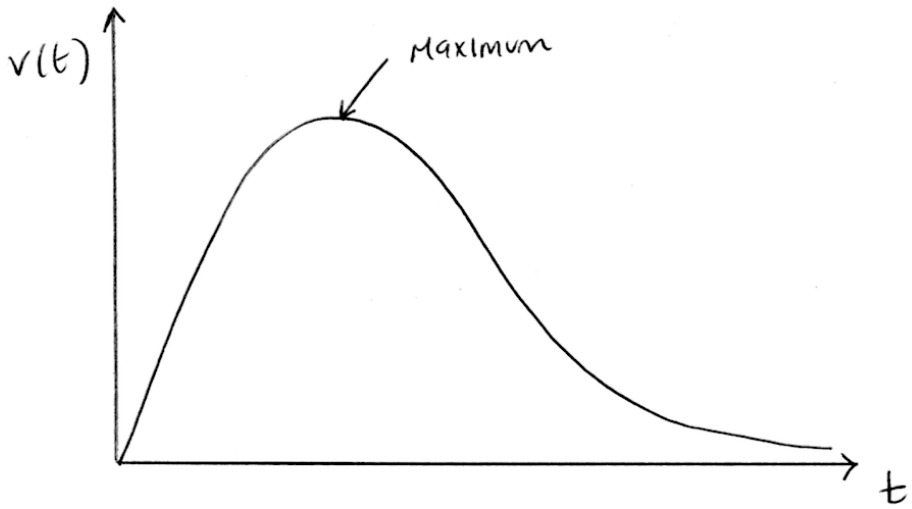
$$-\int_2^0 \frac{v}{4+v^2}\mathrm{d}v = s \text{ (or equivalent)}$$

Note: Award *M1* for setting up a definite integral and award *A1* for stating correct limits.

$$s=0.347 \text{ (m)} \left(s=\frac{1}{2}\ln 2 \text{ (m)}\right)$$

6 – (MAT/22_HL_Summer_2014_Q14) – *Kinematics*

(a)



A1 for correct shape and correct domain

$(1.41,\ 0.0884)\left(\sqrt{2}, \frac{\sqrt{2}}{16}\right)$

(b) **EITHER**

$u = t^2$

$\frac{\mathrm{d}u}{\mathrm{d}t} = 2t$

OR

$t = u^{\frac{1}{2}}$

$\frac{\mathrm{d}t}{\mathrm{d}u} = \frac{1}{2}u^{-\frac{1}{2}}$

THEN

$\int \frac{t}{12+t^4}\mathrm{d}t = \frac{1}{2}\int \frac{\mathrm{d}u}{12+u^2}$

$= \frac{1}{2\sqrt{12}}\arctan\left(\frac{u}{\sqrt{12}}\right)(+c)$

$= \frac{1}{4\sqrt{3}}\arctan\left(\frac{t^2}{2\sqrt{3}}\right)(+c)$ or equivalent

(c) $\int_0^6 \frac{t}{12+t^4}\,\mathrm{d}t$

$$= \left[\frac{1}{4\sqrt{3}}\arctan\left(\frac{t^2}{2\sqrt{3}}\right)\right]_0^6$$

$$= \frac{1}{4\sqrt{3}}\left(\arctan\left(\frac{36}{2\sqrt{3}}\right)\right)\left(= \frac{1}{4\sqrt{3}}\left(\arctan\left(\frac{18}{\sqrt{3}}\right)\right)\right)\text{(m)}$$

Note: Accept $\frac{\sqrt{3}}{12}\arctan\left(6\sqrt{3}\right)$ or equivalent.

(d) $\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{1}{2\sqrt{s(1-s)}}$

$$a = v\frac{\mathrm{d}v}{\mathrm{d}s}$$

$$a = \arcsin\left(\sqrt{s}\right)\times\frac{1}{2\sqrt{s(1-s)}}$$

$$a = \arcsin\left(\sqrt{0.1}\right)\times\frac{1}{2\sqrt{0.1\times 0.9}}$$

$$a = 0.536 \ \left(\mathrm{ms}^{-2}\right)$$

7 – (MAT/20_HL_Winter_2014_Q4) – *Kinematics*

METHOD 1

attempt to set up (diagram, vectors)

correct distances $x = 15t,\ y = 20t$

the distance between the two cyclists at time t is $s = \sqrt{(15t)^2 + (20t)^2} = 25t$ (km)

$\frac{\mathrm{d}s}{\mathrm{d}t} = 25\ (\mathrm{km\,h^{-1}})$

hence the rate is independent of time

METHOD 2

attempting to differentiate $x^2 + y^2 = s^2$ implicitly

$2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t} = 2s\frac{\mathrm{d}s}{\mathrm{d}t}$

the distance between the two cyclists at time t is $\sqrt{(15t)^2 + (20t)^2} = 25t$ (km)

$2(15t)(15) + 2(20t)(20) = 2(25t)\frac{\mathrm{d}s}{\mathrm{d}t}$

Note: Award ***M1*** for substitution of correct values into their equation involving $\frac{\mathrm{d}s}{\mathrm{d}t}$.

$\frac{\mathrm{d}s}{\mathrm{d}t} = 25\ (\mathrm{km\,h^{-1}})$

hence the rate is independent of time

METHOD 3

$s = \sqrt{x^2 + y^2}$

$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{x\frac{\mathrm{d}x}{\mathrm{d}t} + y\frac{\mathrm{d}y}{\mathrm{d}t}}{\sqrt{x^2 + y^2}}$

Note: Award ***M1*** for attempting to differentiate the expression for s.

$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{(15t)(15) + (20t)(20)}{\sqrt{(15t)^2 + (20t)^2}}$

Note: Award ***M1*** for substitution of correct values into their $\frac{\mathrm{d}s}{\mathrm{d}t}$.

$\frac{\mathrm{d}s}{\mathrm{d}t} = 25\ (\mathrm{km\ h^{-1}})$

hence the rate is independent of time

8 – (MAT/20_HL_Winter_2014_Q8) – *Kinematics*

(a) $3-\frac{t}{2}=0 \Rightarrow t=6\,(\text{s})$

Note: Award ***A0*** if either $t=-0.236$ or $t=4.24$ or both are stated with $t=6$.

(b) let d be the distance travelled before coming to rest

$$d=\int_0^4 5-(t-2)^2\,\mathrm{d}t+\int_4^6 3-\frac{t}{2}\mathrm{d}t$$

Note: Award ***M1*** for two correct integrals even if the integration limits are incorrect. The second integral can be specified as the area of a triangle.

$$d=\frac{47}{3}\,(=15.7)\,(\text{m})$$

attempting to solve $\int_6^T\left(\frac{t}{2}-3\right)\mathrm{d}t=\frac{47}{3}$ (or equivalent) for T

$T=13.9\,(\text{s})$

9 – (MAT/22_HL_Summer_2015_Q12) – *Kinematics*

(a) **METHOD 1**

$$s = \int \left(9t - 3t^2\right) \mathrm{d}t = \frac{9}{2}t^2 - t^3 (+c)$$

$t = 0,\ s = 3 \Rightarrow c = 3$

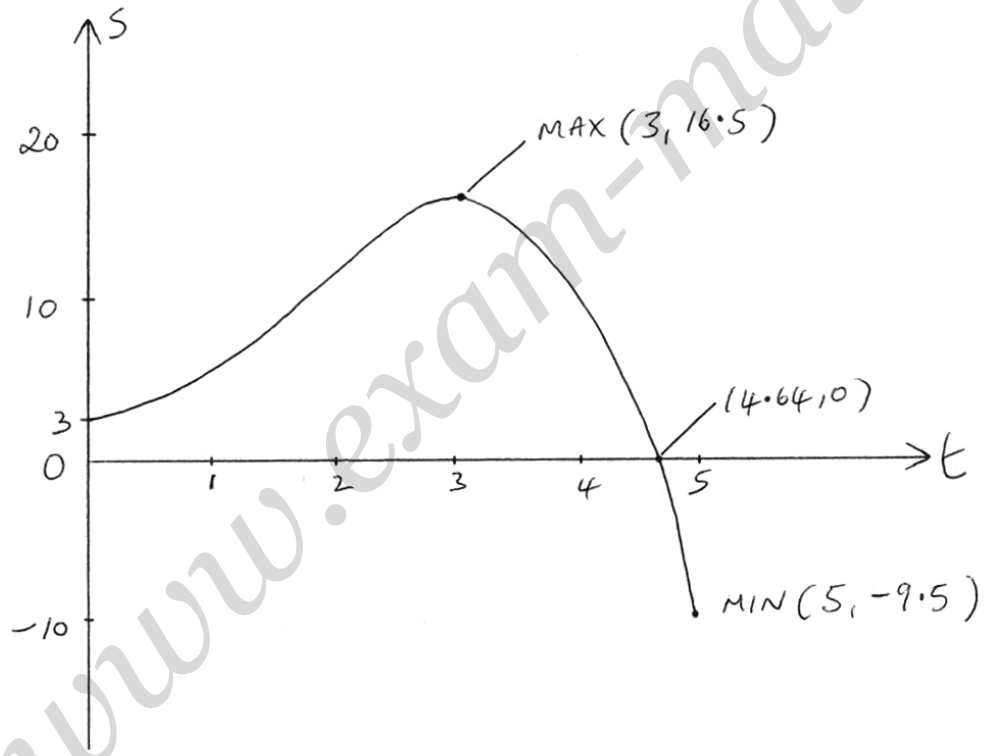
$t = 4 \Rightarrow s = 11$

METHOD 2

$$s = 3 + \int_0^4 \left(9t - 3t^2\right) dt$$

$s = 11$

(b) $s = 3 + \frac{9}{2}t^2 - t^3$



correct shape over correct domain

maximum at $(3, 16.5)$

t intercept at 4.64, *s* intercept at 3

minimum at $(5, -9.5)$

(c) $-9.5 = a + b\cos 2\pi$

$16.5 = a + b\cos 3\pi$

Note: Only award ***M1*** if two simultaneous equations are formed over the correct domain.

$a = \frac{7}{2}$

$b = -13$

(d) at t_1 :

$3 + \frac{9}{2}t^2 - t^3 = 3$

$t^2\left(\frac{9}{2} - t\right) = 0$

$t_1 = \frac{9}{2}$

solving $\frac{7}{2} - 13\cos\frac{2\pi t}{5} = 3$

GDC $\Rightarrow t_2 = 6.22$

10 – (MAT/21_HL_Summer_2015_Q13) – *Kinematics*

(a) (i) $a(t)=\frac{\mathrm{d}v}{\mathrm{d}t}=-10\ (\mathrm{m\,s^{-2}})$

(ii) $t=10 \Rightarrow v=-100\ (\mathrm{m\,s^{-1}})$

(iii) $s=\int -10t\,\mathrm{d}t=-5t^2\ (+c)$

$s=1000$ for $t=0 \Rightarrow c=1000$

$s=-5t^2+1000$

at $t=10,\ s=500\ (\mathrm{m})$

Note: Accept use of definite integrals.

(b) $\frac{\mathrm{d}t}{\mathrm{d}v}=\frac{1}{(-10-5v)}$

(c) **METHOD 1**

$t=\int \frac{1}{-10-5v}\mathrm{d}v=-\frac{1}{5}\ln(-10-5v)(+c)$

Note: Accept equivalent forms using modulus signs.

$t=10,\ v=-100$

$10=-\frac{1}{5}\ln(490)+c$

$c=10+\frac{1}{5}\ln(490)$

$t=10+\frac{1}{5}\ln 490-\frac{1}{5}\ln(-10-5v)$

Note: Accept equivalent forms using modulus signs.

$t=10+\frac{1}{5}\ln\left(\frac{98}{-2-v}\right)$

METHOD 2

$$t=\int\frac{1}{-10-5v}\,\mathrm{d}v=-\frac{1}{5}\int\frac{1}{2+v}\,\mathrm{d}v=-\frac{1}{5}\ln|2+v|(+c)$$

Note: Accept equivalent forms.

$t=10,\ v=-100$

$$10=-\frac{1}{5}\ln|-98|+c$$

Note: If $\ln(-98)$ is seen do not award further A marks.

$$c=10+\frac{1}{5}\ln 98$$

$$t=10+\frac{1}{5}\ln 98-\frac{1}{5}\ln|2+v|$$

Note: Accept equivalent forms.

$$t=10+\frac{1}{5}\ln\left(\frac{98}{-2-v}\right)$$

Note: Accept use of definite integrals.

(d) $5(t-10)=\ln\dfrac{98}{(-2-v)}$

$$\frac{2+v}{98}=-\mathrm{e}^{-5(t-10)}$$

$$v=-2-98\mathrm{e}^{-5(t-10)}$$

(e) $\dfrac{\mathrm{d}s}{\mathrm{d}t}=-2-98\mathrm{e}^{-5(t-10)}$

$$s=-2t+\frac{98}{5}\mathrm{e}^{-5(t-10)}(+k)$$

at $t=10,\ s=500\Rightarrow 500=-20+\dfrac{98}{5}+k\Rightarrow k=500.4$

$$s=-2t+\frac{98}{5}\mathrm{e}^{-5(t-10)}+500.4$$

(f) $t = 250$ for $s = 0$

11 – (MAT/20_HL_Winter_2015_Q9) – *Kinematics*

(a) $t_1 = 1.77(\text{s}) \left(= \sqrt{\pi}(\text{s})\right)$ and $t_2 = 2.51(\text{s}) \left(= \sqrt{2\pi}(\text{s})\right)$

(b) (i) attempting to find (graphically or analytically) the first $t_{\max}$

$t = 1.25(\text{s}) \left(= \sqrt{\frac{\pi}{2}}(\text{s})\right)$

(ii) attempting to find (graphically or analytically) the first $t_{\min}$

$t = 2.17(\text{s}) \left(= \sqrt{\frac{3\pi}{2}}(\text{s})\right)$

(c) distance travelled $= \left|\int_{1.772\ldots}^{2.506\ldots} 1 - \mathrm{e}^{-\sin t^2}\,\mathrm{d}t\right|$ (or equivalent)

$= 0.711(\text{m})$

12 – (MAT/21_HL_Summer_2016_Q3) – *Kinematics*

(a) $v = \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{e}^{-t}}{2 - \mathrm{e}^{-t}} \left(= \frac{1}{2\mathrm{e}^{t} - 1} \text{ or } -1 + \frac{2}{2 - \mathrm{e}^{-t}}\right)$

(b) $a = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = \frac{-\mathrm{e}^{-t}\left(2 - \mathrm{e}^{-t}\right) - \mathrm{e}^{-t} \times \mathrm{e}^{-t}}{\left(2 - \mathrm{e}^{-t}\right)^2} \left(= \frac{-2\mathrm{e}^{-t}}{\left(2 - \mathrm{e}^{-t}\right)^2}\right)$

Note: If simplified in part (a) award ***(M1)A1*** for $a = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = \frac{-2\mathrm{e}^{t}}{\left(2\mathrm{e}^{t} - 1\right)^2}$.

Note: Award ***M1A1*** for $a = -\mathrm{e}^{-t}\left(2 - \mathrm{e}^{-t}\right)^{-2}\left(\mathrm{e}^{-t}\right) - \mathrm{e}^{-t}\left(2 - \mathrm{e}^{-t}\right)^{-1}$.

(c) $a = -2\ (\text{ms}^{-2})$

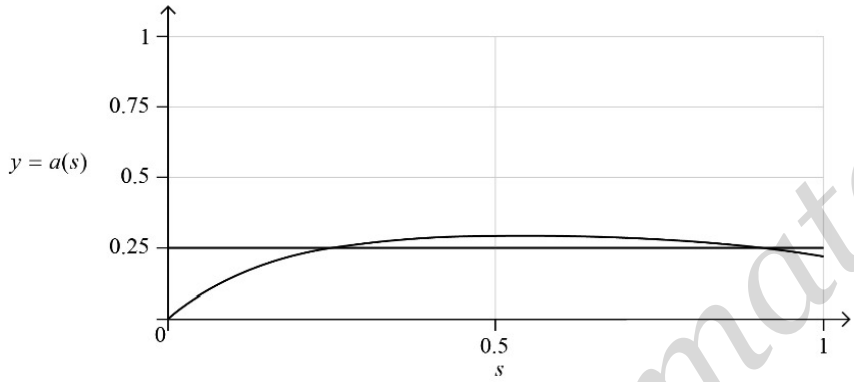
13 – (MAT/22_HL_Summer_2016_Q8) – *Kinematics*

(a) $$\frac{dv}{ds} = \frac{\cos s}{\sin^2 s + 1}$$

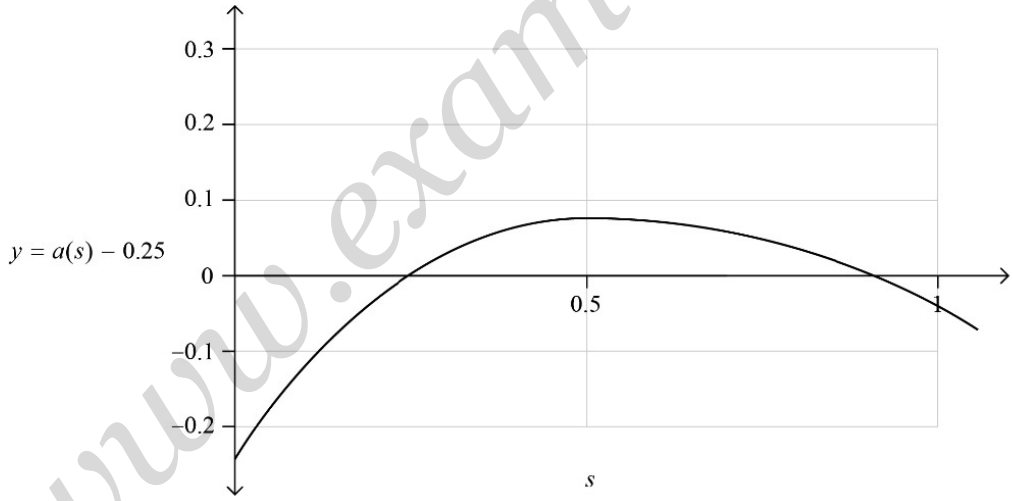
$$a = v\frac{dv}{ds}$$

$$a = \frac{\arctan(\sin s)\cos s}{\sin^2 s + 1}$$

(b) **EITHER**



OR



THEN

$s = 0.296,\ 0.918$ (m)

14 – (MAT/21_HL_Summer_2017_Q11) – *Kinematics*

(a) $v(15) = \dfrac{98}{\sqrt{1+(15-10)^2}}$ ***(M1)***

$v(15) = 19.2\ (\text{ms}^{-1})$ ***A1***

[2 marks]

(b) $\int_0^{10} 9.8t\,\mathrm{d}t$ ***(M1)***

$= 490\ (\text{m})$ ***A1***

[2 marks]

(c) $\dfrac{98}{\sqrt{1+(t-10)^2}} = 2.8$ ***(M1)***

$t = 44.985\ldots(\text{s})$ ***A1***

$h = 490 + \int_{10}^{44.9\ldots} \dfrac{98}{\sqrt{1+(t-10)^2}}\,\mathrm{d}t$ ***(M1)(A1)***

$h = 906\,(\text{m})$ ***A1***

[5 marks]

15 – (MAT/22_HL_Summer_2018_Q7) – *Kinematics*

(a) attempt to solve $v(t) = 0$ for t or equivalent ***(M1)***

$t_1 = 0.441(\text{s})$ ***A1***

[2 marks]

(b) (i) $a(t) = \dfrac{\mathrm{d}v}{\mathrm{d}t} = -\mathrm{e}^{-t} - 16t\mathrm{e}^{-2t} + 16t^2\mathrm{e}^{-2t}$ ***M1A1***

Note: Award ***M1*** for attempting to differentiate using the product rule.

(ii) $a(t_1) = -2.28\left(\text{ms}^{-2}\right)$ ***A1***

[3 marks]

16 – (MAT/22_HL_Summer_2019_Q6) – *Kinematics*

attempt to integrate a to find v ***M1***

$v = \int a \, \mathrm{d}t = \int (2t - 1) \, \mathrm{d}t$

$= t^2 - t + c$ ***A1***

$s = \int v \, \mathrm{d}t = \int (t^2 - t + c) \, \mathrm{d}t$

$= \frac{t^3}{3} - \frac{t^2}{2} + ct + d$ ***A1***

attempt at substitution of given values ***(M1)***

at $t = 6$, $18.25 = 72 - 18 + 6c + d$

at $t = 15$, $922.75 = 1125 - 112.5 + 15c + d$

solve simultaneously: ***(M1)***

$c = -6$; $d = 0.25$ ***A1***

$\Rightarrow s = \frac{t^3}{3} - \frac{t^2}{2} - 6t + \frac{1}{4}$

[6 marks]

17 – (MAT/20_HL_Winter_2020_Q11) – *Kinematics, Sequences - Series*

(a) $\frac{\pi}{6}(=0.524)$ ***A1***

$\frac{\pi}{3}(=1.05)$ ***A1***

[2 marks]

(b) attempt to use integration by parts ***M1***

$s=\int e^{-3t}\sin 6t\ dt$

EITHER

$=-\frac{e^{-3t}\sin 6t}{3}-\int -2e^{-3t}\cos 6t\ dt$ ***A1***

$=-\frac{e^{-3t}\sin 6t}{3}-\left(\frac{2e^{-3t}\cos 6t}{3}-\int -4e^{-3t}\sin 6t\ dt\right)$ ***A1***

$=-\frac{e^{-3t}\sin 6t}{3}-\left(\frac{2e^{-3t}\cos 6t}{3}+4s\right)$

$5s=\frac{-3e^{-3t}\sin 6t-6e^{-3t}\cos 6t}{9}$ ***M1***

OR

$=-\frac{e^{-3t}\cos 6t}{6}-\int \frac{1}{2}e^{-3t}\cos 6t\ dt$ ***A1***

$=-\frac{e^{-3t}\cos 6t}{6}-\left(\frac{e^{-3t}\sin 6t}{12}+\int \frac{1}{4}e^{-3t}\sin 6t\ dt\right)$ ***A1***

$=-\frac{e^{-3t}\cos 6t}{6}-\left(\frac{e^{-3t}\sin 6t}{12}+\frac{1}{4}s\right)$

$\frac{5}{4}s=\frac{-2e^{-3t}\cos 6t-e^{-3t}\sin 6t}{12}$ ***M1***

THEN

$s=-\frac{e^{-3t}(\sin 6t+2\cos 6t)}{15}(+c)$ ***A1***

at $t=0, s=0 \Rightarrow 0=-\frac{2}{15}+c$ ***M1***

$c=\frac{2}{15}$ ***A1***

$$s = \frac{2}{15} - \frac{e^{-3t}\left(\sin 6t + 2\cos 6t\right)}{15}$$

[7 marks]

(c) **EITHER**

substituting $t = \frac{\pi}{6}$ into their equation for s ***(M1)***

$$\left(s = \frac{2}{15} - \frac{e^{-\frac{\pi}{2}}\left(\sin \pi + 2\cos \pi\right)}{15} \right)$$

OR

using GDC to find maximum value ***(M1)***

OR

evaluating $\int_0^{\frac{\pi}{6}} v\,dt$ ***(M1)***

THEN

$$= 0.161 \left(= \frac{2}{15}\left(1 + e^{-\frac{\pi}{2}}\right) \right)$$ ***A1***

[2 marks]

distance required $= \int_0^{1.5} \left| e^{-3t} \sin 6t \right| \, dt$ ***(M1)***

OR

distance required $= \int_0^{\frac{\pi}{6}} e^{-3t} \sin 6t \, dt + \left| \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} e^{-3t} \sin 6t \, dt \right| + \int_{\frac{\pi}{3}}^{1.5} e^{-3t} \sin 6t \, dt$ ***(M1)***

$(= 0.16105... + 0.033479... + 0.006806...)$

THEN

$= 0.201$ (m) ***A1***

METHOD 2

using successive minimum and maximum values on the displacement graph ***(M1)***

$0.16105... + (0.16105... - 0.12757...) + (0.13453... - 0.12757...)$

$= 0.201$ (m) ***A1***

[2 marks]

(e) (i) valid attempt to find $\frac{dv}{dt}$ using product rule and set $\frac{dv}{dt} = 0$ ***M1***

$\frac{dv}{dt} = e^{-3t} 6\cos 6t - 3e^{-3t}\sin 6t$ ***A1***

$\frac{dv}{dt} = 0 \Rightarrow \tan 6t = 2$ ***AG***

(ii) attempt to evaluate t_1, t_2, t_3 in exact form ***M1***

$6t_1 = \arctan 2 \left(\Rightarrow t_1 = \frac{1}{6}\arctan 2 \right)$

$6t_2 = \pi + \arctan 2 \left(\Rightarrow t_2 = \frac{\pi}{6} + \frac{1}{6}\arctan 2 \right)$

$6t_3 = 2\pi + \arctan 2 \left(\Rightarrow t_3 = \frac{\pi}{3} + \frac{1}{6}\arctan 2 \right)$ ***A1***

Note: The ***A1*** is for any two consecutive correct, or showing that $6t_2 = \pi + 6t_1$ or $6t_3 = \pi + 6t_2$.

showing that $\sin 6t_{n+1} = -\sin 6t_n$

eg $\tan 6t = 2 \Rightarrow \sin 6t = \pm\frac{2}{\sqrt{5}}$ ***M1A1***

showing that $\frac{e^{-3t_{n+1}}}{e^{-3t_n}} = e^{-\frac{\pi}{2}}$ ***M1***

eg $e^{-3\left(\frac{\pi}{6}+k\right)} \div e^{-3k} = e^{-\frac{\pi}{2}}$

Note: Award the ***A1*** for any two consecutive terms.

$\frac{v_3}{v_2} = \frac{v_2}{v_1} = -e^{-\frac{\pi}{2}}$ ***AG***

[7 marks]

18 – (MathAA/21_HL_Summer_2021_Q4) – *Kinematics, Relative Velocity*

(a) recognising $v = 0$ ***(M1)***

$t = 6.74416...$

$= 6.74$ (sec) ***A1***

Note: Do not award ***A1*** if additional values are given.

[2 marks]

(b) $\int_0^{10} |v(t)| \mathrm{d}t$ **OR** $-\int_0^{6.74416...} v(t)\,\mathrm{d}t + \int_{6.74416...}^{9.08837...} v(t)\,\mathrm{d}t - \int_{9.08837...}^{10} v(t)\,\mathrm{d}t$ ***(A1)***

$= 37.0968...$

$= 37.1$ (m) ***A1***

[2 marks]

(c) recognising acceleration at $t = 7$ is given by $v'(7)$ ***(M1)***

acceleration $= 5.93430...$

$= 5.93$ (ms^{-2}) ***A1***

[2 marks]

19 – (MathAA/21_HL_Summer_2021_Q9) – *Differentiation, Kinematics, Relative Velocity*

(a) $\tan\theta = \dfrac{50}{y-x}$ OR $\cot\theta = \dfrac{y-x}{50}$ ***A1***

$y = x + 50\cot\theta$ ***AG***

Note: $y - x$ may be identified as a length on a diagram, and not written explicitly.

[1 mark]

(b) attempt to differentiate with respect to t ***(M1)***

$\dfrac{dy}{dt} = \dfrac{dx}{dt} - 50(\operatorname{cosec}\theta)^2\dfrac{d\theta}{dt}$ ***A1***

attempt to set speed of B equal to double the speed of A ***(M1)***

$2\dfrac{dx}{dt} = \dfrac{dx}{dt} - 50(\operatorname{cosec}\theta)^2\dfrac{d\theta}{dt}$

$\dfrac{dx}{dt} = -50(\operatorname{cosec}\theta)^2\dfrac{d\theta}{dt}$ ***A1***

$\theta = \arctan 5 (= 1.373\ldots = 78.69\ldots°)$ OR $\operatorname{cosec}^2\theta = 1 + \cot^2\theta = 1 + \left(\dfrac{1}{5}\right)^2 = \dfrac{26}{25}$ ***(A1)***

Note: This ***A1*** can be awarded independently of previous marks.

$\dfrac{dx}{dt} = -50\left(\dfrac{26}{25}\right) \times -0.1$

So the speed of boat A is 5.2 (ms^{-1}) ***A1***

Note: Accept 5.20 from the use of inexact values.

[6 marks]

20 – (MathAA/21_HL_Summer_2022_Q4) – *Kinematics, Relative Velocity*

(a) recognizing at rest $v = 0$ ***(M1)***

$t = 3.34692...$

$t = 3.35$ (seconds) ***A1***

Note: Award ***(M1)A0*** for any other solution to $v = 0$ eg $t = -0.205$ or $t = 6.08$.

[2 marks]

(b) recognizing particle changes direction when $v = 0$ OR when $t = 3.34692...$ ***(M1)***

$a = -4.71439...$

$a = -4.71\,(\text{ms}^{-2})$ ***A2***

[3 marks]

(c) distance travelled = $\int_0^6 |v|\,\mathrm{d}t$ OR

$$\int_0^{3.34...}\left(e^{\sin(t)} + 4\sin(t)\right)\mathrm{d}t - \int_{3.34...}^{6}\left(e^{\sin(t)} + 4\sin(t)\right)\mathrm{d}t \quad (= 14.3104... + 6.44300...)$$ ***(A1)***

$= 20.7534...$

$= 20.8$ (metres) ***A1***

[2 marks]

21 – (MathAA/22_HL_Summer_2022_Q5) – *Kinematics, Relative Velocity*

(a) recognises the need to find the value of t when $v = 0$ ***(M1)***

$t = 1.5707...\ \left(= \frac{\pi}{2}\right)$

$t = 1.57\ \left(= \frac{\pi}{2}\right)$ (s) ***A1***

[2 marks]

(b) recognises that $a(t) = v'(t)$ ***(M1)***

$t_1 = 2.2627...,\ t_2 = 2.9573...$

$t_1 = 2.26,\ t_2 = 2.96$ (s) ***A1A1***

Note: Award ***M1A1A0*** if the two correct answers are given with additional values outside $0 < t < 3$.

[3 marks]

(c) speed is greatest at $t = 3$ ***(A1)***

$a = -1.8377...$

$a = -1.84$ (m s^{-2}) ***A1***

[2 marks]